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Improved Yokota algorithm for Egyptian fractions

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Abstract

The Yokota algorithm [Yok 88] ***is one of the existing algorithms for generating Egyptian fractions. It defines*** N_k ***as*** $N_k = \underset{i=1}{\overset{k}{\pi}} S_i$ ***where*** $S = \{p^{2^k} \mid k \geq 0 \text{ and } p \text{ is prime}\}$ ***and*** $S_i =$ ***ith smallest element of S. In this paper we define*** N_k ***as*** $N_k = \underset{i=1}{\overset{k}{\pi}} S_i^2$ ***and redesign the algorithm. We discuss the observed changes in the length and denominators of the resulting expansion.***

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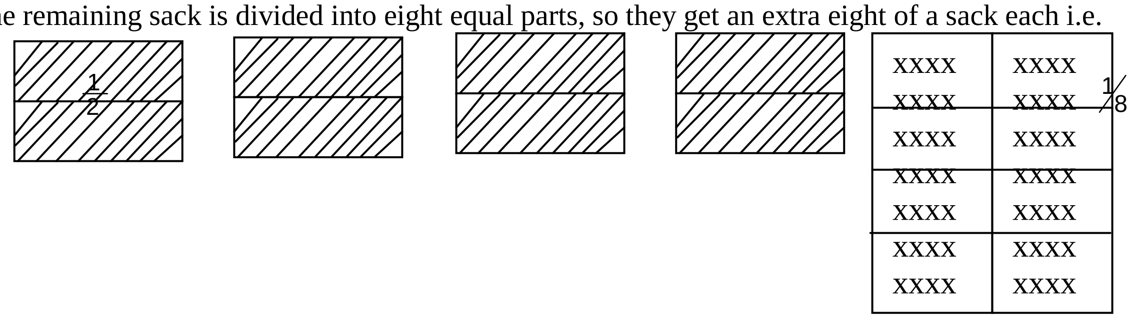
1.0 Introduction:

A sum of positive (usually) distinct unit fractions, that is, an expression of the sum of unit fractions like $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\ldots$, where the denominators a, b, c,… are increasing is called an Egyptian fraction.

The theory of Egyptian fractions gained prominence as early as 4000BC, when ancient Egyptians carried out computations and division of Agricultural product, using Egyptian fractions (Unit fractions). Consider a practical example of a farmer, with five sacks of grain to be shared among eight people working on his farm. First the farmer gives all eight of them half a sack each, with one sack left. Next the remaining sack is divided into eight equal parts, so they get an extra eight of a sack each i.e.



and $\frac{5}{8} = \frac{1}{2} + \frac{1}{8}$. The concept of Egyptian fractions can be generalized for all fractions $\frac{p}{q}, \quad q \neq 0$

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***Theorem* 1**

Every fraction $\frac{p}{q} < 1$ has Egyptian fraction representation and the representation terminates at some stage.

Proof

Consider $\frac{p}{q} < 1$ and if $p = 1$, the problem is solve since $\frac{p}{q}$ is already unit fraction, so our interest is in fractions whose numerators are greater than 1. Our method is to find the biggest unit fraction we can and take it from $\frac{p}{q}$. With what is left, we repeat the process. We will show that this series of unit fractions always decreases, never repeats a fraction and eventually terminates. Let,

$$\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \dots + \frac{1}{n_n} \tag{1.1}$$

where $n_1 < n_2 < n_3 < \dots < n_n$. Chose the largest n_1 at each stage, this implies that $\frac{1}{n_1} < \frac{p}{q}$ but that $\frac{1}{n_1}$ is the largest such fraction. More generally, if $\frac{1}{n_1}$ is the largest unit fraction less than $\frac{p}{q}$ then

$$\frac{1}{n_1 - 1} > \frac{p}{q} \tag{1.2}$$

since $p > 1$, neither $\frac{1}{n_1}$ nor $\frac{1}{n_1 - 1}$ equal $\frac{p}{q}$.

The remainder will then be

$$\frac{p}{q} - \frac{1}{n_1} = \frac{pn_1 - q}{qn_1} \tag{1.3}$$

Also since

$$\frac{1}{n_1 - 1} > \frac{p}{q} \tag{1.4}$$

Multiplying both sides by q we have

$$\frac{q}{n_1 - 1} > p \tag{1.5}$$

or multiplying both sides by $(n_1 - 1)$ and expanding the brackets, then adding p and subtracting q to both sides, we have $\frac{1}{n_1} > \frac{p}{q}$, $\frac{1}{n_1} . n_1 - 1 > \frac{p}{q} . n_1 - 1$, $1 > \frac{p}{q}(n_1 - 1) \Rightarrow q > P(n_1 - 1)$, $q > pn_1 - p$,

$q + p > pn_1 - p + p$, $q + p > pn_1$, $q + p - q > pn_1 - q$

$$p > pn_1 - q \tag{1.6}$$

Observe, $pn_1 - q$ which is the numerator for the remainder is smaller than the original numerator *p*. If the remainder in its lowest term is a unit fraction, we are finished. Otherwise, we can repeat the process on the remainder which has a smaller denominator and so the remainder when we take off the largest unit fraction gets smaller still. Since *p* is a whole (positive) number, this process must inevitably terminate with a numerator of 1 at some stage.

***Theorem* 2**

$\frac{m}{n} = \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_k}$ if and only if there exist positive integers *M* and *N* and divisors $D_1, D_2,...D_k$ of N such that $\frac{M}{N} = \frac{m}{n}$ and $D_1, D_2,...D_k$ =0 (*mod M*). Also, the last condition can be replace by $D_1, D_2,...D_k = M$, and the condition ($D_1, D_2,...D_k$) = 1 may be added without affecting the validity of the theorem [2].

2.0 Definition/Notation

(i) A practical number is an integer *N* > 0 such that for all integers 0 < *n* < *N*, *n* can be written as the sum of distinct divisors of *N*.

For instance, it is easily seen that if *p* can be written as the sum of divisors of *q*, then $\frac{p}{q}$ can be expanded with no denominator greater that q itself. Thus $\frac{9}{20} = \frac{4+5}{20} = \frac{4}{20} = \frac{5}{20} = \frac{1}{5} + \frac{1}{4}$.

***Theorem* 3**

If $N_k = \pi_{i=1}^{k} S_i$ and $r < 2N_k$, then *r* can be written as the sum of distant divisors of N_k.

Proof

The theorem is easily shown to be true for k = 0, 1, or 2. For example if *k* = 2, we have $N_k = 6$ and 2 N_k = 12. So note that 4 = 1 +3, 5 = 2 + 3, 7 = 6 + 1, 8 = 6 + 2. Now suppose the theorem is true for 0, 1, 2, …, k – 1. If $n < 2N_{k-1}$ we are clearly done. So assume $2N_{k-1} \le n < 2N_k$. Note that $2N_k = 2N_{k-1} \,.\, S_k$ so, find S, *r* such that $n = S.S_k + r$ with $S_k \le r < 2S_k$, clearly $r < 2S_k < 2N_{k-1}$ for k > 2 and $S \le 2(N_{k-1} - 1) < 2N_{k-1}$ so we can write $S = \sum d_i^{/}$ and $r = \sum d_i$ where d_i and $d_i^{/}$ are divisors of N_{k-1} and the $d_i^{/}$ and the d_i are distinct. But then, since S_k is not divisible by N_{k-1} we have that $n = \sum (S_k d_i) + \sum d_i$

which is the desired representation of *n*.

(ii) For convenience, we will also define $p_i = ith$ prime number, where $p_1 = 2$,

$(p_2 = 3,\ p_3 = 5, ...)$, $\pi_k = p_1 \,.\, p_2 \,...\, p_k$, $S = \{p^{2k} | k \ge 0 \text{ and } p \text{ is prime}\}$, $S_i = ith$ smallest element of S.

3.0 The Improved Yokota Algorithm:

Define

$$N_k = \underset{i=1}{\overset{k}{\pi}} S_i^2 \tag{3.1}$$

Given: rational $\frac{p}{q} < 1$ in lowest terms

Step 1

We find *k* such that

$$N_{k-1} \le q < N_k$$

Step 2

If $q \mid N_k$ then $\frac{p}{q} = \frac{b}{N_k}$, $[q \mid N_k$ implies q is a divisor of $N_k]$. We can write

$$b = \sum d_i \text{ where all } d_i \mid N_k$$

Step 3

If not, then

$$\frac{p}{q} = \frac{PN_k}{qN_k} = \frac{Sq + r}{qN_k} = \frac{S}{N_k} + \frac{r}{qN_k} \tag{3.2}$$

where

$$\left(1 - \frac{2}{\sqrt{S_k}}\right) N_k \le r < 2N_k \text{ and } \left(1 \le S < N_k\right)$$

The term $\frac{S}{N_k}$ can be done as with $\frac{b}{N_k}$, we can find an expansion for *r* and multiply the denominators by *q*.

Next we shall consider the following examples using first, the Yokota algorithm and then our improved Yokota algorithm. After which we compare the performance of the algorithms based on the length and denominators of the Egyptian fractions produced.

Example

Given the set $S = \{2,3,4,5,7,9,11,13,16,...\}$. Write out the Egyptian fraction for (i) $\frac{16}{17}$, (ii) $\frac{13}{15}$

3.1 Using Yokota Algorithm:

$N_k = \underset{i=1}{\overset{k}{\pi}} S_i \Rightarrow N_1 = 2,\ N_2 = 6,\ N_3 = 24,\ N_4 = 120,\ N_5 = 840$, and so forth. $\frac{16}{17}$: Thus k = 3 and $N_k = 2.3.4 = 24$. So, $\frac{16}{17} = \frac{16(24)}{17(24)} = \frac{22(17) + 10}{17(24)}$.

Note $\left(1 - \frac{2}{\sqrt{S_3}}\right) N_3 = 0$, so this is what we want.

Continuing

$= \frac{22}{24} + \frac{10}{17(24)}$. Now $\frac{22}{24} = \frac{12 + 8 + 2}{24} = \frac{1}{2} + \frac{1}{3} + \frac{1}{12}$ and $\frac{10}{17(24)} = \frac{8 + 2}{17(24)} = \frac{1}{51} + \frac{1}{204}$.

Therefore $$\frac{16}{17}=\frac{1}{2}+\frac{1}{3}+\frac{1}{12}+\frac{1}{51}+\frac{1}{204} \qquad (3.4)$$

3.2 **Using improved Yokota Algorithm:**

$N_k = \overset{k}{\underset{i=1}{\pi}} S_i^2 \Rightarrow N_1 = 4,\ N_2 = 36,\ N_3 = 576$ and so forth. $\frac{16}{17}$: Thus k = 2 and $N_k = 36$, so

$\frac{16}{17}=\frac{16(36)}{17(36)}=\frac{33(17)+15}{17(36)}$

Note $\left(1-\frac{2}{\sqrt{S_3}}\right)N_2 \le 15 < 2N_2$ i.e $-5 \le 15 < 72$ as required. Continuing

$=\frac{33}{36}+\frac{15}{17(36)}$. Again $=\frac{33}{36}=\frac{18+12+3}{36}=\frac{1}{2}+\frac{1}{3}+\frac{1}{12}$ and $\frac{15}{17(36)}=\frac{9+6}{17(36)}=\frac{1}{68}+\frac{1}{102}$

Thus $$\frac{16}{17}=\frac{1}{2}+\frac{1}{3}+\frac{1}{12}+\frac{1}{68}+\frac{1}{102} \qquad (3.5)$$

3.3 **Using Yokota Algorithm:**

$\frac{13}{15}$: Thus, $k > 3,\ N_k = 24$, so $\frac{13}{15}=\frac{13(24)}{15(24)}=\frac{20(15)+12}{15(24)}$

Note $\left(1-\frac{2}{\sqrt{S_3}}\right)N_k \le r < 2N_k$.

Continuing

$=\frac{20}{24}+\frac{12}{15(24)}$. Now $=\frac{20}{24}=\frac{12+6+2}{24}=\frac{1}{2}+\frac{1}{4}+\frac{1}{12}$ and $\frac{12}{15(24)}=\frac{8+4}{15(24)}=\frac{1}{45}+\frac{1}{90}$

Thus $$\frac{13}{15}=\frac{1}{2}+\frac{1}{4}+\frac{1}{12}+\frac{1}{45}+\frac{1}{102} \qquad (3.6)$$

3.4 **Using Improved Yokota Algorithm:**

$\frac{13}{15}$: Thus $k > 2,\ N_k = 36$, so $\frac{13}{15}=\frac{13(36)}{15(36)}=\frac{31(15)+3}{15(36)}$.

Note $\left(1-\frac{2}{\sqrt{S_3}}\right)N_k \le r < 2N_k$. Continuing,

$=\frac{31}{36}+\frac{3}{15(36)}$. Now $\frac{31}{36}=\frac{18+9+4}{36}=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}$ and $\frac{3}{15(36)}=\frac{1}{180}$. Thus

$$\frac{13}{15}=\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{180} \qquad (3.7)$$

4.0 Conclusion

From (3.5) and (3.6) both definitions yield the same number of terms, but $N_k = \prod_{i=1}^{k} S_i^2$ yields solutions with smaller denominators. While in (3.7) $N_k = \prod_{i=1}^{k} S_i^2$ gave the least number of terms. Since the performance, of an Egyptian fraction algorithm is based on the length and denominators of the unit fractions produced. The compassion above shows that the definition

$N_k = \prod_{i=1}^{k} S_i^2$ perform better than $N_k = \prod_{i=1}^{k} S_i$.

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