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A 5-step maximal order method for direct solution of second order Ordinary Differential Equations

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Abstract

In this work, we propose a direct solution of second order ordinary differential equations without reduction to systems of first order equations. The method is based on collocating the differential system arising from a polynomial basis function at selected grid points x_{n+i}, i = 0(1)5, which yields a five-step continuous method. The computational burden and computer time wastage involved in the usual reduction of second order problems into system of first order equations are avoided by this approach. The method is symmetric, consistent and of order nine. The interval of absolute stability of the method is sufficient for moderately stiff problems. The accuracy of the method is shown with some test examples.

Key words: Collocation, differential system, basis function, symmetric, continuous coefficient.
C.R. Categories: G 1.7

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1.0 Introduction

It has been observed in literature that solution of general higher order ordinary differential equations of the form

$$y^{(m)} = f(t, y, y^{1}, y^{11}, ..., y^{(m-1)}), \; y^{(s)}(a) = y_0^{(s)}, s = 0(1)m-1 \tag{1.1}$$

involves the reduction of the problem (1.1) into a system of first order equations

$$y' = f(x, y), \; y(a) = \mu, \; f \in C^1[a, b], y, x \in R^n \tag{1.2}$$

Any appropriate numerical or analytical methods are then adopted for solving the resulting equations (1.2). [See Spiegel (1971) [11], Lambert (1973) [10], Goult et al (1973) [5], Jain (1984) [8], lxaru (1984) [6], Jaeques and Judd (1987) [7], Fatunla (1988) [4], Bun and Vasil' Yer (1992) [3], Awoyemi (1992) [1], Jaun (2001) [9]]. Awoyemi and Kayode (2003) [2] highlighted some of the direct methods for solving (1.1) for m = 2 when the derivative is present in the right side. In this work, we propose numerical techniques for a direct solution of initial value problems (1.1) for $m = 2$. The accuracy of the method is compared with Awoyemi and Kayode (2003) [2] for k = 4.

2.0 Derivation of the Method

In this section, we discuss the development of our collocation methods for the solution of second order ordinary differential equations (1.1) directly without reducing it to first order system of equations. We develop a maximal order linear multistep method with continuous coefficients of the form

$$y(t) = \sum_{j=0}^{k-1} \alpha_j(t) y_{n+j} + \sum_{j=2}^{k} \beta_j(t) f_{n+j}, \; t \in (0,1], \tag{2.1}$$

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The order of accuracy is maximized to enhance significant accuracy.
We propose a power series of a single variable x as a basis function in the form

$$P(x)=\sum_{j=0}^{\infty}a_j x^j . \tag{2.2}$$

Also, let the approximate solution to problem (1.1) be given as

$$y(x)=\sum_{j=0}^{2k}a_j x^j \tag{2.3}$$

$a_j \in \Re,\ j=0(1)2k,\ y\in C^m(a,b)\subset P(x)$.

The first and second derivatives of (2.3) are

$$y'(x)=\sum_{j=1}^{2k} j a_j x^{j-1}, \tag{2.4}$$

$$y''(x)=\sum_{j=2}^{2k} j(j-1)a_j x^{j-2}, \tag{2.5}$$

It is assumed that problem (1.1) satisfies the hypotheses of existence and uniqueness theorem of Lambert (1973 p. 2).[10].
From (1.1) and (2.5), we have

$$\sum_{j=2}^{2k} j(j-1)a_j x^{j-2} = f(x, y(x), y'(x)) \tag{2.6}$$

Collocating (2.6) at $x_{n+i},\ i=0(1)k$, and interpolating (2.3) *at* $x_{n+i},\ i=0(1)k-1$, give

$$\sum_{j=2}^{2k} j(j-1)a_j x_{n+i}^{j-2} = f_{n+i},\ i=0(1)k \tag{2.7}$$

$$\sum_{j=0}^{2k} a_j x_{n+i}^{j} = y_{n+i},\ i=0(1)k-1 \tag{2.8}$$

where $f_{n+i}=f(x_{n+i}, y_{n+i}, y'_{n+i})$, y_{n+i} *is the numerical approximation to* $y(x_{n+i})$ at x_{n+i}, and $x_{n+i}=x_n+ih.$ Solving equations (2.7) and (2.8) for a_j's, j = 0(1)10, and substituting for the values of a_j's in equation (2.3) for k = 5, and by letting

$$t=\frac{1}{h}(x-x_{n+k-1}) \text{ with } \frac{dt}{dx}=\frac{1}{h},\ t\in(0,1], \tag{2.9}$$

we obtain, after some algebraic simplification, the continuous implicit methods (2.1) in power series of t, where the coefficients α_j, β_j are obtained to be

$$\alpha_4(t)=-\frac{1}{470208}\{-470208-\frac{11262704}{5}t+4089696t^3+2060344t^4-\frac{4856481}{5}t^5$$
$$-888251t^6-142098t^7+32802t^8+11887t^9+961t^{10}\}$$

$$\alpha_3(t)=\frac{1}{14694}\{-\frac{193536}{5}t+124032t^3+168272t^4+\frac{279216}{5}t^5-41440t^6-40716t^7$$
$$-13659t^8-2097t^9-124t^{10}\}$$

$$\alpha_2(t) = \frac{1}{235104}\{-\frac{8688384}{5}t + 1940832t^3 - 2156424t^4 - \frac{14906241}{5}t^5 + 13125t^6 + 1009134t^7 + 447138t^8 + 77727t^9 + 4929t^{10}\}$$

$$\alpha_1(t) = \frac{1}{14694}\{\frac{332288}{5}t - 103488t^3 + 22280t^4 + \frac{418572}{5}t^5 + 11648t^6 - 21756t^7 - 10815t^8 - 1939t^9 - 124t^{10}\}$$

$$\alpha_0(t) = \frac{1}{470208}\{334800t - 449376t^3 + 275528t^4 + 525357t^5 + 38843t^6 - 161262t^7\ 78306t^8 - 14415t^9 - 961t^{10}\}$$

$$\beta_5(t) = \frac{h^2}{312480}\{\frac{3456}{5}t + 336t^3 + 4382t^4 + \frac{28959}{5}t^5 + 3220t^6 + 902t^7 + 126t^8 + 7t^9\}$$

$$\beta_4(t) = \frac{h^2}{49371840}\{-1408896t + 24685920t^2 + 48834520t^3 + 20338430t^4 - 9759309t^5 9644761t^6 - 2410010t^7 - 81060t^8 + 47215t^9 + 4991t^{10}\}$$

$$\beta_3(t) = \frac{h^2}{12342960}\{-42353280t + 94404240t^3 + 39806550t^4 - 30329985t^5 - 19815124t^6\ 69590t^7 + 2135490t^8 + 512855t^9 + 37324t^{10})$$

$$\beta_2(t) = \frac{h^2}{24685920}\{-130529664t + 204655920t^3 - 41865950t^4 - 164964261t^5 - 24292093t^6 + 42577870t^7 + 21573720t^8 + 3931655t^9 + 255843t^{10}\}$$

$$\beta_1(t) = \frac{h^2}{24685920}\{-28918656t + 40383280t^3 - 20197870t^4 - 43026459t^5 - 3740548t^6 + 12776530t^7 + 6212010t^8 + 1133545t^9 + 74648t^{10}\}$$

$$\beta_0(t) = \frac{h^2}{49371840}\{-\frac{8045568}{5}t + 2108232t^3 - 1445346t^4 - \frac{13041777}{5}t^5 - 177401t^6 + 814454t^7 + 396732t^8 + 73759t^9 + 4991t^{10}\} \quad (2.10)$$

The first derivatives of (2.10), using (2.9), are,

$$\alpha_4'(t) = -\frac{1}{470208h}\{-\frac{11262704}{5} + 12269088t^2 + 8241376t^3 - 4856481t^4 - 5329506t^5\ 994686t^6 + - 262416t^7 + 106983t^8 + 9610t^9\}$$

$$\alpha_3'(t) = \frac{1}{14694h}\{-\frac{193536}{5} + 372096t^2 + 673088t^3 + 279216t^4 - 248640t^5 - 285012t^6 - 109272t^7 - 18873t^8 - 1240t^9\}$$

$$\alpha_2'(t) = \frac{1}{235104h}\{-\frac{8688384}{5} + 5822496t^2 - 8625696t^3 - 14906241t^4 + 78750t^5 + 7063938t^6 + 3577104t^7 + 699543t^8 + 49290t^9\}$$

$$\alpha_1'(t) = \frac{1}{14694h}\{\frac{332288}{5} - 310464t^2 + 89120t^3 + 418572t^4 + 69888t^5 - 152292t^6 - 86520t^7 - 17451t^8 - 1240t^9\}$$

$$\alpha_0'(t) = \frac{1}{470208h}\{334800 - 1348128t^2 + 1102112t^3 + 2626785t^4 + 233058t^5 - 1128834t^6 - 626448t^7 - 129735t^8 - 9610t^9\}$$

$$\beta_5'(t) = \frac{h}{312480}\{\frac{3456}{5} + 1008t^2 + 17528t^3 + 28959t^4 + 19320t^5 + 6314t^6 + 1008t^7 + 67t^8\}$$

$$\beta_4'(t) = \frac{h}{49371840}\{-1408896 + 49371840t + 146503560t^2 + 81353720t^3 - 48796545t^4 - 57868566t^5 - 16870070t^6 - 648480t^7 + 424934t^8 + 49910t^9\}$$

$$\beta_3'(t) = \frac{h}{12342960}\{-42353280 + 283212720t^2 + 159226200t^3 - 151649925t^4 - 118890744t^5 - 487130t^6 + 17083920t^7 + 4615695t^8 + 373240t^9\}$$

$$\beta_2'(t) = \frac{h}{24685920}\{-130529664 + 613967760t^2 - 167463800t^3 - 824821305t^4 - 145752558t^5 + 298045090t^6 + 172589760t^7 + 35384895t^8 + 2558430t^9\}$$

$$\beta_1'(t) = \frac{h}{24685920}\{-28918656 + 121149840t^2 - 80791480t^3 - 215132295t^4 - 22443288t^5 + 89435710t^6 + 49696080t^7 + 10201905t^8 + 746480t^9\}$$

$$\beta_0'(t) = \frac{h}{49371840}\{-\frac{8045568}{5} + 6324696t^2 - 5781384t^3 - 13041777t^4 - 1064406t^5 + 5701178t^6 + 3173856t^7 + 663831t^8 + 49910t^9\} \qquad (2.11)$$

Evaluating (2.1) at $x = x_{n+4}$ or at $t = 1$, we have a symmetric discrete scheme

$$31y_{n+5} + 97y_{n+4} - 446y_{n+3} + 446y_{n+2} - 97y_{n+1} - 31y_n = \frac{h^2}{15}\{23f_{n+5} + 665f_{n+4} + 1670f_{n+3} - 1670f_{n+2} - 665f_{n+1} - 23f_n\} \qquad (2.12)$$

of maximal order p = 9 and the principal error constant $c_{p+2} \approx -0.00013484$, interval of absolute stability $x(\theta) = (-7.47, \infty)$, which is P- Stable. The first derivative of (2.12) is given by

$$y_{n+5}' = \frac{1}{146940h}\{-2330081y_{n+4} + 6226558y_{n+3} - 4986558y_{n+2} + 760706y_{n+1} + 329375y_n + \frac{h^2}{336}(\frac{59167208}{5}f_{n+5} + 15211407f_{n+4} + 604522784f_{n+3} - 292042784f_{n+2} - 152111408f_{n+1} - 27916048f_n)\} \qquad (2.13)$$

p = 7, $c_{p+2} \approx -\ 0.0002721$

4.0 Development of Starting Values for the 5-Step Method

To be able to implement our implicit linear 5-step discrete scheme (2.12) and its first derivative (2.13), the following starting values and their derivatives are developed using the same technique for the main method (2.12). Thus at $t = 1$,

$$y_{n+5} = -15y_{n+4} + 50y_{n+3} - 50y_{n+2} + 15y_{n+1} + y_n + \frac{h^2}{3}(8f_{n+4} + 36f_{n+3} - 36f_{n+2} - 8f_{n+1})$$ p = 7, $c_{p+2} = 0.006085$ (3.1)

$$y_{n+5}' = -\frac{45607}{644}y_{n+4} + \frac{217835}{966h}y_{n+3} - \frac{74945}{322h}y_{n+2} + \frac{23591}{322h}y_{n+1} + \frac{1325}{276h}y_n + \frac{h}{7245}\{64814f_{n+4} + 356463f_{n+3} - 432063f_{n+2} - 93164f_{n+1}\} \qquad (3.2)$$

p = 7, c_{p+2} = -0.02948,

$$y_{n+4} = -16y_{n+3} + 34y_{n+2} - 16y_{n+1} - y_n + \frac{h^2}{3}(8f_{n+3} + 44f_{n+2} + 8f_{n+1}) \quad (3.3)$$

p = 6, c_{p+2} = 0.006085,

$$y'_{n+4} = -\frac{1124}{15h}y_{n+3} + \frac{4399}{30h}y_{n+2} - \frac{1012}{15h}y_{n+1} - \frac{127}{30h}y_n + \frac{h}{45}\{400f_{n+3} + 2767f_{n+2} + 508f_{n+1}\}$$

p=5, c_{p+2} = 0.00286, (3.4)

$$y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n + h^2(f_{n+2} - f_{n+1}) \quad (3.5)$$

p = 3, c_{p+2} = 0.0833,

$$y'_{n+3} = \frac{11}{2h}y_{n+2} - \frac{10}{h}y_{n+1} + \frac{9}{2h}y_n + \frac{h}{6}\{11f_{n+2} - 29f_{n+1}\} \quad (3.6)$$

p = 3, c_{p+2} = -0.353

$$y_{n+2} = 2y_{n+1} - y_n + h^2 f_{n+1}, \text{ p = 2, } c_{p+2} = 0.0833, \quad (3.7)$$

$$y'_{n+2} = \frac{1}{h}(y_{n+1} - y_n) + \frac{h}{6}(11f_{n+1} - 2f_n), \text{ p = 2, } c_{p+2} = -0.375 \quad (3.8)$$

$$y_{n+1} = y_n + (jh)y'_n + \frac{(jh)^2}{2!}f_n + \frac{(jh)^3}{3!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + O(h^4) \quad (3.9)$$

$$y'_{n+1} = y'_n + (jh)f_n + \frac{(jh)^2}{2!}\left\{\frac{\partial f_n}{\partial x_n} + y'_n\frac{\partial f_n}{\partial y_n} + f_n\frac{\partial f_n}{\partial y'_n}\right\} + O(h^3) \quad (3.10)$$

The starting values y_n, y'_n are initial values in the given problem.

4.0 Numerical Experiment

In this section, we illustrate the accuracy of our new continuous method developed for direct solution of second order ordinary differential equations by considering the numerical results obtained when applied to solve the following non-linear test problems.

(1) $y''(x) = \frac{(y')^2}{2y} - 2y$, $y(\frac{\Pi}{6}) = \frac{1}{4}$, $y(\frac{\Pi}{6}) = \frac{\sqrt{3}}{2}$, $h = \frac{1}{40}$

Theoretical solution is given as $y(x) = Sin^2 x$.

(2) $y'' - x(y')^2 = 0$, $y(0) = 1$, $y'(0) = \frac{1}{2}$; $h = \frac{1}{40}$

Theoretical solution is $y(x) = 1 + \frac{1}{2}\ln\{\frac{(2+x)}{(2-x)}\}$.

Table 3: Comparison of errors for Problem 2(*i*) for *k* = 4, 5

X	**Errors for *k* = 4**	**Errors for *k* = 5**
1.1	0.2797988606D-06	0.2294958138D-06
1.2	0.2437232373D-06	0.1981423201D-06
1.3	0.1327387861D-06	0.9744664964D-07
1.4	0.6180288070D-07	0.8284189512D-07

1.5	0.3411234970D-06	0.3455922933D-06
1.6	0.6984167915D-06	0.6856775822D-06
1.7	0.1118856739D-05	0.1089959666D-05
1.8	0.1580221977D-05	0.1537887582D-05
1.9	0.2054110707D-05	0.2002690424D-05
2.0	0.2507689327D-05	0.2453116345D-05

Table 4: Comparison of errors for Problem 2(ii) for k = 4, 5

X	Errors for k = 4	Errors for k = 5
1.1	0.1527105109D-09	0.1994382437D-10
1.2	0.1163118268D-08	0.6717315593D-09
1.3	0.3928643633D-08	0.2819816824D-08
1.4	0.9542545865D-08	0.7497457988D-08
1.5	0.1943261863D-07	0.1603567235D-07
1.6	0.3606988486D-07	0.3069697807D-07
1.7	0.6153828203D-07	0.5341827980D-07
1.8	0.1003970409D-06	0.8833716314D-07
1.9	0.1591855572D-06	0.1413690136D-06
2.0	0.2483634625D-06	0.2219141864D-06

Note: Awoyemi and Kayode (2003) [2] and the new method (2.12) are represented in the Tables above by k = 4, 5 respectively**.**

5.0 Conclusion

In this paper, we developed a five-step maximal order method and show its efficiency over an existing method [Awoyemi and Kayode (2003) [2]]. The results obtained from the test problems 1 and 2 show a better accuracy of the new method over Awoyemi and Kayode (2003) [2].

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