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R[2, 4; 2: 6] rational one-step numerical integrator for initial value problems in Ordinary Differential Equations.

F. O. Otunta and **G. C. Nwachukwu**
Department of Mathematics
University of Benin, Benin City. Nigeria.

Abstract

A method of order six is proposed for solving singular initial value problems in ordinary differential equations. It compares favourably with existing schemes.

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1.0 Introduction

Our concern is to derive and implement a rational one-step numerical integrator of order six to solve initial value problems. The problem may be represented by

$$y' = f(x, y),\ y(a) = y_0 \in R^m,\ x \in [a,\ b],\ y(x) \in R^m \tag{1.1}$$

which may be singular, but whose solution we have assumed to have continuous derivatives to the order desired. The points of singularities of the solution of the IVPs (1.1) are the poles of the rational interpolant. Some algorithms designed for this class of IVPs (1.1) are based on rational functions because of their smooth behaviour in the vicinity of singularities compared to polynomial functions. These schemes include those of van Niekerk (1987), Luke et al (1975), Lambert and Shaw (1965), Fatunla (1982, 1986, 1990, 1994) [2, 3, 4, 5], Otunta and Ikhile (1996, 1999) [10, 11], Fatunla and Aashikpelokhai (1994) [6] and others. Van Niekerk (1987) [13], using a local interpolant

$$y_n = a_n + \frac{b_n}{1 + c_n x_n} \tag{1.2}$$

developed one step integration formula of order three. Otunta and Ikhile (1999) [11], constructed a scheme of order four based on the rational approximant

$$y(x) \approx \frac{a_0 + a_1 x}{1 + b_1 x + b_2 x^2 + b_3 x^3} \tag{1.3}$$

Otunta and Nwachukwu (2003) [13], considered the interpolant

$$y(x) \approx \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3} \tag{1.4}$$

and derived a rational one-step numerical integrator of order five.

In this paper, we propose to construct a rational one-step method using the approximant

$$y(x) \approx \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4} \tag{1.5}$$

for its promise of higher accuracy.

2.0 The order conditions

Otunta and Ikhile (1996, 1999) [10, 11] approximated the solution, $y(x)$ of the IVP (1.1) by

the rational interpolant

$$y(x) \approx R_{k-L,k}\ (x) = \frac{\sum_{j=0}^{k-L} a_j x^j}{1 + \sum_{j=1}^{k} b_j x^j}\ ,\ k \geq 1, L = 0(1)k \tag{2.1}$$

$y(x)$ and y_n are the theoretical and computed solutions of (1.1) respectively. $x_n = nh,\ n = 0, 1, 2, \ldots,$ as the mesh points with a constant distance of h between them. $y(x)$ has derivatives to the order desired. The schemes derived are component applicable to systems of ODEs in the sense of Lambert (1974) [8].

The resultant class of one-step rational schemes from (2.1) is the variable order self adjusting integrator

$$y_{n+1} = \frac{P_{k-L}(x_{n+1})}{1 + \sum_{j=1}^{k} b_j x_{n+1}^j}\quad ,\ k - L \geq 0,\ k \geq 1 \tag{2.2}$$

with

$$P_{k-L}(x_{n+1}) = \sum_{j=0}^{k-L} a_j x_{n+1}^j\ , L = 0(1)k \tag{2.3}$$

The method parameters

$$\begin{aligned} a &= (a_1,\ a_2,\ a_3, \ldots, a_{k-L})^T \\ b &= (b_1,\ b_2,\ b_3, \ldots, b_k)^T \end{aligned} \tag{2.4}$$

are obtained by writing (2.2) as

$$y_{n+1} = P_{k-L}(x_{n+1}) \left[1 + \sum_{r=1}^{\infty} \left(\sum_{j=1}^{k} b_j x_{n+1}^j \right)^r (-)^r \right] \tag{2.5}$$

and superimposing it on

$$y(x_{n+1}) = \sum_{j=0}^{\infty} \frac{h^j y_n^{(j)}}{j!},\ y_n^{(0)} = y_n \tag{2.6}$$

the result of this is the order equations

$$a_0 = y_n,\ x_{n+1}(a_1 - a_0 b_1) = h y_n^1,\ x_{n+1}^2 \left(a_2 - a_1 b_1 - a_0 b_2 + a_0 c_{(2)}^{(1)}\right) = \frac{h^2 y_n^{II}}{2!}$$

$$x_{n+1}^3 \left(a_3 - a_1 b_2 - a_2 b_1 - a_0 b_3 + a_0 c_3^{(1)} + a_1 c_2^{(1)} - a_0 c_3^{(2)}\right) = \frac{h^3 y_n^{III}}{3!}$$

$$x_{n+1}^{4}\left(a_4-a_3b_1-a_2b_2-a_1b_3-a_0b_4+a_0c_4^{(1)}+a_1c_3^{(1)}+a_2c_2^{(1)}-a_0c_4^{(2)}-a_1c_3^{(2)}\ +\ a_0c_4^{(3)}\right)=\frac{h^4y_n^{(iv)}}{4!}$$

$$x_{n+1}^{k-L}\Big[\ a_{k-L}-\sum_{j=0}^{k-L}b_{k-L-j}a_j+\sum_{j=0}^{k-L-1}C_{k-L-j}^{(1)}\ a_j$$

$$-\sum_{j=0}^{k-L-2}C_{k-L-j}^{(2)}\ a_j+\sum_{j=0}^{k-L-3}\ C_{k-L-j}^{(3)}\ a_j$$

$$-\ \sum_{j=0}^{k-L-4}C_{k-L-j}^{(4)}\ a_j\ +\ldots+(-)^{k-L-j}\sum_{j=0}^{1}\ C_{k-L-j}^{(k-L-2)}\ a_j+\ (-)^{k-1}\ a_0\ C_{k-L}^{(k-L-1)}\Big]$$

$$=\frac{h^{k-L}y_n^{(k-L)}}{(k-L)!}$$

$$x_{n+1}^{k-L+1}\ \ \Big[-\sum_{j=0}^{k-L+1}\ b_{k-L-j+1}\ a_j\ +\sum_{j=0}^{k-L}\ C_{k-L-j+1}^{(1)}\ a_j$$

$$-\sum_{j=0}^{k-L-1}\ C_{k-L-j+1}^{(2)}\,a_j\ +\sum_{j=0}^{k-l-2}C_{k-L-j+1}^{(3)}\ a_j$$

$$-\sum_{j=0}^{k-L-3}\ C_{k-L-j+1}^{(4)}\ a_j\ +\ldots+(-)^{k-L}\sum_{j=0}^{1}\ C_{k-L-j+1}^{(k-L-j)}\ a_j+(-)^{k-L-+1}\ a_0C_{k-L+1}^{(k-L)}\Big]$$

$$=\frac{h^{k-L+1}y_n^{(k-L+1)}}{(k-L+1)!}$$

$$x_{n+1}^{k}\Big[-\sum_{j=0}^{k}a_{k-j}b_j\ +\sum_{j=0}^{k-1}C_{k-j}^{(1)}a_j\ -\sum_{j=0}^{k-2}\ C_{k-j}^{(2)}\ a_j\ +\ldots$$

$$+\sum_{j=0}^{1}C_{k-j}^{(k-2)}\ a_j\ +(-)^k\ a_0C_{\ k}^{\ (k-1)}\Big]=\frac{h^k\ y_n^{(k)}}{k!}$$

$$x_{n+1}^{k+1}\Big[-\sum_{j=0}^{k+1}\ a_{k-j+1}b_j\ +\sum_{j=0}^{k}C_{k-j+1}^{(1)}a_j-\sum_{j=0}^{k-1}\ C_{k-j+1}^{(2)}a_j+\ldots$$

$$+(-)^k\sum_{j=0}^{1}\ C_{k-j+1}^{(k-1)}a_j\ +(-)^{k+1}a_0C_{k+1}^{(k)}\Big]=\frac{h^{k+1}y_n^{(k+1)}}{(k+1)!}$$

$$x_{n+1}^{2k-L}\Big[-\sum_{j=0}^{2k-L}\ a_{2k-L-j}\,b_j\ +\sum_{j=0}^{2k-L-1}C_{2k-L-j}^{(1)}\ a_j-\sum_{j=0}^{2k-L-2}C_{2k-L-j}^{(2)}\,a_j+\ldots$$

$$+(-)^{2k-L-1}\sum_{j=0}^{1}\ C_{2k-L-j}^{(2k-L-2)}\ a_j+(-)^{2k-L}\ \ a_0\,C_{2k-L}^{(2k-L-1)}\Big]\ =\ \frac{h^{2k-L}y_n^{(2k-L)}}{(2k-L)!} \qquad (2.7)$$

where the coefficients $C_r^{(s)}$, $r=2(1)P$, $s=1(1)\ P-1,\ \ r-s>0$ are such that

$$L_{K(q+1)}(x)=\left[\sum_{j=1}^{k}b_jx^j\right]^{q+1}=\sum_{j=q+1}^{k(q+1)}C_j^{(q)}x^j\ ,q\ \ \geq 1 \qquad (2.8)$$

and with the understanding that

$$a_s = 0 \; ; \; s \geq k - L + 1, \; s < 0 \qquad (2.9)$$

$$b_s = 0 \; ; \; s \geq k + 1, \; s \leq 0 \, , \; C_r^{(s)} = 0 \; ; \; r - s \leq 0$$

More expositions as to the recursive computation of the coefficients $C_r^{(s)}$ are found in Otunta and Ikhile (1996). It holds that the attainable order of the rational scheme (2.2) is at least $p = 2k - L$, since from (2.7) the local truncation error, Lte_n, is

$$Lte_n = x_{n+1}^{2k-L+1}\left[-\sum_{j=0}^{2k-L+1} a_{2k-L-j+1}\, b_j + \sum_{j=0}^{2k-L} C_{2k-L-j+1}^{(1)}\, a_j - \sum_{j=0}^{2k-L-1} C_{2k-L-j+1}^{(2)}\, a_j + \ldots + (-)^{2k-L} \sum_{j=0}^{1} C_{2k-L-j+1}^{(2k-L-1)}\, a_j + (-)^{2k-L+1}\, a_0\, C_{2k-L+1}^{(2k-L)} \right] - \frac{h^{2k-L+1} y_n^{(2k-L+1)}}{(2k - L + 1)!} + 0\left(h^{2k-L+2}\right)$$

3.0 An L-stable rational integrator

Otunta and Ikhile (1999) [11], characterised the class of rational methods (2.2) as $R[k - L, k; \; L : 2k - L]$ where the parameters have the meaning given in (2.2) and (2.3) respectively.

Niekerk (1987), Otunta and Ikhile (1999) [11], Otunta and Nwachukwu (2003) [12] considered the special cases of R[1, 2; 1:3], R[1, 3; 2:4] and R[2, 3; 1:5] respectively.

We shall now consider the special case of R[2, 4; 2:6]. That is

$$y(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4} \qquad (3.1)$$

To obtain the method parameters $a = (a_0, a_1, a_2)^T$ *and* $b = (b_1, b_2, b_3, b_4)^T$ it suffices to solve the order equations

$$a_0 = y_n \, , \; x_{n+1}(a_1 - a_0 b_1) = h\, y_n^I \, , \; x_{n+1}^2 \left(a_2 - a_1 b_1 - a_0 b_2 + a_0 C_2^{(1)}\right) = h^2 \, y_n^{II} / 2!$$

$$x_{n+1}^3 \left(-a_1 b_2 - a_2 b_1 - a_0 b_3 + a_0 C_3^{(1)} + a_1 C_2^{(1)} - a_0 C_3^{(2)}\right) = h^3 \, y_n^{III} / 3! \qquad (3.2)$$

$$x_{n+1}^4 \left(-a_2 b_2 - a_1 b_3 - a_0 b_4 + a_0 C_4^{(1)} + a_1 C_3^{(1)} + a_2 \, C_2^{(1)} - a_0 C_4^{(2)} - a_1 C_3^{(2)} + a_0 C_4^{(3)}\right) = h^4 \, y_n^{(IV)} / 4!$$

$$x_{n+1}^5 \left(- a_2 b_3 - a_1 b_4 + a_0 C_5^{(1)} + a_1 C_4^{(1)} + a_2 C_3^{(1)} - a_0 C_5^{(2)} - a_1 C_4^{(2)} - a_2 C_3^{(2)} + a_0 C_3^{(3)} + a_1 C_4^{(3)} - a_0 C_5^{(4)}\right) = h^5 \, y_n^{(V)} / 5!$$

$$x_{n+1}^6 \left(-a_2 b_4 + a_0 C_6^{(1)} + a_1 C_5^{(1)} + a_2 \, C_4^{(1)} - a_0 C_6^{(2)} - a_1 C_5^{(2)} - a_2 C_4^{(2)} + a_0 C_6^{(3)} + a_1 C_5^{(3)} + a_2 C_4^{(3)} - a_0 C_6^{(4)} - a_1 C_5^{(4)} + a_0 C_6^{(5)}\right) = h^6 y_n^{(VI)} / 6!$$

where

$$
\begin{aligned}
C_v^{(1)} &= \sum_{j=1}^{v} b_{v-j}\, b_j\,,\ v = 2(1)6 \\
C_s^{(q)} &= \sum_{j=1}^{s} C_{s-j}^{(q-1)}\, b_j\,,\ q \geq 2,\ s = (1+q)(1)6
\end{aligned} \tag{3.3}
$$

Fixing (3.3) into (3.2) and writing the equation as a linear combination of each other, we have then that

$$
\begin{bmatrix}
\frac{h^2 y_n^{II}}{2!\ x_{n+1}^2} & \frac{h\, y_n^{I}}{x_{n+1}} & y_n & 0 \\
\frac{h^3 y_n^{III}}{3!\ x_{n+1}^3} & \frac{h^2\, y_n^{II}}{2!\, x_{n+1}^2} & \frac{h y_n^{I}}{x_{n+1}} & y_n \\
\frac{h^4 y_n^{(IV)}}{4!\ x_{n+1}^4} & \frac{h^3\, y_n^{III}}{3!\, x_{n+1}^3} & \frac{h^2 y_n^{II}}{2!\, x_{n+1}^2} & \frac{h y_n^{I}}{x_{n+1}} \\
\frac{h^5 y_n^{(V)}}{5!\ x_{n+1}^5} & \frac{h^4\, y_n^{(IV)}}{4!\, x_{n+1}^4} & \frac{h^3 y_n^{III}}{3!\, x_{n+1}^3} & \frac{h^2 y_n^{II}}{2!\ x_{n+1}^2}
\end{bmatrix}
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}
= -
\begin{bmatrix}
\frac{h^3 y_n^{III}}{3!\ x_{n+1}^3} \\
\frac{h^4 y_n^{(IV)}}{4!\ x_{n+1}^4} \\
\frac{h^5 y_n^{(V)}}{5!\ x_{n+1}^5} \\
\frac{h^6 y_n^{(VI)}}{6!\ x_{n+1}^6}
\end{bmatrix} \tag{3.4}
$$

The solution of this system is readily obtain as

$$
b = T^{-1} \left(T_1\,,\quad T_2\,,\quad T_3,\quad T_4\right)^T \tag{3.5}
$$

where

$$
\begin{aligned}
T = {} & \frac{h^8 y_n^{II}}{8\ x_{n+1}^8} \left\{ y_n^{II}\left(\frac{y_n^{II} y_n^{II}}{2} - \frac{y_n^{I} y_n^{III}}{3}\right) - \frac{y_n^{I}}{3}\left(y_n^{II} y_n^{III} - \frac{y_n^{I} y_n^{(IV)}}{2}\right) + \frac{y_n}{3}\left(\frac{y_n^{III} y_n^{III}}{3} - \frac{y_n^{II} y_n^{(IV)}}{4}\right)\right\} \\
& - \frac{h^8 y_n^{I}}{12\ x_{n+1}^8} \left\{ y_n^{III}\left(\frac{y_n^{II} y_n^{II}}{2} - \frac{y_n^{I} y_n^{III}}{3}\right) - \frac{y_n^{I}}{3}\left(\frac{y_n^{II} y_n^{(IV)}}{2} - \frac{y_n^{I} y_n^{(V)}}{5}\right) + \frac{y_n}{4}\left(\frac{y_n^{(IV)} y_n^{III}}{3} - \frac{y_n^{II} y_n^{(V)}}{5}\right)\right\} \\
& + \frac{h^8 y_n}{24\ x_{n+1}^8} \left\{ \frac{y_n^{III}}{3}\left(y_n^{II} y_n^{II} - \frac{y_n^{I} y_n^{(IV)}}{2}\right) - \frac{y_n^{II}}{2}\left(\frac{y_n^{II} y_n^{(IV)}}{2} - \frac{y_n^{I} y_n^{(V)}}{5}\right) + \frac{y_n}{6}\left(\frac{y_n^{(IV)} y_n^{(IV)}}{4} - \frac{y_n^{III} y_n^{(V)}}{5}\right)\right\}
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
T_1 = {} & \frac{-h^9 y_n^{III}}{24\ x_{n+1}^9} \left\{ y_n^{II}\left(\frac{y_n^{II} y_n^{II}}{2} - \frac{y_n^{I} y_n^{III}}{3}\right) - \frac{y_n^{I}}{3}\left(y_n^{II} y_n^{III} - \frac{y_n^{I} y_n^{(IV)}}{2}\right) + \frac{y_n}{3}\left(\frac{y_n^{III} y_n^{III}}{3} - \frac{y_n^{II} y_n^{(IV)}}{4}\right)\right\} \\
& - \frac{h^9 y_n^{I}}{48\ x_{n+1}^9} \left\{ -y_n^{(IV)}\left(\frac{y_n^{II} y_n^{II}}{2} - \frac{y_n^{I} y_n^{III}}{3}\right) - \frac{y_n^{I}}{5}\left(-y_n^{II} y_n^{(V)} + \frac{y_n^{I} y_n^{(VI)}}{3}\right) + \frac{y_n}{15}\left(-y_n^{III} y_n^{(V)} + \frac{y_n^{II} y_n^{(VI)}}{2}\right)\right\}
\end{aligned} \tag{3.7}
$$

$$+\frac{h^9 y_n}{96\ x_{n+1}^9}\left\{-\frac{y_n^{(IV)}}{3}\left(y_n^{II}y_n^{III}-\frac{y_n^I y_n^{(IV)}}{2}\right)-\frac{y_n^{II}}{5}\left(-y_n^{II}y_n^{(V)}+\frac{y_n^I y_n^{(VI)}}{3}\right)+\frac{y_n}{15}\left(-\frac{y_n^{(V)}y_n^{(IV)}}{2}+\frac{y_n^{III}y_n^{(IV)}}{3}\right)\right\}$$

$$T_2=\frac{h^{10}y_n^{II}}{96\ x_{n+1}^{10}}\left\{-y_n^{(IV)}\left(\frac{y_n^{II}y_n^{II}}{2}-\frac{y_n^I y_n^{III}}{3}\right)-\frac{y_n^I}{5}\left(-y_n^{II}y_n^{(V)}+\frac{y_n^I y_n^{(VI)}}{3}\right)+\frac{y_n}{15}\left(-y_n^{III}y_n^{(V)}+\frac{y_n^{II}y_n^{(VI)}}{2}\right)\right\}$$

$$+\frac{h^{10}y_n^{III}}{72\ x_{n+1}^{10}}\left\{y_n^{III}\left(\frac{y_n^{II}y_n^{II}}{2}-\frac{y_n^I y_n^{III}}{3}\right)-\frac{y_n^I}{2}\left(\frac{y_n^{II}y_n^{(IV)}}{2}-\frac{y_n^I y_n^{(V)}}{5}\right)+\frac{y_n}{4}\left(\frac{y_n^{III}y_n^{(IV)}}{3}-\frac{y_n^{II}y_n^{(V)}}{5}\right)\right\} \quad (3.8)$$

$$+\frac{h^{10}y_n}{288\ x_{n+1}^{10}}\left\{\frac{y_n^{III}}{5}\left(-y_n^{II}y_n^{(V)}+\frac{y_n^I y_n^{(VI)}}{3}\right)+\frac{y_n^{(IV)}}{2}\left(\frac{y_n^{II}y_n^{(IV)}}{2}-\frac{y_n^I y_n^{(V)}}{5}\right)+\frac{y_n}{10}\left(-\frac{y_n^{(IV)}y_n^{(VI)}}{6}+\frac{y_n^{(V)}y_n^{(V)}}{5}\right)\right\}$$

$$T_3=\frac{h^{11}y_n^{II}}{192\ x_{n+1}^{11}}\left\{\frac{y_n^{II}}{5}\left(-y_n^{II}y_n^{(V)}+\frac{y_n^I y_n^{(VI)}}{3}\right)+\frac{y_n^{(IV)}}{3}\left(y_n^{II}y_n^{III}-\frac{y_n^I y_n^{(IV)}}{2}\right)+\frac{y_n}{15}\left(-\frac{y_n^{III}y_n^{(VI)}}{3}+\frac{y_n^{(IV)}y_n^{(V)}}{2}\right)\right\}$$

$$-\frac{h^{11}y_n^I}{288\,x_{n+1}^{11}}\left\{\frac{y_n^{III}}{5}\left(-y_n^{II}y_n^{(V)}+\frac{y_n^I y_n^{(VI)}}{3}\right)+\frac{y_n^{(IV)}}{2}\left(\frac{y_n^{II}y_n^{(IV)}}{2}-\frac{y_n^I y_n^{(V)}}{5}\right)+\frac{y_n}{10}\left(-\frac{y_n^{(IV)}y_n^{(VI)}}{6}+\frac{y_n^{(V)}y_n^{(V)}}{5}\right)\right\} \quad (3.9)$$

$$-\frac{h^{11}y_n^{III}}{144\ x_{n+1}^{11}}\left\{\frac{y_n^{III}}{3}\left(y_n^{II}y_n^{III}-\frac{y_n^I y_n^{(IV)}}{2}\right)-\frac{y_n^{II}}{2}\left(\frac{y_n^{II}y_n^{(IV)}}{2}-\frac{y_n^I y_n^{(V)}}{5}\right)+\frac{y_n}{6}\left(\frac{y_n^{(IV)}y_n^{(IV)}}{4}-\frac{y_n^{III}y_n^{(V)}}{5}\right)\right\}$$

$$T_4=\frac{h^{12}y_n^{II}}{567\ x_{n+1}^{12}}\left\{\frac{y_n^{II}}{5}\left(-\frac{y_n^{II}y_n^{(VI)}}{2}+y_n^{III}y_n^{(V)}\right)-\frac{y_n^I}{5}\left(-\frac{y_n^{III}y_n^{(VI)}}{3}+\frac{y_n^{(IV)}y_n^{(V)}}{2}\right)-y_n^{(IV)}\left(\frac{y_n^{III}y_n^{III}}{3}-\frac{y_n^{II}y_n^{(IV)}}{4}\right)\right\}$$

$$-\frac{h^{12}y_n^I}{288\,x_{n+1}^{12}}\left\{\frac{y_n^{III}}{15}\left(-\frac{y_n^{II}y_n^{(V)}}{2}+y_n^{III}y_n^{(V)}\right)-\frac{y_n^I}{10}\left(-\frac{y_n^{(IV)}y_n^{(VI)}}{6}+\frac{y_n^{(V)}y_n^{(V)}}{5}\right)-\frac{y_n^{(IV)}}{4}\left(\frac{y_n^{III}y_n^{(IV)}}{3}-\frac{y_n^{II}y_n^{(V)}}{5}\right)\right\} \quad (3.10)$$

$$\frac{h^{12}y_n}{576\ x_{n+1}^{12}}\left\{\frac{y_n^{III}}{15}\left(-\frac{y_n^{III}y_n^{(VI)}}{3}+\frac{y_n^{(IV)}y_n^{(V)}}{2}\right)-\frac{y_n^{II}}{10}\left(-\frac{y_n^{(IV)}y_n^{(VI)}}{6}+\frac{y_n^{(V)}y_n^{(V)}}{5}\right)-\frac{y_n^{(IV)}}{6}\left(\frac{y_n^{(IV)}y_n^{(IV)}}{4}-\frac{y_n^{III}y_n^{(V)}}{5}\right)\right\}$$

$$+\frac{h^{12}y_n^{III}}{144\ x_{n+1}^{12}}\left\{\frac{y_n^{III}}{3}\left(\frac{y_n^{III}y_n^{III}}{3}-\frac{y_n^{II}y_n^{(IV)}}{4}\right)-\frac{y_n^{II}}{4}\left(\frac{y_n^{III}y_n^{(IV)}}{3}-\frac{y_n^{II}y_n^{(V)}}{5}\right)+\frac{y_n^I}{6}\left(\frac{y_n^{(IV)}y_n^{(IV)}}{4}-\frac{y_n^{(V)}y_n^{III}}{5}\right)\right\}$$

From (2.7), we have

$$a_1=\frac{h\,y_n^1}{x_{n+1}}+y_n\ b_1 \quad (3.11)$$

$$a_2=\frac{h^2\ y_n^{II}}{2!\,x_{n+1}^2}+\frac{h\ y_n^I}{x_{n+1}}\ b_1+y_n\ b_2 \quad (3.12)$$

The error on truncation is

$$Lte_n=x_{n+1}^7\left(a_0\sum_{j=1}^{6}C_7^{(j)}(-)^{j+1}+a_1\sum_{j=1}^{5}C_6^{(j)}(-)^{j+1}+a_2\sum_{j=1}^{4}C_5^{(j)}(-)^{j+1}\right)-\frac{h^7y_n^{(VII)}}{7!}+0\left(h^8\right)$$

The order of the method is $p=6$

4.0 The stability function of the rational integrator

On the application of (3.1) on the scalar test problem

$$y^1 = \lambda y \; , \; \mathrm{Re}(\lambda) < 0 \qquad (4.1)$$

with the method parameters as defined in (3.5 - 3. 12) we have that $a_1 = \frac{z \; y_n}{3x_{n+1}}, \; a_2 = \frac{z^2 y_n}{30 \, x_{n+1}^2}$

$$B_1 = \frac{-2 \; z}{3x_{n+1}}, \; B_2 = \frac{z^2}{5 \, x_{n+1}^2} \qquad (4.2)$$

$$B_3 = \frac{-z^3}{30x_{n+1}^3}, \; B_4 = \frac{z^4}{360x_{n+1}^4}, \; z = \lambda h, \mathrm{Re}(\lambda) < 0$$

so that

$$y_{n+1} = \left[\frac{1+\frac{1}{3}z + \frac{1}{30}z^2}{1-\frac{2}{3}z + \frac{1}{5}z^2 - \frac{1}{30}z^3 + \frac{1}{360}z^4} \right] y_n \qquad (4.3)$$

$$n = 0, 1, \; 2, \; 3, \cdots, k \, .$$

where

$$\mu(z) = \frac{1+\frac{1}{3}z + \frac{1}{30}z^2}{1-\frac{2}{3}z + \frac{1}{5}z^2 - \frac{1}{30}z^3 + \frac{1}{360}z^4} \qquad (4.4)$$

we notice that the above stability polynomial satisfies the conditions of L-stability.

$$|\mu(z)| \leq 1 \; , \; \mathrm{Re}(z) < 0$$
$$\lim_{|z| \to \infty} \mu(z) = 0 \qquad (4.5)$$

5.0 Numerical Experiments

We present results on two test problems. The first is an order test. The second is a singularity test.

***Problem* 1**

Evans and Sangui (1987) [1], Otunta and Ikhile (1999) [11], Otunta and Nwachukwu (2003) [12].

$$y^1 = -y, \; y(0) = 1, \quad 0 \leq x \leq 1$$

The numerical results to this problem are presented in Table I and Table II. The results obtained via our proposed scheme (3.1) exhibit remar kable improvement over existing methods.

Problem **2**

Fatunla (1986) [3]; Niekerk (1987) [13]; Lambert and Shaw (1965) [7]; Luke et al (1975) [9]; Otunta and Ikhile (1999) [11]; Otunta and Nwachwkwu (2003) [12].

$$y^1 = 1 + y^2, \; y(0) = 1, \quad 0 \leq x \leq 1; \quad y(x) = \tan\left(x + \frac{\pi}{4}\right)$$

The numerical results to this problem are presented in Table III. The method (3.1) compares favourably with existing methods both within the neighborhood of the singular point $x = \frac{\pi}{4}$ out without it.

Table I: $y^1 = -y$, $y(0) = 1$, $h = 0.1$, $0 \le x \le 1$ |Error|

x	**RKM**	**Evans and Sangui (1987)**	**Otunta and Ikhile (1999)**	**Otunta and Nwachukwu (2003)**	**Formula (3.1)**
0.1	8.537769 (-8)	1.993220 (-7)	1.84352654 (-8)	1.366 (-10)	1.28526804578 (-12)
0.2	1.44768 (-7)	3.509680 (-7)	4.5222131(-8)	2.732 (-10)	2.57048946047 (-12)
0.3	2.046044 (-7)	4.844737 (-7)	5.391025 (-8)	4.098 (-10)	3.85571214472 (-12)
0.4	2.399892 (-7)	5.776375 (-7)	6.153260 (-8)	5.464 (-10)	5.14085963419 (-12)
0.5	2.755226 (-7)	6.574187 (-7)	6.684795 (-8)	6.830 (-10)	6.42597190571 (-12)
0.6	2.95112 (-7)	7.121759 (-7)	7.008194 (-8)	8.195 (-10)	7.71131460603 (-12)
0.7	3.166163 (-7)	7.543546 (-7)	7.208025 (-8)	9.561 (-10)	8.99640394589 (-12)
0.8	3.270209 (-7)	7.796861(-7)	7.369261 (-8)	1.0927 (-9)	1.028156420546 (-11)
0.9	3.314901 (-7)	7.922771(-7)	7.384228 (-8)	1.2293 (-9)	1.156673312446 (-11)
1.0	3.315419 (-7)	7.948056 (-7)	7.406362 (-8)	1.3659 (-9)	1.285187429027 (-11)

Table II $y^1 = -y$, $y(0) = 1$, $h = 0.01$, $0 \le x \le 1$ |Error|

x	**Otunta and Ikhile (1999)**	**Otunta and Nwachukwu (2003)**	**Formula (3.1)**
0.1	2.375403 (-10)	0.98 (-11)	0.0000000 (-16)
0.2	9.206934 (-11)	1.96 (-11)	0.0000000 (-16)
0.3	1.911267 (-10)	2.94 (-11)	1.4986443 (-16)
0.4	1.519854 (-10)	3.92 (-11)	3.3125162 (-16)
0.5	1.539043 (-10)	4.90 (-11)	1.8304483 (-16)
0.6	7.519851 (-11)	5.87 (-11)	4.0459164 (-16)
0.7	1.584570 (-10)	6.85 (-11)	5.5892865 (-16)
0.8	9.454559 (-11)	7.83 (-11)	6.1771168 (-16)
0.9	1.231553 (-10)	8.81 (-11)	5.4614159 (-16)
1.0	4.138562 (-11)	9.79 (-11)	6.0357980 (-16)

Table III: $y^1 = 1 + y^2$, $y(0) = 1$, $h = 0.05$, $0 \le x \le 1$ |Error|

x	**Theoretical solution**	**Fatunla (1986)** P=1	P=2	P=3	P=4	P=5	P=9
0.00	1.000,000,000						
0.10	1.223048880	1.228(-2)	8.277(-4)				
0.20	1.508497647	2.99(-2)	1.894(-3)	2.298(-4)	5.953(-5)		
0.30	1.895765123	5.580(-2)	3.520(-3)	5.507(-4)	5.434(-5)	2.303(-5)	
0.40	2.464962757	1.045(-1)	6.473(-3)	6.726(-4)	1.685(-4)	2.408(-5)	
0.50	3.408223442	2.134(-1)	1.312(-2)	1.754(-3)	2.238(-4)	7.234(-5)	1.337(-6)
0.60	5.331855223	5.562(-1)	3.405(-2)	4.206(-3)	7.541(-4)	1.272(-4)	4.552(-6)
0.70	11.681373800	3.092(0)	1.762(-1)	2.016(-2)	3.322(-3)	7.849(-1)	7.349(-5)
0.80	-68.479668346	3.776(1)	8.284(0)	7.037(-1)	1.282(-1)	2.622(-2)	1.585(-2)
0.90	-8.687629546	1.166(0)	1.952(-1)	6.093(-3)	2.108(-2)	6.158(-3)	4.204(-1)
1.00	-4.588037825	3.421(-1)	2.34(-1)	9.291(-3)	1.610(-2)	1.054(-2)	2.127(0)

x Theoretical solution		Niekerk (1987)		Lambert and Shaw (1965)	Luke et al (1975)	Otunta and Ikhile (1999)	Otunta and Nwachukwu (2003)	Formula (3.1)
		P=1	P=2					
0.00	1.000,000,000							
0.10	1.223048880	1.0(-4)	2.0(-6)	8 (-8)	2 (-4)	2.420 (-7)	4.43301 (-8)	4.460393447050195 (-8)
0.20	1.508497647	4.0 (-4)	8.0 (-6)	8 (-7)	5 (-4)	2.893 (-7)	4.69359 (-8)	4.746470559009062 (-8)
0.30	1.895765123	1.0 (-3)	2.0 (-5)	4 (-7)	1 (-3)	6.972 (-7)	5.21695 (-8)	5.297316414964577 (-8)
0.40	2.464962757	2.0 (-3)	5.0 (-5)	7 (-7)	2 (-3)	1.601 (-6)	6.16053 (-8)	6.275687152517258 (-8)
0.50	3.408223442	5.0 (-3)	1.0 (-4)	2 (-6)	5 (-3)	3.970 (-6)	7.92477 (-8)	8.092770849906523 (-8)
0.60	5.331855223	1.0 (-2)	5.0 (-4)	4 (-6)	1 (-2)	1.562 (-5)	1.179648 (-7)	1.2067514376316450 (-7)
0.70	11.681373800	7.0 (-2)	3.0 (-3)	2 (-5)	8 (-2)	6.886 (-5)	2.512394 (-7)	2.5728283231602810 (-7)
0.80	-68.479668346					2.828 (-3)	1.4619233 (-6)	1.49767978700874800 (-6)
0.90	-8.687629546					5.382 (-5)	1.880853 (-7)	1.9160611775376290 (-7)
1.00	-4.588037825					1.807 (-5)	1.027867 (-7)	1.0461534818915210 (-7)

6.0 Conclusion

Our proposed scheme compares favourably with existing methods. It does not require the solution of system of linear equations at every integration step unlike the methods of Fatunla (1982) [2], and Luke et al (1975) [9]. Moreover, from our numerical results, our present method performs better than Otunta and Ikhile (1999) [11], which enjoys similar advantages. The method is L-stable and it has the additional advantage of resolving singularities.

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