Journal of the Nigerian Association of Mathematical Physics, Volume 9 (November 2005)

Comparing parallel square root method with extended Bell's class of iteration method for simultaneous determination of polynomial zeros.

Stephen Ehidiamhen Uwamusi
Department of Mathematics,
University of Benin, Benin City, Nigeria.
e-mail address: uwamusi 2000@yahoo.com

Abstract

We extend the Bell's polynomial disks appearing as correction terms in the generalized Wang ***and*** Zheng ***iterative formula for simultaneous determination of zeros of polynomial. We synchronize our numerical result as a comparison with result obtained from the q-step parallel square root iteration formula. A significant conclusion is drawn from the results of these two methods.***

Keywords: Polynomial Zeros, Circular complex arithmetic Bell's Polynomial disks, Parallel square root iteration.

pp 307 - 316

1.0 Introduction

One often uses interval arithmetic for the construction of inclusion methods. As initial values of such methods, one often assumes, that, given inclusion intervals $\zeta_1 \in z_1^{(0)},\ \zeta_2 \in Z_2^{(0)},\ \ldots,\ \zeta_s \in Z_s^{(0)}$ for the roots which are to be iteratively and simultaneous improved with the desired limit given by $\lim\limits_{k\to\infty} Z_i^{(k)} = \zeta_i$, $(1 \le i \le s)$, interval method has no problem proving that the solution exists.

Readers are expected to be familiar with the operation of machine interval arithmetic. Petkovic [9] is an excellent monograph behind such reference. The basic properties of such applications of interval methods are in

(a) inclusion monotonicity,
(b) inclusion property,
(c) convergence, and
(d) estimation of the R-order of convergence.

One major advantage which interval arithmetic enjoys is inclusion isotonicity property (see e.g. Carstensen and Petkovic [2]). This means that if $z \in Z$

Then
$$\Phi(z) \in \Phi(Z), \tag{1.1}$$
where Φ is an operational weight of the iterative process.

An interval extension, Moore [7], can be performed by the substitution of all argument in the expression of formulas by the intervals and carrying out all occurring operations in interval arithmetic. Such interval extensions usually have such characteristics

$$d\left(\Phi\left(Z_1, Z_2, \ldots, Z_{i-1}, z_i, Z_{i+1}, \ldots, Z_s, p(z_i), p'(z_i) \ldots, p^{(m-1)}(Z_i)\right)\right)$$
$$\le c d(Z_i)^p \left(\sum_{\substack{j=1\\ j\ne i}}^{s} d(Z_j)\right)^{\beta},$$

where

$$p,\ \beta \in N \setminus \{0\},\ 1 \le i \le s. \qquad (1.2)$$

In method (1.2), $d(Z) = d([Z_1, Z_2]) = z_1 - z_2$ is the width of interval Z. For details, see Moore [7], Alefeld and Herzberger [1], and Petkovic [9].

An iterative process approaching z^* must terminate when all the significant digits of $p(z)$ are lost. Thus the total number of significant digits lost in evaluating $p(z)$ is the sum of the number of digits lost in evaluating each of individual factor of $p(z)$.

Let p be the order of an iterative process converging to z^* and m be multiplicity of the zeros of $p(z)$ in the sense of Lagouanelle's limiting formula (Farmer and Loizou [3]). Then for $p \le m$ it can be shown in the sense of Alefeld and Herzberger [1] that

$$\left|Z^{(k+1)} - z^*\right| \le \left(1 - \frac{p-1}{(m+p-2)}\right) \left|z^{(k)} - z^*\right| \qquad (1.3)$$

converges linearly to z^*.

Thus the iterative process comes to an end when $Z^{(k)}$ has s/m more correct digits than does $Z^{(0)}$ and this is approximately estimated to be

$$\frac{\left|Z^{(k)} - z^*\right|}{\left|Z^{(0)} - z^*\right|} \cong t^{-s/m}, \qquad (1.4)$$

and t is the base arithmetic (see e.g. Stoer and Bulirsch [8], Chapter one for details). In the case of simple zeros, i.e., $m = 1$ the relationships (1.3) and (1.4) hold verbatim.

We remark that all algorithms discussed in this paper are based on fixed point relation which enables the construction of interval methods. Since these algorithms make use of disk in the form of complex plane, we will give a brief exposition of theoretical foundation of circular complex arithmetic due to Gargantini and Henrici [4]. For this, we review:

Let

$$Z = \{z : |z - c| \le r\}$$

with centre $c = mid(Z)$ and radius $r = rad(Z)$. We denote a disk Z by the parametric notation

$$Z = \{c, r\}. \text{ If } Z_i = \{c_i, r_i\}, (i = 1, 2)$$

then as in Gargantini and Henrici [4], we have

$$Z_1 \pm Z_2 = \{c_1 \pm c_2, r_1 + r_2\},$$

$$Z_1 \cdot Z_2 = \{c_1 c_2,\ |c_1| r_2 + |c_2| r_1 + r_1 r_2\},$$

$$Z^{-1} = \{c, r\}^{-1} = \frac{\{\bar{c}, r\}}{|c|^2 - r^2},\ (|c| > r),$$

and

$$(0 \notin Z);\ Z_1 \div Z_2 = Z_1 \cdot Z_2^{-1} \quad (0 \notin Z_2).$$

Finally we note that $\{c_1, r_1\} \subseteq \{c_2, r_2\}$ if and only if $|c_1 - c_2| \le r_1 - r_2$. We will adopt also in our work the disk inversion due to Carstensen and Petkovic [2] as follows

$$Z^{-1}=\{c,\ r\}^{-1}=\left\{\frac{1}{c\left(1-\frac{r^2}{|c|^2}\right)},\ \frac{r}{|c|^2-r^2}\right\},\ Z^{I_1}=\{c,\ r\}^{I_1}=\left\{\frac{1}{c},\ \frac{r}{|c|\left(|c|-r\right)}\right\},$$

$$Z^{I_2}=\{c,\ r\}^{I_2}=\left\{\frac{1}{c},\frac{2r}{|c|^2-r^2}\right\}.$$

Thus the inclusion monotonicity of $Z^{-1}\subseteq Z^{I_1}\subseteq Z^{I_2}$ is valid for any disk inversion. In this paper we will only use Z^{I_2} for our purpose. Finally, we remark that if $c\in Z$ it implies that $|c|\le|mid(Z)|+rad(Z)$. The organization of the paper is as follows: In section two, we give theoretical backgrounds of Bell's class of polynomials which appear as the correction term of Wang and Zheng iterative Formula [11] for the simultaneous determination of zeros of polynomial. In section three, an extension of Bell's polynomials is made using certain technique to obtain higher order methods from the generalized Wang and Zheng method [11]. In section four, the parallel square root method and its modification have been introduced as comparison with our method in section three. Finally, numerical example is illustrated with these methods and a significant conclusion is drawn.

2.0 Theoretical Backgrounds.

The preliminary exposition of Bell's class of polynomials can be found in Wang and Zheng [11], Petkovic [9], Kolbig and Stramp [5]. However for easy orientation we mention briefly the characteristics of Bell polynomial given by

$$B_{t,i}\left(Z_{1,i},\ \ldots,\ Z_{t,i}\right)=\sum_{h=1}^{t}\ \sum\frac{1}{q_1!\ q_2!\ldots\ q_n!}\left(\frac{Z_{1,i}}{1}\right)^{q_1}\ldots\left(\frac{Z_{h,i}}{h}\right)^{q_h},\ \left(B_{0,i}=1\right).\qquad(2.1)$$

The second term on the right of (2.1) runs over h-partitions $q_1,\ q_2,\ldots,q_t$ of t which satisfies the pair of equations $q_1+2q_2+\ldots+hq_h=t,\ q_1+q_2+\ldots+q_h=t, 1\le h\le t$.

We introduce the function

$$\delta: D_0\subset Z\to\frac{\Delta_{t-1}(z)}{\Delta_t(z)},\qquad(2.2)$$

$(t=1,\ 2)$ as Wang and Zheng [11] iterative method as follows for the third order Halley's formula

$$\Delta_t(z)=\frac{1}{p(z)}\sum_{h=1}^{t}(-1)^{h+1}\frac{p^{(t)}(z)}{t!}\Delta_{t-h}(z),\ \Delta_0(z)=1.\qquad(2.3)$$

The $p^{(t)}(z)$ in method (2.3) signifies the t^{th} derivative $\frac{d^t\ p(z)}{dz^t}$.

For instance, taking $n=1$ and 2 will result to the following expression

$$\Delta_1(z) = p'(z)/p(z),\ \Delta_2(z) = \left(\frac{p'(z)}{p(z)}\right)^2 - \frac{p'(z)}{2p(z)}.$$

We introduce the notation

$$c_{t,j}(z) = \sum_{\substack{j=1 \\ j\neq i}}^{t} \left(z_i - \zeta_j\right)^{-t},\ (t-1,2), \tag{2.4}$$

to represent the logarithmic derivatives of $p(z)$ in the case of simple zeros. Then following procedure of Wang and Zheng [11], an iterative method can be obtained

$$Z_i^{(k+1)} = z_i^{(k)} - \frac{\Delta_{t-1}\left(z_i^{(k)}\right)}{\Delta_t\left(z_i^{(k)}\right) - B_t\left(c_{1,i}(z) + c_{2,i}(z)\right)}. \tag{2.5}$$

A hybrid interval method which combines classical Weierstrass interval methods with a refining point iterative Meahly third order method has been constructed in the work of Uwamusi and Otunta [10].

In this paper we are only concerned with the construction of extended higher Bell's polynomial disk for the simultaneous determination of zeros of the polynomial.

4.0 Experimental

The extended Bell's polynomial disks as correction term for method (2.5). The method under investigation is an extension of Bell's polynomial disks appearing in the correction term of the generalized Wang and Zheng [11] formula. The first two Bell's polynomial disks are

$$B_1(z) = C_{1j}(z) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{\left(z - Z_j\right)} \tag{3.1}$$

$$B_2\left(C_{1,j}(z),\ C_{2,i}(z)\right) = \frac{1}{2}C_{1,i}{}^2(z) + \frac{1}{2}C_{2,i}(z) = \frac{1}{2}\Big(\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{1}{z_i - Z_j}\Big)^2 + \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{1}{(z_i - Z_j)^2} \tag{3.2}$$

We define

$$B_{t,i}(z) = B_{t,i}\left(c_{1,i}(z),\ldots,c_{t,i}(z)\right) = \frac{1}{t}\sum_{h=1}^{t} C_{t,i}(z)\,B_{t-1,i}(z),\ \left(B_0(z) = 1\right) \tag{3.3}$$

Taking t to be 1 and 2 in (3.3) will lead to the following expressions

$$B_{1,i}(z) = C_{1,i}(z)\,B_0(z) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_i - Z_j} \tag{3.4}$$

$$B_{2,i}(z) = \frac{1}{2}\left(C_{2,i}(z)\,B_1(z),\ C_{2,i}(z)\,B_{0,i}(z)\right) = \frac{1}{2}\left(C_{2,i}(z)B_{1,i}(z) + C_{2,i}(z)B_{0,i}(z)\right) \tag{3.5}$$

If we plug equations (3.4) and (3.5) into (2.5) we will obtain.

$$Z_i^{(k+1)} = z_i^{(k)} - \frac{\Delta_1\left(z_i^k\right)}{\Delta_2\left(z_i^{(k)}\right) - \frac{1}{2}\left(\sum_{\substack{j=1\\j\neq i}}^{n} \frac{1}{\left(z_i^{(k)} - Z_j^{(k)}\right)^2} \sum_{\substack{j=1\\j\neq i}}^{n} \frac{1}{\left(z_i^{(k)} - Z_j^{(k)}\right)} + \sum_{\substack{j=1\\j\neq i}}^{n} \frac{1}{\left(z_i^{(k)} - Z_j^{(k)}\right)^2}\right)} \tag{3.6}$$

The error estimate in method (2.5) can be seen from the introduction of variable: Substituting some approximations $z_1, z_2, \ldots, z_n$ instead of the exact zeros $\zeta_1, \zeta_2, \ldots, \zeta_n$ for $Z_1, Z_2, \ldots, Z_n$ in method (2.5) and taking $z = z_i$, a new approximation $\hat{z}$ for ζ_i can be obtained in the form

$$\hat{Z}_i = z_i - \frac{\Delta_{t-1}(z_i)}{\Delta_t(z_i) - B_t\left(C_{1,i}{}^*(z_i)\ldots C_{t,i}(z_i)\right)}, (i = 1, 2, \ldots, n) \tag{3.7}$$

It follows that

$$B_{t,i}^*(z_i) = B_t\left(C_{1,i}{}^*(z_i), \ldots, C_{t,i}^*(z_i)\right) \text{ and } C_{t,i}(z_i) = \sum_{\substack{j=1\\j\neq i}}^{n} \left(z_i - Z_j\right)^{-t}$$

$$C_{t,i}^*(z_i) = \sum_{\substack{j=1\\j\neq i}}^{n} \left(z_i - Z_j\right)^{-t}.$$

Setting $\in = z_i - \zeta_i$, $\hat{\in}_i = \hat{z}_i - \zeta_i$ and taking $|\in| = \max_i |\in_i| = \max_i |\hat{\in}_i|$, after an extensive but elementary analysis we will obtain

$$B_{t,i}^*(z) - B_{t,i}(z) = 0_t\left(\sum_{j\neq i} \varepsilon_j\right) = 0_t(\varepsilon). \tag{3.8}$$

We assume that $\varepsilon_i = 0_t(\varepsilon_j)$ $\forall_{i,i} \in \{1, 2, \ldots, n\}$. But $p(z_i) \sim \varepsilon_i \sim \varepsilon$ and by definition, $\Delta_t(z_i) \sim \frac{1}{\varepsilon_i{}^t} \sim \frac{1}{\varepsilon^t}$. Then it is easy to see by induction that the error term in the correction term of Wang and Zheng [11] formula (2.5) propagates in the form.

$$\frac{\Delta_{t-1}(z_i)}{\Delta_t(z_i)} \sim \varepsilon_i \sim \varepsilon. \tag{3.9}$$

Algebraic manipulation will result in the form

$$\hat{\varepsilon}_i = \hat{z}_i - \varepsilon_i = z_i - \frac{\Delta_{t-1}(z_i)}{\Delta_t(z_i) - B_t^*(z_i)} - z_i + \frac{\Delta_{t-1}(z_i)}{\Delta_t(z_i) - B_t(z_i)}$$

$$= \frac{\Delta_{t-1}(z_i)}{\Delta_t(z_i)} \cdot \frac{B_{t,i}^* - B_{t,i}}{\Delta_t(z_i)\left(1 - \frac{B_{t,i}^*}{\Delta_t(z_i)}\right)\left(1 - \frac{B_{t,i}}{\Delta_t(z_i)}\right)}.$$

Using (3.8) and (3.9) we see that

$$\hat{\varepsilon} \sim \varepsilon_i^{(t+1)} \sum_{j \neq i} \varepsilon_j = \varepsilon_i^{(t+2)} \tag{3.10}$$

Setting α to be a positive constant, then, (3.10) can be written as

$$|\hat{\varepsilon}| = \alpha |\varepsilon_i|^{(t+2)} \tag{3.11}$$

which explains the R-order of convergence of the iterative method (2.5).

4.0 Comparison with other Favourite Interval Method-The Parallel Square Root Method.

We introduce the function at the point $z_k \to h_k(z)$ by the relation

$$h_t(z) = \frac{(-1)^{t-1}}{(t-1)!} \frac{d^t}{dz^t} (\log_e p(z)),$$

$$(t = 1, 2) \tag{4.1}$$

Denoting the second Newton's sum evaluated at the point z by $h_2(z)$ and using the fact that $t \geq 2$ then the first unknown roots ζ can be written in the form.

$$(z - \zeta_i)^2 = \frac{1}{h_2(z) - \sum_{j=2}^{n} \frac{1}{(z - \zeta_j)^2}}. \tag{4.2}$$

Relation (4.2) gives the two complex zeros for ζ_1 when all the remaining distinct zeros $\zeta_2, \zeta_3, \ldots, \zeta_n$ are known. Let z be an approximate value to ζ_1, *i.e*, $z = z^{(0)}$ and replace the point ζ_j by the disk $Z_j^{(0)}$ we obtain from (4.2) the parallel square Root method due to Gargantini and Henrici [4] in the form

$$Z_i^{(k+1)} = z_i^{(k)} - \frac{1}{\left(h_2(z^{(k)}) - \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{1}{z_i^{(k)} - z_j^{(k)}}\right)^2\right)^{1/2}},$$

$$(k = 0, 1, \ldots, i = 1, 2, \ldots, n). \tag{4.3}$$

The choice of the appropriate value of square root to be taken is answered in a monograph by Petkovic [9].

Let

$$r = \max_{1 \leq j \leq n} r_j$$

$$B(v,n) = \begin{cases} 2n \ , \ v = 1 \\ v(n-1), \ v > 1 \end{cases} \quad , \tag{4.4}$$

$$\ell = \min_{\substack{i,j \\ i \neq j}} \{|z| : z \in z_i - z_j\} = \min_{\substack{i,j \\ i \neq j}} \{|z_i - z_j| - r_j\}. \tag{4.5}$$

Let the condition

$\ell > B(v,n)r$ hold, then we have

***Theorem* 1** (Petkovic [9])

If the above condition holds then the value of the t^{th} *root to be chosen in the parallel square root method (4.3) for* $z = z_i$ *is that which satisfies*

$$\left| \frac{p'(z_i)}{p(z_i)} - h_i^{1/t} \right| \leq \frac{n-1}{\ell}.$$

Proof: See Petkovic [9].

We will make a comparison with the modified square root method which uses q-steps earlier proposed by the author else where. Its representation is given by

$$Z_i^{\left(k+\frac{\lambda+1}{q}\right)} = z_i^{\left(k+\frac{\lambda}{q}\right)} - \frac{1}{\left[h_2 z_i^{\left(k+\frac{\lambda}{q}\right)} - \sum_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{1}{z_i^{(k+\frac{\lambda}{q})} - Z_j^{(k+\frac{\lambda+1}{q})}} \right)^2 \right]^{1/2}} \tag{4.7}$$

$(k = 0, 1, \ldots, \ \lambda = 0,1\ldots,q-1, \ q = 2,3,\ldots,v, \ and \ v$ is a positive integer$)$. Usually, we take $q = 2$. For $t = 2$, following Petkovic [9] the following results are inductive and valid

$$\left| mid\left(\frac{1}{z_i - Z_j} \right)^2 - \frac{1}{(z_i - Z_j)^2} \right| + rad \ \frac{1}{(z_i - Z_j)^2} = \left| \frac{(z - z_j)^2}{\left(|z_i - z_j|^2 - r_j^z \right)} - \frac{1}{(z_i - z_j)^2} \right|$$

$$+ \sum_{\lambda=1}^{2} \frac{\binom{2}{\lambda} |z_i - z_i|^{2-\lambda} \ r_j^{\lambda}}{\left(|z_i - z_j|^2 - r_j^2 \right)^2} = \frac{\left(1 - \frac{r}{|z_i - z_j|} \right)^2}{\left(|z_i - z_j| - r_j \right)^2} < 1 - \frac{\left(1 - \frac{r_j}{(\ell - r)^2} \right)}{(\ell - 2r)^2} < \frac{2r_j}{(\ell - r)(\ell - 2r)^2}$$

By inclusion isotonicity property of circular arithmetic we have

$$\left(\frac{1}{z_i - Z_j} \right)^2 \subset \left\{ \frac{1}{(z_i - z_j)^2} \ , \ \frac{2r_j}{(\ell - r)(\ell - 2r)^2} \right\}$$

Therefore,

$$\left(\frac{1}{z_i - Z_j} \right)^2 \subset \left\{ \frac{1}{(z_i - z_j)^2} \ , \ \frac{2r_j}{\ell^3} \right\}.$$

The implication of this is that multiplication of two disk is, in general a region enlarged to be a disk. Thus in carrying out the square of inverse operation of a disk it is recommended that the inverse operation should be performed before squaring in order to avoid unwanted zeros.

5.0 Numerical Experiment

Consider the problem taken from Gargantini and Henrici [4].

***Problem* 1**:

$$p(z) = z^3 - 2z^2 - z + 2$$

with $Z_1^{(0)} = [2.2,\ 0.3]$, $Z_2^{(2)} = [0.9,\ 0.2]$, $Z_3^{(0)} = [-0.9,\ 0.3]$.

We present our numerical findings in the following tables.

Table 1: q-steps for square Root method

$Z_1^{1/2}$	= {2.00025768, -7.81835 E^{-4}}
$Z_2^{1/2}$	= {1.000024602, -5.0497 E^{-5}}
$Z_3^{1/2}$	= {-0.999999985, -3.9 E^{-8}}
$Z_1^{(1)}$	= {2.000000001, 8.631845019 E^{-16}}
$Z_2^{(1)}$	= {1.000000000 , 0}
$Z_3^{(1)}$	= {-1.000014985, 0}

Table 2: The Extended Bell's polynomial as correction term for method (2.5).

$Z^{(1)}$	= {2.002091863, -5.350009E^{-3}}
$Z_2^{(1)}$	= {0.999924336, 6.09662E^{-4}}
$Z_3^{(1)}$	= {-0.999398679, 2.80296E^{-4}}
$Z_1^{(2)}$	= {2.000000355, 6.536922434 E^{-10}}
$Z_2^{(2)}$	= {1.000075737, 4.905671975 E^{-10}}
$Z_3^{(2)}$	= {-0.99939904, 3.109281601 E^{-17}}

***Problem* 2:**

Consider the exponential polynomial taken from Makrelov and Semendziev [6]:

$$E_n(z) = a_0 + \sum_{j=1}^{n} \left(a_j\, e^{-iz} + b_j e^{iz}\right)$$

which reduces to algebraic polynomial

$$E_n(\omega) = \omega^{2n} + C_{2n-1}\,\omega^{2n-1} + \ldots + C_1\omega + C_0$$

by substituting $e^z = \omega$. The coefficients C_j are given by

$$C_j = \frac{a_{n-1}}{b_n} \quad , \; j = 0, 1, \ldots, n\,,\; C_{n+j} = \frac{b_j}{b_n} \quad , \; j = 1,\; 2\ldots\,.$$

The case $n = 2$ will give

$$E_2(z) = a_0 + a_1 e^{-z} + b_1 e^z + a_2 e^{-2z} + b_2 e^{-2z}\,,$$

where $a_0 = e^3 + e^{-3} + pq$,

$a_1 = -\left(e^{7/2}p + e^{1/2}q\right)$, $b_1 = -\left(e^{-7/2}p + e^{-1/2}q\right)$, $a_2 = e^4$, $b_2 = e^{-4}$, $p = 2\cosh\dfrac{3}{2}$

$q = 2\cosh\dfrac{1}{2}$. The transformed polynomial after normalization is given by

$$E_2(\omega) = \omega^4 - 82.44062249\omega^3 + 1678.667985\omega^2 - 8709.524030\omega + 2980.957987$$

The initial approximations are given by

$$\omega^{(0)} = 1,\; \omega_2^{(0)} = 10,\; \omega_3^{(0)} = 20,\; \omega_4^{(0)} = 60\,.$$

The exact zeros of E_2 are.

$\zeta_1 = -1,\; \zeta_2 = 2,\; \zeta_3 = 3,\; \zeta_4 = 4$, we set $\zeta_i = z = \log_e(\omega_i)$.

The following results are presented in table 3.

Table 3: Extended Bell's polynomial as correction term for method (2.5).

$Z_1^{(1)}$	= 0.377568637
$Z_2^{(1)}$	= 7.461537202
$Z_3^{(1)}$	= 20.08551979
$Z_4^{(1)}$	= 54. 20644948
$Z_1^{(2)}$	= 0.367879461
$Z_2^{(2)}$	= 7.389058437
$Z_3^{(2)}$	= 20.08553099
$Z_4^{(2)}$	= 54.59801742
$Z_1^{(3)}$	= 0.367879477
$Z_2^{(3)}$	= 7.389058303
$Z_3^{(3)}$	= 20.08553099
$Z_4^{(3)}$	= 54.59802612

6.0 Conclusion

The interval square root method presented as method (4.7) which uses q-step is more efficient compared to (3.6) with respect problem1 but was disastrously tested with problem 2. In problem 2, our method (3.6) gave quite accurate result. Thus method (3.6) is the most efficient in the class of iterative interval procedures for the improvement of all polynomial zeros simultaneously. Furthermore, the algorithm can compete favourably with the hybrid point interval algorithms (see for instance, Uwamusi and Otunta [10]. The main disadvantage of square root method lies in the fact that the real zeros of the polynomial which are being sought may branch into complex plane even though the roots of $p(z)$ are real. It is advisable to halt the iteration when the termination accuracy is a little greater than the limit precision of the employed arithmetic. This means that the interval procedure is repeated for $k+1, k+2, \ldots$, until for some iteration index the termination criterion given by the inequalities

$$rad\, Z_i^{(u_0)} < \varepsilon$$

is satisfied for all $k = 0, 1, \ldots, n$ and the given accuracy ε. Then behold, the remaining disk $Z_i^{(u_0)}$ will contain the wanted zeros ζ_i $(i = 1, 2, \ldots, n)$.

References

[1] G. Alefeld and J. Herzberger, Introduction to Interval Computation, Academic Press, New York (1983).

[2] C. Carstensen and M.S. Petkovic, An improvement of Gargantini's Simultaneous inclusion method for polynomial roots by Schroder's correction. Applied Numerical Mathematics 13 (1994) 453-468.

[3] M. R. Farmer and G. Loizou, An algorithm for the total, or Partial, factorization of a Polynomial, Math. Proc. Cambridge Philos. Soc. 82 (1977) 427-437.

[4] I. Gargantini and P. Henrici, Circular arithmetic and the determination of Polynomial zeros; Numer. Math. 18 (1972) 305-320.

[5] K. S. Kolbig and W. Stramp, Some infinite integrals with powers of logarithms and the complete Bell Polynomials. JCAM 69 (1996) 39-47.

[6] I.Makrelov and K.I. Semendziev, On the convergence of two methods for the simultaneous finding of all roots of exponential equation..IMA J.Numer.Anal.5(1985)191-200.

[7] R.E. Moore., Interval analysis, Englewood Cliffs, Prentice Hall, New York (1966).

[8] J. Stoer and R. Bulirsch, Introduction to Numerical Analysis, Springer, Berlin (1980).

[9] M. S. Petkovic, Iterative Methods for simultaneous inclusion of Polynomial zeros, Springer, Berlin (1989)

[10] S. E. Uwamusi and F. O. Otunta, A hybrid method for guaranteed simultaneous determination of zeros of Polynomial equations. Arabian J. for Science and Engineering, 27, 2A (2002) 207- 211.

[11] X. Wang and S. Zheng, A family of Parallel and interval iterations for finding Simultaneously all roots of a polynomial with rapid convergence. International J. Compute. Math 4 (1984) 70- 76.