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A fourth order exponentially-fitted multiderivative method for Stiff initial value problems

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Abstract

A class of fourth order exponential–fitted multiderivative method for the numerical integration of stiff initial value problems is designed. The method is derived with certain free parameters 'a' and 'b', which allow it to be fitted automatically to exponential functions. The formula has been implemented and preliminary numerical results indicate that the approach compares favourably with other existing methods that have solved the same set of stiff problems. Finally the graphical comparison of the numerical results is displayed after each problem.

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1.0 Introduction

We shall consider Initial Value Problems in ordinary differential equation of the type,

$$y' = f(x, y),\ y(x_0) = y_0\,;\ a \le x \le b, \quad y \in R \tag{1.1}$$

It is assumed that problem (1.1) has a unique solution y(x) ϵ R.
The derivation of our scheme is based on the idea of exponentially fitted formula.

***Definition* 1**

A numerical integration method is said to be exponentially fitted at a complex value $\lambda = \lambda_0$, if when the method is applied to the scalar test problem $y' = \lambda y$, $y(0) = y_0$ with exact initial conditions, the characteristic equation $\phi(\lambda h)$ satisfies the relation;

$$\phi(\lambda_0 h) = e^{\lambda_0 h} \tag{1.2}$$

The idea of using exponentially fitted scheme for the approximate numerical integration of certain classes of first order initial value problems in ordinary differential equations of the form (1.1) above has received considerable attention in recent years.

The basic reason behind exponentially fitting, which was originally proposed by Liniger and Willoughby (1970) [10], is to derive integration schemes containing free parameters.

These parameters are chosen so that a given exponential function e^q, where q is real, satisfies the integration formula exactly. As a result of this idea, Cash (1981) [4], attempted using second derivative formulas with step number k = 1, and 2, respectively, to derive exponentially fitted schemes, applying scalar test problem;

$$y' = \lambda y, \quad y(0) = 1,\ \lambda \in C,\ \mathrm{Re}(\lambda) < 0. \tag{1.3}$$

and set $q = \lambda h$

Okunuga (1994, 1997) [[11, 12], derived exponentially fitted schemes using second derivatives formula. His work showed that the methods are stable. Herein, we shall derive exponentially fitted scheme using third derivative formula, with adequate stability characteristics to

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cope with stiff systems. Our proposed scheme employs the procedure of predictor –corrector forms.

2.0 Development of the Integration Formula

The general multiderivative multistep method is given by,

$$\sum_{i=0}^{k} \alpha_i y_{n+1} = \sum_{j=1}^{s} h^j \sum_{i=0}^{k} \gamma_{ji} f_{n+i}^{(j-1)}, \quad n = 0, 1, 2 \tag{2.1}$$

where $f_{n+i}^{(j)}$ is the j^{th} derivative of f(x, y) evaluated at (x_{n+i}, y_{n+i}) ,α_I and $\gamma_{j,i}$ are real constants with $\alpha_k \neq 0$ and y_{n+i} is the approximate numerical solution evaluated at the point x_{n+i}. In order to remove the arbitrary constant in (2.1) we shall always assume that $\alpha_k = 1$.

Bearing in mind that our objective in this paper is to consider a third derivative exponential-fitted formula, then equation (2.1) is reduced to

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \gamma_{1i} f_{n+i} + h^2 \sum_{i=0}^{k} \gamma_{2,i} f_{n+i} + h^3 \sum_{i=0}^{k} \gamma_{3,i} f_{n+i}^{2} \tag{2.2}$$

Let $\beta_i = \gamma_{1,i}$, $\phi_i = \gamma_{2,I}$ $\omega_i = \gamma_{3,I}$ so that (2.2) now becomes

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \phi_i g_{n+i} + h^3 \sum_{i=0}^{k} \omega_i v_{n+i} \tag{2.3}$$

where $f_{n+i} = f[(x_{n+i}, y(x_{n+i})] = y'_{n+I}$ *and* $g_{n+I} = f'[(x_{n+i}, y(x_{n+i})] = y''_{n+I}$
$v_{n+i} = f''[x_{n+i}, y(x_{n+i})] = y'''_{n+i}$ are respectively the first, second and third derivatives of y_{n+i}.

The implementation of our proposed scheme involved a pair of formulae. Thus equation (2.3) serves as the predictor and while equation (2.4) below serves as the corrector.

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{i=0}^{k+1} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \phi_i g_{n+i} + h^3 \sum_{i=0}^{k} \omega_i v_{n+i} \tag{2.4}$$

When deriving exponentially fitted method, the approach is to allow both (2.3) and (2.4) to posses free parameters other than the mesh size '*h*'. We choose free parameters in (2.3) and (2.4), so that both formulas are fit for exponentially fitting condition.

2.1 Derivation of a Fourth Order Exponential-fitted Method

The procedure employed here is that we first derive a third order predictor scheme and then move further to derive a fourth order corrector scheme. The third order predictor formulae is obtained from (2.3) as,

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} = h[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2}] + h^2[\phi_0 g_n + \phi_1 g_{n+1} + \phi_2 g_{n+2}] + h^3[\omega_0 v_n + \omega_1 v_{n+1} + \omega_2 v_{n+2}] \tag{2.5}$$

The corresponding corrector formula from (2.4) is given as,

$$\alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} = h[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_3 f_{n+3}] + h^2[\phi_0 g_n + \phi_1 g_{n+1} + \phi_2 g_{n+2}] + h^3[\omega_0 v_n + \omega_1 v_{n+1} + \omega_2 v_{n+2}] \tag{2.6}$$

Now to derive a fourth order predictor of our scheme, we set the coefficients of y, f, g and v at x_{n+1} to zero, Also, we let $\beta_2 = a$ as - free parameter and $\alpha_k = \alpha_2 = +1$.

We then obtain the set of equations below by expanding equation (2.5) by Taylor series and equating their coefficients.

$$\left.\begin{aligned} &\alpha_0 + 1 = 0 \\ &2 - a - \beta_0 = 0 \\ &2 - 2a - \phi_0 - \phi_2 = 0 \\ &4/3 - 2a - 2\phi_2 - (\omega_0 + \omega_2) = 0 \end{aligned}\right\} \tag{2.7}$$

When we solve equation (2.7) above, we have; $\alpha_0 = -1$, $\beta_0 = 2 - a$, $\phi_0 = \frac{4}{3} - a + \omega_2$, $\omega_0 = \omega_2$, $\phi_2 = \frac{2}{3} - a - \omega_2$

When the values of the parameters are substituted into equation (2.5), we obtain.

$$Y_{n+2} - y_n = h\,[af_{n+2} + (2-a)f_n] + h^2\left[\left(\frac{2}{3} - a - \omega_2\right) g_{n+2} + \left(\frac{4}{3} - a + \omega_2\right)g_n\right) + h^3(\omega_2 v_{n+2} + \omega_2 v_n) \qquad (2.8)$$

But $y^1 = f(x,y)$, $y'' = g(x,y)$ and $y''' = v(x,y)$

Then equation (2.7) now becomes the third derivative predictor formula, given as

$$y_{n+2} - y_n = h[(ay'_{n+2}) + (2 - ay'_n)] + h^2\left[\left(\frac{2}{3} - a - \omega^2\right)y''_{n+2} + \left(\frac{4}{3} - a + \omega^2\right)y''_n\right] + h^3[\omega_2 y''''_{n+2} + \omega_2 y''''_n] \qquad (2.9)$$

For the purpose of exponential fitting condition, we apply (2.9) to scalar test function (1.3) to obtain the stability function of our fourth order predictor scheme.

$$\frac{\bar{y}_{n+2}}{y_n} = \frac{1 + (2-a)q + (4/3 - a)q^2 + \omega_2 A}{1 - aq - (2/3 - a)q^2 - \omega_2 B}, = R(\bar{q}) \qquad (2.10)$$

where $A = q^3 + q^2$, $B = q^3 - q^2$ and $q = \lambda h$

By the solution of (1.3) we have

$$\frac{y_{n+1}}{y_n} = \frac{y\begin{pmatrix} x+h \\ n \end{pmatrix}}{y(x_n)} = \frac{e^{\lambda}\begin{pmatrix} x+h \\ n \end{pmatrix}}{e^{\lambda_n^x}} = \frac{e^{\lambda xn}}{e^{\lambda xn}} \cdot e^{\lambda h} = e^{\lambda h} = e^{q} \qquad (2.11)$$

as $\lambda h = q$.

For the purpose of computations and stability analysis of our method, we substitute (2.10) into (2.9), to obtain the free parameter 'a' as;

$$a = \frac{1 + 2q + \frac{4}{3}q^2 + \frac{1}{3}\left(2q^2e^{2q} - 3e^{2q}\right) + \omega_2\left(A - Be^{2q}\right)}{qe^{2q}(q-1) + q(q+1)} \qquad (2.12)$$

Again to derive the corresponding corrector formula, we impose the same condition as in the predictor. That is by setting $\alpha_1 = \phi_1 = \beta_1 = \omega_1 = 0$ in equation (2.6) to obtain the following set of simultaneous equations;

$$\left.\begin{aligned} &\alpha_1 + 1 = 0 \\ &2 - b - \beta_2 = \beta_0 \\ &2 - 3b - 2\beta_2 - \phi_2 = \phi_0 \\ &\frac{4}{3} - \frac{9}{2}b - 2\beta_2 - (\omega_0 + \omega_2) = 2\phi_2 \\ &\tfrac{2}{3} - \tfrac{9}{2}b - \tfrac{4}{3}\beta_3 - 2\phi_2 = 2\omega_2 \end{aligned}\right\} \qquad (2.13)$$

We also let $\beta_3 = b$ as free parameter and $\alpha_2 = +1$. Solving equation (2.13), we obtain the values of the unknown parameters as;

$\alpha_0 = -1$, $\beta_0 = 1 - b$, $\phi_0 = \frac{1}{3} - \frac{3}{4}b + \omega_2$, $\beta_2 = 1$, $\phi_2 = -(\frac{1}{3} + \frac{9}{4}b + \omega_2,)$ d, 0 $= \omega_2$

When the values of the parameters are substituted into equation (2.6), we have the fourth order corrector formula as.

$$y_{n+2} - y_n = h\left[by_{n+3} + y'_{n+2} + (1-b)y'_n\right] + h^2\left[-\left(\tfrac{1}{3} + \frac{9}{4}b + \omega_2\right)y'_{n+2} + \left(\tfrac{1}{3} - \tfrac{3}{4}b + \omega_2\right)y''_n\right] + h^3\left[\omega_2 y'''_{n+2} + \omega_2 y'''_n\right] \qquad (2.14)$$

We apply our corrector formula in equation (2.13) to test function (1.3) to obtain;

$$\frac{y_{n+2}}{y_n} = \frac{1 + (1-b)q + bq\frac{y_{n+3}}{y_n} + \left(\frac{1}{3} - \frac{3}{4}b\right)q^2 + \omega_2 A}{1 - q + \left(\frac{1}{3} + \frac{9}{4}b\right)q^2 - \omega_2 B} \qquad (2.15)$$

We need to determine the ratio $\frac{y_{n+3}}{y_n}$ in (2.13)

$$\frac{y_{n+3}}{y_n} = \frac{\bar{y}_{n+2}}{y_n}.\frac{y_{n+1}}{y_n} = \left(e^{2q}\right)^{\frac{3}{2}} = \left[\frac{y_{n+2}}{y_n}\right]^{\frac{3}{2}} = \left[R(\bar{q})\right]^{\frac{3}{2}} \qquad (2.16)$$

By equation (2.14), equation (2.13) now becomes

$$\frac{y_{n+2}}{y_n} = \frac{1 + (1-b)q + bq\left[R(\bar{q})\right]^{\frac{3}{2}} + \left(\frac{1}{3} - \frac{3}{4}b\right)q^2 + \omega_2 A}{1 - q + \left(\frac{1}{3} + \frac{9}{4}b\right)q^2 - \omega_2 B} = R(q) \qquad (2.17)$$

Equation (2.15) now unites both our predictor and corrector schemes, which is capable of solving stiff systems for which exponential fitting is applicable.

Solving for b in (2.15) we have,

$$b = \frac{1 + q + \frac{q^2}{3} - e^{2q}\left(1 + \frac{q^2}{3} - q\right) + \omega_2\left(A - Be^{2q}\right)}{\frac{3}{4}q^2\left(3e^{2q} + 1\right) - q\left(e^{3q} - 1\right)} \qquad (2.18)$$

3.0 Stability analysis

Stability is the property of a numerical method to keep the errors bounded as the calculation advances.

***Definition* 2**

A method is said to be A-stable if the stability region associated with that formula contains the open left half plane.

***Definition* 3**

A method is said to be zero-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root with modulus one is simple.

Now to investigate the stability criteria of our method, the determination of the values of parameters *a* and *b* in the open left – half plane $(-\infty, 0]$ are of interest.

Therefore, it is straight forward to find the criteria which *a* and *b* need to satisfy such that

$$\left|\frac{y_{n+2}}{y_n}\right| < 1 \text{ .for all q with Re } (q) = 0.$$

Necessary and sufficient conditions for this inequality to hold are given by the application of the maximum, modulus theorem.

***Theorem* (3.1) [maximum modulus theorem]**

Let f be analytic and not constant in a domain M. Then | f | cannot have a local maximum in M.

The implications of this theorem are;

(i) $|\text{Re}(q)| \le 1$ on Re (q) = 0 (3.1)

(ii) R(q) is analytic in Re(q) < 0

If condition (i) holds, it follows that R (q) is analytic as $q \to -\infty$ and thus (i) and (ii) will guarantee A-stability by the maximum modulus theorem (3.1).

Now consider
$$\left|\frac{y_{n+2}}{y_n}\right| < 1, \tag{3.2}$$

in which case $\left|\frac{y_{n+2}}{y_n}\right| - 1 \prec 0.$ From (2.16) if we assume $\omega_2 = 0$ throughout, condition (i) gives $\frac{1}{3} + \frac{9}{4}b \prec 0 \Rightarrow b \prec -\frac{4}{27}.$ Whenever $\omega_2 = 0$, we obtain from condition (ii) that a < 2/3. Furthermore, we can show analytically that a and b have finite limits. From equation (2.12) we have $\lim_{q\to 0} a = 1,\ \lim_{q\to -\infty} a = \frac{4}{3}.$ Similarly, from equation (2.16) $\lim_{q\to 0} b = 0$, and $\lim_{q\to -\infty} b = \frac{4}{9}.$ However, by numerically plotting the values of a and b for a large sample N of q Є (-∞, 0], we observe that for various values of q the corresponding values of a and b are within the range above. That is a Є (1, 4/3) and b Є (0, 4/9). We further use a numerical procedure to examine the behaviour of our parameters by plotting the values $a(q)$ and $b(q)$ for a large sample N values of q in the range (- ∞, 0]. It was found that as the values of q decreases, the corresponding values of a and b are monotonically increasing as shown in table (3.1).

Table 3.1: **Parameter values *a* and *b* associated with fourth order scheme.**

q	*a*	*b*
-1.0	1.0596	0.01683
-2.0	1.1178	0.2171
-10.0	1.2689	0.3744
-100.0	1,3254	0.4371
-200.0	1.3278	0.4407
-1000.0	1.3313	0.4437

This set of values in Table 3.1 above, suggest that our integration formula will be A–stable within the range of values specified by the choices of parameters a and b.

However, since R(q) gives the stability region as a > 0, and b < 4/9, then, $|\tau| \prec 1$ for all q, thus zero stability is satisfied, where $\tau^2 = \frac{1 + (2-a)q + (4/3 - a)q^2 + \omega_2 A}{1 - aq - (2/3 - a)q^2 - \omega_2 B}.$ Furthermore, the combined predictor-corrector stability polynomial of our scheme gives

$$\tau^2 = \frac{1+(1-b)q + bq[R(\bar{q})]^{\frac{3}{2}} + \left(\frac{1}{3}-\frac{3}{4}b\right)q^2 + \omega_2 A}{1-q+\left(\frac{1}{3}+\frac{9}{4}b\right)q^2 + \omega_2 B}$$

Testing for values of q ∈ (- ∞,0], we established from Table (3.1) that | τ | < 1 for all q. That is the predictor- corrector formula (2.15) fitted to stiff scalar problem (1.2) is absolutely stable for all choices of free parameters.

This formula derived so far in section 2 is coded in Fortran 77 language to solve several stiff systems of ordinary differential equations.
Such problems are discussed in the next section.

4.0 Application and numerical results

The aim of the numerical results presented in this section is first to show the accuracy of our scheme when compared with the exact solutions. Secondly, how our scheme compare with various existing other methods. All numerical experiments are coded in Fortran 77 and implemented on digital computer, VIA Samuel 2 Processor.

The following problems are considered.

***Problem* 1**

Jackson and Kenue (1974) [8], Cash (1981) [4], Okunuga (1997) [12].

$$\left.\begin{array}{ll} y' = -y + 95z; & y(0)=1 \\ z' = -y - 97z; & z(0)=1 \end{array}\right\} x \in [0,1] \tag{4.1}$$

The eigenvalues of the Jacobian matrix of the system are λ_1 = -2, and λ_2 = - 96 and the general solution is of the form

$$\begin{aligned} y(x) &= Ae^{\lambda_1 x} + Be^{\lambda_2 x} \\ Z(x) &= Ce^{\lambda_1 x} + De^{\lambda_2 x} \end{aligned} \tag{4.2}$$

Hence the true solution is given as

$$\begin{aligned} y &= \left(95e^{-2x} - 48e^{-96x}\right)/47 \\ z &= \left(48e^{-96x} - e^{-2x}\right)/47 \end{aligned} \tag{4.3}$$

***Problem* 2 (Test problem)**

From Enright and Pryce (1983) [5], Gear (1967) [7] and Okunuga (1997) [12].

$$\begin{array}{ll} y_1' = 0\cdot 013y_1 + 1000y_1y_2; & y_1(0)=1 \\ y_2' = -2500y_2y_3; & y_2(0)=1 \\ y_3' = 0\cdot 013y_1 - 1020y_1y_3 - 2500y_2y_3; & y(0)=1 \end{array} \tag{4.4}$$

The eigenvalues of the system (Problem 2) are given by $\lambda_1 = 0, \lambda_2 = -0\cdot 00928572$ and $\lambda_3 = 3500\cdot 002714$. The exact solution is given by

$$y_1 = C_j + D_j e^{\frac{\lambda}{x}}, \; j = 1,2,3 \tag{4.5}$$

C_j and D_j are determined using the initial value condition. Problem 3, Test problem from Enright and Pryce (1983) [5], $y' = My$

where $$M = \begin{pmatrix} 10^4 & 100 & -10 & 1 \\ 0 & -1000 & 10 & -10 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 0\cdot 1 \end{pmatrix} y \qquad (4.6)$$

$y(0) = [1, \quad 1, \quad 1, \quad 1]^T, \quad 0 \le x \le 11$

The eigenvalues of the Jacobian matrix is given as $\lambda_1 = -0.1,\ \lambda_2 = -1,\ \lambda_3 = 1000,\ \lambda_4 = -10000$.

The exact solution is of the form

$$y_j(x) = w_j e^{\lambda 1x} + v_j e^{\lambda 2x};\ j = 1, \cdots 4 \qquad (4.7)$$

where

$$W_1 = \frac{-9909 - \lambda_2}{\lambda_1 - \lambda_2};\ V_1 = \frac{9909 + \lambda_1}{\lambda_1 - \lambda_2}$$

$$W_2 = \frac{-1000 - \lambda_2}{\lambda_1 - \lambda_2};\ V_2 = \frac{9909 + \lambda_1}{\lambda_1 - \lambda_2}$$

$$W_3 = \frac{-9 + \lambda_1}{\lambda_1 - \lambda_2};\ V_3 = \frac{9 - \lambda_1}{\lambda_1 - \lambda_2}$$

$$W_4 = \frac{-0.1 - \lambda_2}{\lambda_1 - \lambda_2};\ V_4 = \frac{0.1 + \lambda_1}{\lambda_1 - \lambda_2}$$

Problem 3 was considered by Enright and Pryce (1983) [5] and the error tolerance was fixed at $10^{-5.}$

We shall compare the result obtains by our order 4 scheme with that of Cash (1981) [4], Jackson and Keunue (1974) [[8] and Okunuga (1997) [12]. We denotes the methods proposed by Jackson and Kunue as J-K, cash orders 4 and 5 formula as CH4 and CH5 Okunuga as OK 4 orders 4 while EXPN4 represent our order 4 scheme. Table 4.1 clearly confirms the competitive nature of our methods with the existing ones, as shown below.

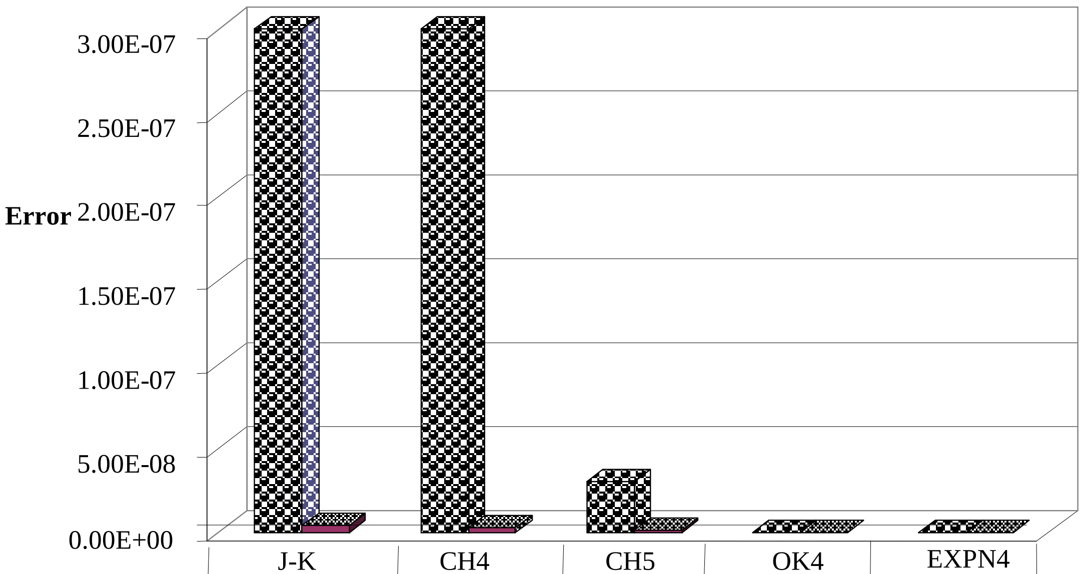
Table 4.1: **Comparative analysis of result of Problem 1**

Step h	Method	$Y(1)$	$Z(1)$ x10^{-2}	Error (y)	Error (z)
0.0625	J-K	0.2735503	-0.287477	3x10^{-7}	4x10^{-9}
	CH4	0.2735498	-0.2879471	3x10^{-7}	3x10^{-9}
	CH5	0.27355005	-0.287942	3x10^{-18}	1x10^{-9}
	OK4	0.273550041	-0.287947411	3x10^{-16}	3x10^{-18}
	EXPN4	0.273550041	-0.28794741	3x10^{-17}	3x10^{-18}
0.125	OK4	0.27355005	-0.287947402	1x10^{-16}	3x10^{-17}
	EXPN4	0.27355004	-0.287947401	0.0000000	0.0000000
0.05	EXPN 4	0.27355003	-0.287947403	3x10^{-16}	3x10^{-18}
0.25	EXPN 4	0.27355004	-0.28794741	0.0000000	0.000000
True	Solution	0.27355004	-0.28794741	-	-

We observe that with h = 0.25 0ur order 4 integration formula needs only two iterations to evaluate y (1) and z(1) respectively, yet is very accurate when compared with other methods.

The graphical representation of the Absolute Errors of the numerical results is displayed below.

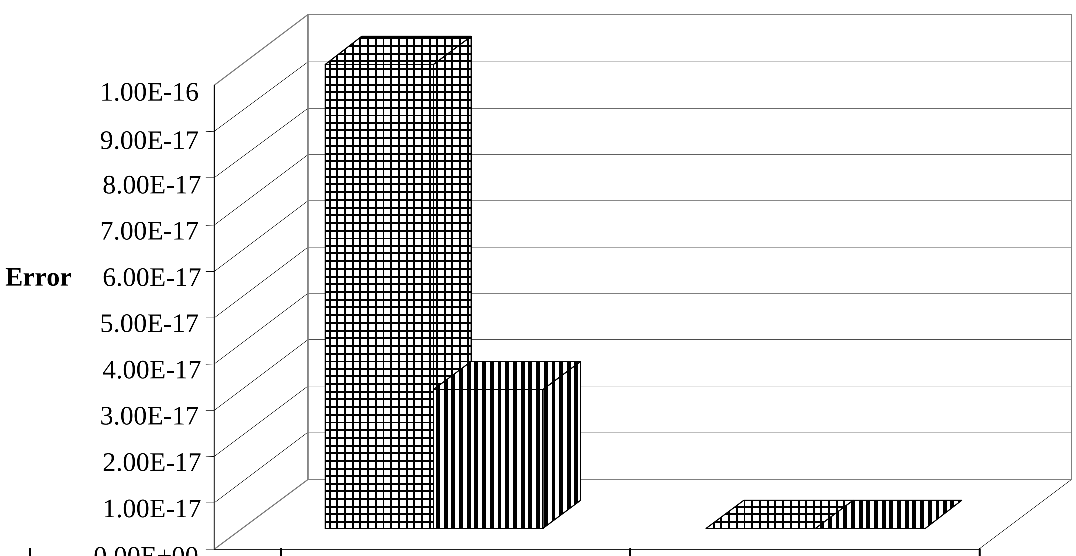
Figure 1: Error of Numerical Values of y(x) for Problem 1 when h = 0.0625



	J-K	CH4	CH5	OK4	EXPN4
Error (y)	3.00E-07	3.00E-07	3.00E-08	3.00E-16	3.00E-17
Error (z)	4.00E-09	3.00E-09	1.00E-09	3.00E-18	3.00E-18

Method

Figure 2: Error of Numerical values of $y(x)$ for Problem 1 when h = 0.125



	OK4	EXPN4
Error (y)	1.00E-16	0.00E+00
Error (z)	3.00E-17	0.00E+00

Method

Figure 3: Absolute Error of Numerical values of $y(x)$ for h = 0.5 and 0.25

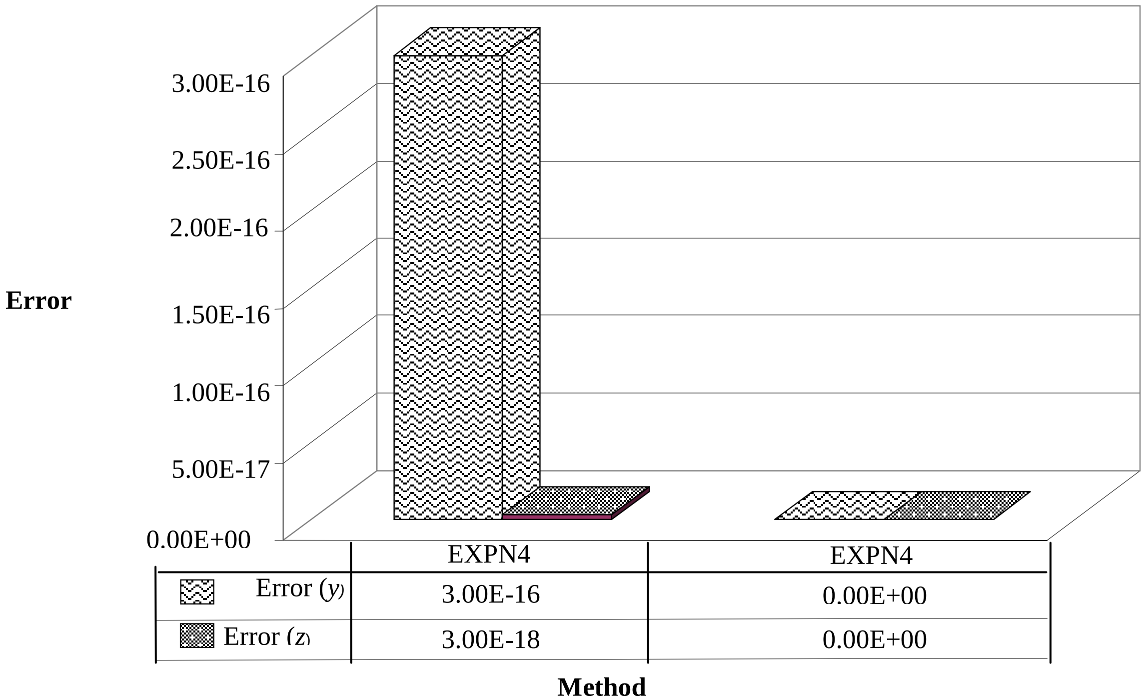


Table 4.2: Performance of our order on Problem 2

Step h	Method	$Y_1^{(1)}$	$Y_2^{(1)}$	$Y_3^{(1)}$	$ER\ (y_1)$	$ER\ (y_2)$	$ER\ (y_3)$
0.0625	OK4	0.5908282	1.00924009	-2.79412225	$4.4\text{x}10^{-16}$	$4.4\text{x}10^{-16}$	$8.9\text{x}10^{-16}$
	EXPN 4	0.5907599	1.00924036	-2.79146048	0.0000	0.0000	0.0000
0.1	OKA4	0.59076	1.00924005	-2.77412225	$6.6\text{x}10^{-16}$	$6.6\text{x}10^{-16}$	$1.8\text{x}10^{-15}$
	EXPN4	0.5907599	1.00924036	-2.79146048	$6.7\text{x}10^{-16}$	$8.9\text{x}10^{-16}$	$2.7\text{x}10^{-15}$
True	Solution	0.5907599	1.00924036	-2.79146048	-	-	-

From Table 4.2 we observe that both OK4 and EXPN4 are identical for y_1, y_2, y_3 for a step length of h=0.0625. They are very accurate when compared to the exact solution.

We also observed from our numerical results that the error tolerance can be raised to 10^{-16} as against 10^{-5} suggested by Enright and Pryce (1983) [5].

We represent the absolute errors of the numerical results of Problem 2 in the graph below.

Figure 4: Error of Numerical values of $y(x)$ for problem 2 when h = 0.0625

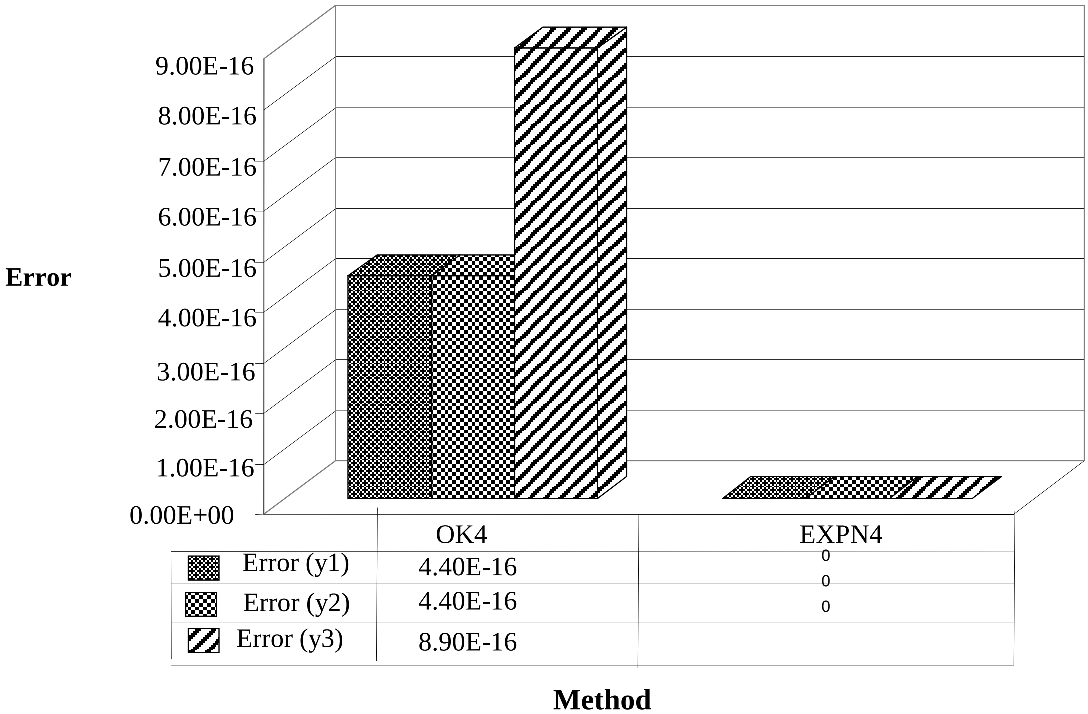


Table 4.3: Accuracy table for Problem 3

Step	Method step h	Error (y_1(1)	Error $_2$(1)	Error $_3$(1)	Error y(1)$_4$
	OK4	3.6x10-12	$3.4\text{x}10^{-13}$	$3.1\text{x}10^{-15}$	$1\text{x}10^{-16}$
0.1	EXPN4	$2.6\text{x}10^{-12}$	$3.4\text{x}10^{-13}$	$2.7\text{x}10^{-15}$	$2.8\text{x}10^{-17}$
0.0625	EXPN4	$3.6\text{x}10^{-12}$	$3.4\text{x}10^{-13}$	$4.4\text{x}10^{-15}$	$4.1\text{x}0^{-17}$

The true solution: Y_1 = 59 10.942866
Y_2 = 595.6555
Y_3 = 6.334079
Y_4 = 0.90483242

From the table the efficient of our order 4 scheme is demonstrated, with the result of h = 0.0625 which is slightly more accurate than h = 0.l. It is also clear from the graphical representation of the absolute errors of the numerical result.

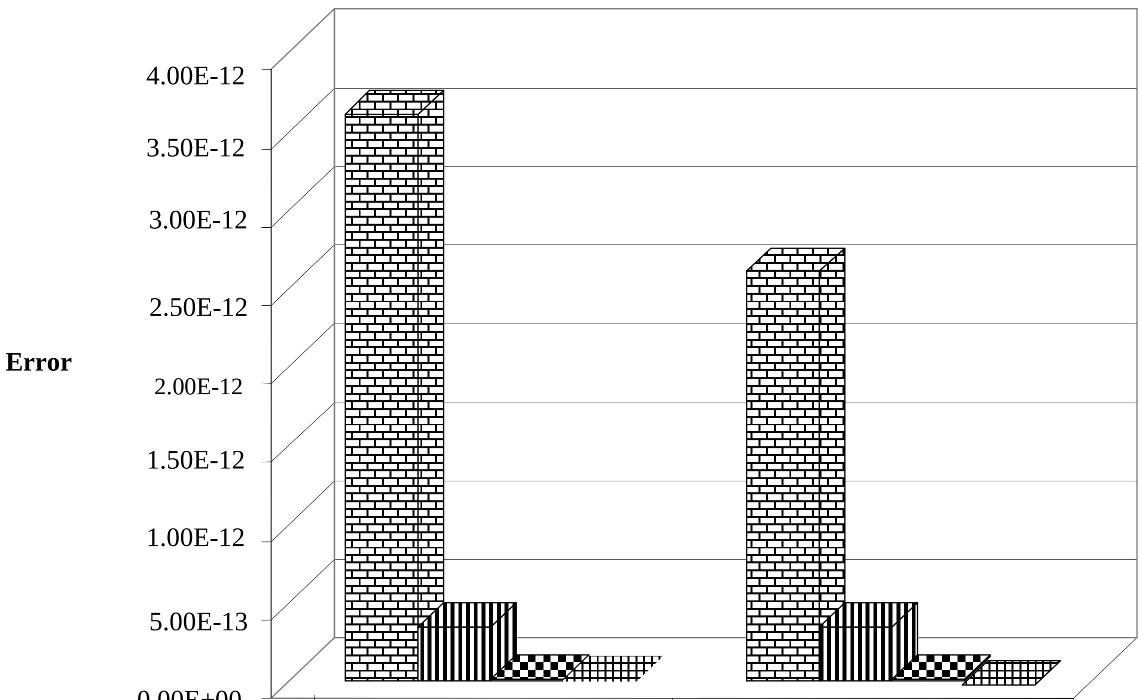
5.0 Conclusion

The numerical results of Problems 1, 2 and 3 in this paper shows that any of the scheme derived for order 4 could be used for solution of stiff initial value problems.

5.1 Remark

Our fourth order exponential fitted scheme is relatively stable for $0 < \omega_2 \leq 0$. Our numerical result indicate that for $\omega_2 = 0$ our scheme is far more accurate when compared with the exact solution of our kind of problems considered in this paper. It is thus believed that the method derived in this paper represent useful addition to the library of algorithms or methods for solving stiff problems of which exponential fitted is applicable.

Figure 5: Error of Numerical values of $y(x)$ for Problem 3 when h = 0.1



	OK4	EXPN4
Error (y1(1)	3.60E-12	2.60E-12
Error 2(1)	3.40E-13	3.40E-13
Error 3(1)	3.10E-15	2.70E-15
Error y(1)4	1.00E-16	2.80E-17

Method

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