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Successive substitution one-leg hybrid P-stable LMM for initial value problems in Second Order Ordinary Differential Equations.

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Abstract

This paper derives P-stable successive substitution one-leg hybrid linear multistep methods for the numerical solution of second order initial value problems in ordinary differential equations without explicit first order derivative. The methods are demonstrated by a numerical example also considered by Fatunla, et al (1997) [8], Fatunla (1985) [7], Cash (1981) [4] ***and*** Lambert and Watson (1976) [17].

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1.0 Introduction

This paper considers the second order initial value problem of the form

$$y'' = f(x,y); \quad |y|,|y'| < \infty, \quad y(a) = y_0,\ y'(a) = y_0', \quad a \le x \le b \tag{1.1}$$

without explicit first order derivative. This problem arises in orbital mechanics where it is required to model the path of an object in space. The solution of this differential equation is often oscillatory. The linear multistep method for the numerical solution is of the form

$$\sum_{j=0}^{k} \sigma_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}; \qquad \sigma_k \neq 0, \quad k \geq 2 \tag{1.2}$$

This generates the discrete solution $\{y_n\}_{n=0}$ on the discrete point set $\{x_n / x_n = a + nh,\ n = 0,1,...\}$, of the specific second order IVP (1.1) and can be restated as

$$\rho(\mathrm{E}) y_n = h^2 \sigma(\mathrm{E}) f_n \tag{1.3}$$

where E is the shift operator $\rho(\mathrm{E})$ and $\sigma(\mathrm{E})$ are characteristics polynomials

$$E^j y_n = y_{n+j} \tag{1.4}$$

and $$\rho(\mathrm{E}) = \sum_{j=0}^{k} \alpha_j \mathrm{E}^j, \quad \sigma(\mathrm{E}) = \sum_{j=0}^{k} \beta_j \mathrm{E}^j, \quad k \geq 2 \tag{1.5}$$

respectively. Fatunla (1985) [7], has proposed the application of the integration formulae

$$\begin{aligned} y_{n+2} - 2y_{n+1} + y_n &= h^2 f\left(\theta_n, \sum_{j=0}^{2} \beta_j y_{n+j} + \gamma y_{n+r}\right) \\ y_{n+r} + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} &= h^2 f\left(\theta_0', \sum_{j=0}^{2} \beta_j'' y_{n+j}\right) \end{aligned} \tag{1.6}$$

where $$\theta_n = \sum_{j=0}^{2} \beta_j x_{n+j} + \gamma x_{n+r}, \; \theta'_n = \sum_{j=0}^{2} \beta' x_{n+j} \tag{1.7}$$

with $$\sum_{j=0}^{2} \beta_j^1 = 1, \; \sum_{j=0}^{2} \beta_j + \gamma = 1 \tag{1.8}$$

The resultant examples are

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f\left(\theta_n, \frac{1}{9} y_n + \frac{14}{18} y_{n+1} + \frac{1}{9} y_{n+2}\right) \quad \left(c_4 = \frac{1}{36}\,;\ p = 2\right) \tag{1.9}$$

and

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f\left(\theta_n, \frac{1}{12} y_n + \frac{5}{6} y_{n+1} + \frac{1}{12} y_{n+2}\right) \quad \left(c_6 = -\frac{1}{240}\,;\ p = 2\right) \tag{1.10}$$

Fatunla, Ikhile and Otunta (1997) [8], considered the application of the integration formulae of general form

$$y_{n+r} + \alpha'_1 y_{n+1} + \alpha'_2 y_{n+2} = h^2 f\left(\sum_{j=0}^{2} \beta'_j x_{n+j}, \sum_{j=0}^{2} (\beta'_j y_{n+j} + h^2 \delta'_j f_{n+j})\right)$$

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f\left(\sum_{j=0}^{2} \bar{\beta}_j x_{n+j}, \sum_{j=0}^{2} (\bar{\beta}_j y_{n+j} + h^2 \bar{\delta}_j f_{n+j}) + h^2 \bar{\delta}_3 f_{n+r}\right) \tag{1.11}$$

such that $$\sum_{j=0}^{2} \beta'_j = \sum_{j=o}^{2} \bar{\beta}_j = 1, \; 0 < r < 2 \tag{1.12}$$

with an example given as

$$\begin{aligned} y_{n+2} - 2y_{n+1} + y_n = h^2 f\Bigg(& x_{n+1}, \frac{1}{40} y_n + \frac{19}{20} y_{n+1} + \frac{1}{40} y_{n+2} \\ & + h^2\left(-\frac{1}{600} f_n + \frac{1}{800} f_{n+1} - \frac{1}{600} f_{n+2}\right)\Bigg) \end{aligned} \tag{1.13}$$

The linear multistep methods are symmetric by requirement of P-stability. See Lambert and Watson (1975) [17]. More methods are in Avdelas et al (2001) [1], Hairer etal (2004) [9], Petzold (1981) [14] and Chawla et al (1985) [4].

2.0 A Class of P-Stable Methods

In this paper, consider the application of the integration formulae

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f\left(\sum_{j=0}^{2} \beta_j x_{n+j}, \sum_{j=0}^{2} (\beta_j y_{n+j} + h^2 \delta_j f_{n+j}) + h^2 \delta_3 F_{n+r_m}\right)$$

$$Y_{n+r_k} + \alpha_1^{(k)} y_{n+1} + \alpha_2^{(k)} y_{n+2} = h^2 f(\sum_{j=0}^{2} \beta_j^{(k)} x_{n+j}, \sum_{j=0}^{2} (\beta_j^{(k)} y_{n+j} + h^2 \delta_j^{(k)} f_{n+j}) + h^2 \delta_3^{(k)} F_{n+r_{k-1}})$$

$$Y_{n+r_o} + \alpha_1^{(o)} y_{n+1} + \alpha_2^{(o)} y_{n+2} = h^2 f(\sum_{j=0}^{2} \beta_j^{(o)} x_{n+j}, \sum_{j=0}^{2} (\beta_j^{(o)} y_{n+j} + h^2 \delta_j^{(o)} f_{n+j})) \qquad (2.1)$$

on the IVP (1.1) where $k = 1,2,\ldots,m$ and $F_{n+r_o} = f\left(x_{n+r_o}, Y_{n+r_o}\right)$ such that

$$\sum_{j=0}^{2} \beta_j^{(o)} = \sum_{j=0}^{2} \beta_j^{(k)} = \sum_{j=o}^{2} \beta_j = 1, \quad 0 < r_k < 2$$

The development of P-stable hybrid LMM (2.1) can be obtained by Pade approximation method, in this regard, the stability polynomial of (2.1) is given by

$$\pi(w,z) = \sum_{j=0}^{k} Q_j\left(z^2\right) w^{k-j} \qquad (2.2)$$

where $Q_j\left(z^2\right)$ is a polynomial in z^2. For the method (2.1) to be P-stable, the characteristics polynomial need take the form [for more see Butcher (2002) [3]]

$$\pi(w,z,s) = \left(p_s(-z)w - p_s(z)\right)\left(p_s(z)w - p_s(-z)\right) \qquad (2.3)$$

$$= p_s(z)p_s(-z)w^2 - \left(p_s(z)^2 + p_s(-z)^2\right)w + p_s(z)p_s(-z) \qquad (2.4)$$

where $$p_s(z) = 1 + \frac{sz}{2s} + \frac{s(s-1)z^2}{2s(2s-1)2!} + \cdots + \frac{s(s-1)\cdots 2.1.z^s}{2s(2s-1)\cdots(s+1)s!} \qquad (2.5)$$

is a polynomial of degree s. Thus the rational expression is $$R_{s,s}(z) = \frac{p_s(z)}{p_s(-z)} \qquad (2.6)$$

is an (s,s)-Pade approximation to the exponential function.

3.0 The one-leg Hybrid P-Stable LMM

The interest is in the integration formulae for m=1, which can be written alternatively as, Ayo [2005] [2].

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f(\sum_{j=0}^{2} \beta_j x_{n+j}, \sum_{j=0}^{2} (\beta_j y_{n+j} + h^2 \delta_j f_{n+j}) + h^2 \delta_3 F_{n+r})$$

$$Y_{n+r} + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} = h^2 f(\sum_{j=0}^{2} \beta'_j x_{n+j}, \sum_{j=0}^{2} (\beta'_j y_{n+j} + h^2 \delta'_j f_{n+j}) + h^2 \delta_4 F_{n+k})$$

$$Y_{n+k} + \mu_1 y_{n+1} + \mu_2 y_{n+2} = h^2 f(\sum_{j=0}^{2} \overline{\beta}_j x_{n+j}, \sum_{j=0}^{2} (\overline{\beta}_j y_{n+j} + h^2 \overline{\delta}_j f_{n+j})) \qquad (3.1)$$

such that $\sum_{j=0}^{2}\beta_j \sum_{j=0}^{2}\beta'_j = \sum_{j=0}^{2}\overline{\beta}_j = 1,\ 0 < k, r < 2$ this implies

$$y_{n+2} - 2y_{n+1} + y_n = h^2 f(\sum_{j=0}^{2}\beta_j x_{n+j}, \sum_{j=0}^{2}(\beta_j y_{n+j} + h^2\delta_j f_{n+j}) + h^2\delta_3 f(-\alpha_1 y_{n+1} - \alpha_2 y_{n+2}$$

$$+ h^2 f(\sum_{j=0}^{2}\beta'_j x_{n+j}, \sum_{j=0}^{2}(\beta'_j y_{n+j} + h^2\delta'_j f_{n+j}) + h^2\delta_4 f(-\mu_1 y_{n+1} - \mu_2 y_{n+2}$$

$$+ h^2 f(\sum_{j=0}^{2}\overline{\beta}_j x_{n+j}, \sum_{j=0}^{2}(\overline{\beta}_j y_{n+j} + h^2\overline{\delta}_j f_{n+j}))))))\qquad (3.2)$$

Applying the scalar test problem $y'' = -\lambda^2 y$, we have,

$$y_{n+2} - 2y_{n+1} + y_n = -\lambda^2 h^2(\sum_{j=0}^{2}(\beta_j y_{n+j} - \lambda^2 h^2\delta_j y_{n+j}) - \lambda^2 h^2\delta_3(-\alpha_1 y_{n+1} - \alpha_2 y_{n+2}$$

$$-\lambda^2 h^2(\sum_{j=0}^{2}(\beta'_j y_{n+j} - \lambda^2 h^2\delta'_j y_{n+j}) - \lambda^2 h^2\delta_4(-\mu_1 y_{n+1} - \mu_2 y_{n+2} - \lambda^2 h^2$$

$$(\sum_{j=0}^{2}\overline{\beta}_j y_{n+j} - \lambda^2 h^2\overline{\delta}_j y_{n+j})))))\qquad (3.3)$$

The result is the polynomial

$$\begin{aligned}\pi(w,z) = (1 &- z^2\beta_2 - z^4(\delta_2 - \delta_3\alpha_2) - z^6\delta_3\beta'_2 - z^8(\delta_3\delta'_2 - \delta_3\delta_4\mu_2)\\ &- z^{10}\delta_3\delta_4\overline{\beta}_2 - z^{12}\delta_3\delta_4\overline{\delta}_2)w^2 - (2 + z^2\beta_1 + z^4(\delta_1 + \delta_3\alpha_1)\\ &+ z^6\delta_3\beta'_1 + z^8(\delta_3\delta'_1 + \delta_3\delta_4\mu_1) + z^{10}\delta_3\delta_4\overline{\beta}_1 + z^{12}\delta_3\delta_4\overline{\delta}_1)w \qquad (3.4)\\ &+ (1 - z^2\beta_o - z^4\delta_o - z^6\delta_3\beta'_o - z^8\delta_3\delta'_o - z^{10}\delta_3\delta_4\overline{\beta}_o - z^{12}\delta_3\delta_4\overline{\delta}_o)\end{aligned}$$

Again the concept of Pade approximation is applied in deriving the coefficients of the scheme. The necessary polynomial is of degree s = 6 given as

$$p_6(z) = 1 + \frac{z}{2} + \frac{5z^2}{44} + \frac{z^3}{66} + \frac{z^4}{792} + \frac{z^5}{15840} + \frac{z^6}{665280}\qquad (3.5a)$$

$$p_6(-z) = 1 - \frac{z}{2} + \frac{5z^2}{44} - \frac{z^3}{66} + \frac{z^4}{792} - \frac{z^5}{15840} + \frac{z^6}{665280}\qquad (3.5b)$$

But
$$\pi(w,z,s) = p_s(z)p_s(-z)w^2 - (p_s(z)^2 + p_s(-z)^2)$$
$$+ p_s(z)p_s(-z)\qquad (3.6)$$

then

$$\pi(w,z,6)=(1-\frac{z^2}{44}+\frac{5z^4}{17424}-\frac{z^6}{365904}+\frac{z^8}{43908480}$$
$$-\frac{z^{10}}{5269017600}+\frac{z^{12}}{442597478400})w^2-(2+\frac{21z^2}{22}+\frac{533z^4}{8712}$$
$$+\frac{533z^6}{457380}+\frac{169z^8}{21954240}+\frac{41z^{10}}{2634508800}+\frac{z^{12}}{221298739200})w$$
$$+(1-\frac{z^2}{44}+\frac{5z^4}{17424}-\frac{z^6}{365904}+\frac{z^8}{43908480}-\frac{z^{10}}{5269017600}$$

$$+\frac{z^{12}}{442597478400}) \tag{3.7}$$

Comparing coefficients of z in equations (3.4) and (3.7) we have the corresponding system of equations given as

$$\beta_2=\frac{1}{44};\ \delta_2-\delta_3\alpha_2=-\frac{5}{17424};\ \delta_3\beta_2'=\frac{1}{365904};\ \delta_3\delta_2'-\delta_3\delta_4\mu_2=-\frac{1}{43908480};$$
$$\delta_3\delta_4\overline{\beta}_2=\frac{1}{5269017600}\ \delta_3\delta_4\overline{\delta}_2=-\frac{1}{442597478400};\ \beta_1=\frac{21}{22};\delta_1+\delta_3\alpha_1=\frac{533}{8712};$$
$$\delta_3\beta_1'=\frac{533}{457380};\ \delta_3\delta_1'+\delta_3\delta_4\mu_1=\frac{169}{21954240};\ \delta_3\delta_4\overline{\beta}_1=\frac{41}{2634508800}$$
$$\delta_3\delta_4\overline{\delta}_1=\frac{1}{221298739200};\ \beta_o=\frac{1}{44};\delta_o=-\frac{5}{17424};\ \delta_3\beta_o'=\frac{1}{365904};$$
$$\delta_3\delta_o'=-\frac{1}{43908480};\ \delta_3\delta_4\overline{\beta}_o=\frac{1}{5269017600};\delta_3\delta_4\overline{\delta}_o=-\frac{1}{4425974784\ 00}$$
$$\beta_o+\beta_1+\beta_2=1;\beta_o'+\beta_1'+\beta_2'=1;\overline{\beta}_o+\overline{\beta}_1+\overline{\beta}_2=1;\ \delta_o+\delta_1+\delta_2+\delta_3=0;$$
$$\delta_o'+\delta_1'+\delta_2'+\delta_4=0;\ \overline{\delta}_o+\overline{\delta}_1+\overline{\delta}_2=0 \tag{3.8}$$

set $$\alpha_1=2-r,\ 0<r<2;\ \alpha_2=1-r\ ,0<k<2 \tag{3.9}$$

and $$\mu_1=2-k,\ \mu_2=1-k \tag{3.10}$$

This equations are solve to give the following

$$\beta_o=\frac{1}{44},\beta_1=\frac{21}{22},\beta_2=\frac{1}{44};\beta_o^{'}=\frac{5}{2142},\beta_1^{'}=\frac{2132}{2142},\beta_2^{'}=\frac{5}{2142}\overline{\beta}_o=\frac{1}{84},\overline{\beta}_1=\frac{82}{84},\overline{\beta}_2=\frac{1}{84}$$
$$\delta_o=\frac{-5}{17424},\delta_1=\frac{107644}{1829520}+\frac{2142r}{1829520},\delta_2=\frac{1617}{1829520}-\frac{2142r}{1829520},\delta_3=\frac{2142}{1829520}$$
$$\delta_o^{'}=\frac{-1}{51408},\delta_1^{'}=\frac{40392}{6168960}+\frac{84k}{6168960};\delta_2^{'}=\frac{-36}{6168960}-\frac{84k}{6168960},\delta_4=\frac{84}{6168960}$$
$$\overline{\delta}_o=-\frac{1}{7056},\overline{\delta}_1=\frac{1}{3528},\overline{\delta}_2=-\frac{1}{7056} \tag{3.11}$$

The P-stable one-leg hybrid method with coefficients given from the solution of the above equations gives the scheme

$$y_{n+2}=2y_{n+1}-y_n+h^2f(x_{n+1},\frac{1}{44}y_{n+2}+\frac{42}{44}y_{n+1}+\frac{1}{44}y_n+h^2((\frac{1617}{1829520}$$
$$-\frac{2142r}{1829520})f_{n+2}+(\frac{107644}{1829520}+\frac{2142r}{1829520})f_{n+1}$$
$$+(\frac{-5}{17424})f_n)+h^2(-\frac{36}{6168960}-\frac{84k}{6168960})F_{n+r})$$
$$Y_{n+r}=-(2-r)y_{n+1}-(1-r)y_{n+2}$$
$$+h^2f(x_{n+1},\frac{5}{2142}y_{n+2}+\frac{2132}{2142}y_{n+1}+\frac{5}{2142}y_n+h^2f((-\frac{36}{6168960}$$

$$-\frac{84k}{6168960})f_{n+2}+(\frac{40392}{6168960}+\frac{84k}{6168960})f_{n+1}+(\frac{-1}{51408})f_n)+h^2(\frac{84}{6168960})F_{n+k})$$
$$Y_{n+k}=-(2-k)y_{n+1}-(1-k)y_{n+2}+h^2f(x_{n+1},\frac{1}{84}y_{n+2}+\frac{82}{84}y_{n+1}+\frac{1}{84}y_n$$
$$+h^2(-\frac{1}{7056}f_{n+2}+\frac{1}{3528}f_{n+1}-\frac{1}{7056}f_n \tag{3.12}$$

Now consider for

$$y_{n+j}=\mathrm{E}^jy_n=e^{jhD}y_n;\, y''_{n+j}=e^{jhD}D^2y_n$$

by calculus of finite difference with $D=\frac{d}{dx}$, the difference operator is given as

$$L[y(x_n);h]=\Bigg\{-e^{2hD}+2e^{hD}-1+h^2D^2\left(\sum_{j=0}^{2}\beta_je^{jhD}-h^2D^2\delta_je^{jhD}\right)-h^2D^2\delta_3\Big(-\alpha_1e^{hD}-\alpha_2e^{2hD}$$
$$-h^2D^2\left(\sum_{j=0}^{2}\beta_j^{'}e^{jhD}-h^2D^2\delta_j^{'}e^{jhD}\right)-h^2D^2\delta_4(-\mu_1e^{hD}-\mu_2e^{2hD}-h^2D^2\left(\sum_{j=0}^{2}\overline{\beta_j}e^{jhD}-h^2D^2\overline{\delta_j}e^{jhD}\right)\Big)\Bigg)\Bigg\}y(x_n) \tag{3.13}$$

The result from the above constants give rise to a method which order can be determined from the order conditions from (3.13)

$h^2:1-\beta_2-\beta_1-\beta_0=0$, $h^3:\frac{4}{3}-\frac{1}{3}-2\beta_2-\beta_1=0$

$$h^4: \frac{2}{3} - \frac{1}{12} - 2\beta_2 - \delta_2 - \frac{1}{2}\beta_1 - \delta_1 - \delta_0 + \delta\alpha_1 + \delta\alpha_2 = 0$$

$$h^5: \frac{4}{15} - \frac{1}{60} - \frac{4}{3}\beta_2 - 2\delta_2 - \frac{1}{6}\beta_1 - \delta_1 + \delta\alpha_1 + 2\delta\alpha_2 = 0$$

$$h^6: \frac{4}{45} - \frac{1}{360} - \frac{2}{3}\beta_2 - 2\delta_2 - \frac{1}{24}\beta_1 + \delta_1 \frac{1}{2} + \frac{1}{2}\delta_3\alpha_1 + 2\delta_3\alpha_2 - 2\delta\beta_2 - \delta_3\beta_1^{'} - \delta_3\beta_0^{'} = 0$$

$$h^7: \frac{8}{315} - \frac{1}{2520} - \frac{4}{15}\beta_2 - \frac{4}{3}\delta_2 - \frac{1}{120}\beta_1 - \frac{1}{6}\delta_1 + \frac{1}{6}\delta_3\alpha_1 + \frac{4}{3}\delta_3\alpha_2 - 2\delta_3\beta_2^{'} - \delta_3\beta_1^{'} = 0$$

$$h^8: \frac{2}{315} - \frac{1}{20160} - \frac{4}{45}\beta_2 - \frac{2}{3}\delta_2 - \frac{1}{720}\beta_1 - \frac{1}{24}\delta_1 + \frac{1}{24}\delta_3\alpha_1 + \frac{2}{3}\delta_3\alpha_2 - 2\delta_3\beta_2^{'} - \delta_3\delta_2^{'} - \frac{1}{2}\delta_3\beta_1^{'} - \delta_3\delta_1^{'}$$
$$- \delta_3\delta_0^{'} + \delta_3\delta_4\mu_1 + \delta_3\delta_4\mu_2 = 0$$

$$h^9: \frac{4}{2835} - \frac{1}{181440} - \frac{8}{315}\beta_2 - \frac{4}{15}\delta_2 - \frac{1}{5040}\beta_1 - \frac{4}{120}\delta_1 + \frac{1}{120}\delta_3\alpha_1 + \frac{4}{15}\delta_3\alpha_2 - \frac{4}{3}\delta_3\beta_2^{'} - 2\delta_3\delta_2^{'}$$
$$- \frac{1}{6}\delta_3\beta_1^{'} - \delta_3\delta_1^{'} + \delta_3\delta_4\mu_1 + 2\delta_3\delta_4\mu_2 = 0$$

$$h^{10}: \frac{4}{14175} - \frac{1}{1814400} - \frac{2}{315}\beta_2 - \frac{4}{45}\delta_2 - \frac{1}{40320}\beta_1 - \frac{1}{720}\delta_1 + \frac{1}{720}\delta_3\alpha_1 + \frac{4}{45}\delta_3\alpha_2 - \frac{2}{3}\delta_3\beta_2^{'} - 2\delta_3\delta_2^{'}$$
$$- \frac{1}{24}\delta_3\beta_1^{'} - \frac{1}{2}\delta_3\delta_1^{'} + \frac{1}{2}\delta_3\delta_4\mu_1 + 2\delta_3\delta_4\mu_2 - \delta_3\delta_4\overline{\beta_2} - \delta_3\delta_4\overline{\beta_1} - \delta_3\delta_4\overline{\beta_0} = 0$$

$$h^{11}: \frac{8}{155925} - \frac{1}{19958400} - \frac{4}{2855}\beta_2 - \frac{8}{315}\delta_2 - \frac{1}{362880}\beta_1 - \frac{1}{5040}\delta_1 + \frac{1}{5040}\delta_3\alpha_1 + \frac{8}{315}\delta_3\alpha_2 - \frac{4}{15}\delta_3\beta_2^{'}$$
$$- \frac{4}{3}\delta_3\delta_2^{'} - \frac{1}{120}\delta_3\beta_1^{'} - \frac{1}{6}\delta_3\delta_1^{'} + \frac{1}{6}\delta_3\delta_4\mu_1 + \frac{4}{3}\delta_3\delta_4\mu_2 - 2\delta_3\delta_4\overline{\beta_2} - \delta_3\delta_4\overline{\beta_1} = 0$$

$$h^{12}: \frac{4}{467775} - \frac{1}{239500800} - \frac{4}{14175}\beta_2 - \frac{2}{315}\delta_2 - \frac{1}{3628800}\beta_1 - \frac{1}{40320}\delta_1 + \frac{1}{40320}\delta_3\alpha_1 + \frac{2}{315}\delta_3\alpha_2$$
$$- \frac{4}{45}\delta_3\beta_2^{'} - \frac{2}{3}\delta_3\delta_2^{'} - \frac{1}{720}\delta_3\beta_1^{'} - \frac{1}{24}\delta_3\delta_1^{'} + \frac{1}{24}\delta_3\delta_4\mu_1 + \frac{2}{3}\delta_3\delta_4\mu_2 - 2\delta_3\delta_4\overline{\beta_2} - \delta_3\delta_4\overline{\delta_2}$$
$$- \delta_3\delta_4\overline{\beta_1} - \delta_3\delta_4\overline{\delta_1} - \delta_3\delta_4\overline{\delta_0} = 0$$

$$h^{13}: \frac{8}{6081075} - \frac{1}{3113510400} - \frac{8}{155925}\beta_2 - \frac{4}{2835}\delta_2 - \frac{1}{39916800}\beta_1 - \frac{1}{362880}\delta_1 + \frac{1}{362880}\delta_3\alpha_1$$
$$+ \frac{4}{2835}\delta_3\alpha_2 - \frac{8}{315}\delta_3\beta_2^{'} - \frac{4}{15}\delta_3\delta_2^{'} - \frac{1}{5040}\delta_3\beta_1^{'} - \frac{1}{120}\delta_3\delta_1^{'} + \frac{1}{120}\delta_3\delta_4\mu_1 + \frac{4}{15}\delta_3\delta_4\mu_2$$
$$- \frac{4}{3}\delta_3\delta_4\overline{\beta_2} - 2\delta_3\delta_4\overline{\delta_2} - \frac{1}{6}\delta_3\delta_4\overline{\beta_1} - \delta_3\delta_4\overline{\delta_2} = 0$$

The local truncation error is given by

$$LTE = \Bigg(\frac{8}{42567525} - \frac{1}{43589145600} - \frac{4}{467775}\beta_2 - \frac{4}{14175}\delta_2 - \frac{1}{479001600}\beta_1 - \frac{1}{3628800}\delta_1$$
$$+ \frac{1}{3628800}\delta_3\alpha_1 + \frac{4}{14175}\delta_3\alpha_2 - \frac{2}{315}\delta_3\beta_2^{'} - \frac{4}{45}\delta_3\delta_2^{'} - \frac{1}{40320}\delta_3\beta_1^{'} - \frac{1}{720}\delta_3\delta_1^{'} + \frac{1}{720}\delta_3\delta_4\mu_1$$
$$+ \frac{4}{45}\delta_3\delta_4\mu_2 - \frac{2}{3}\delta_3\delta_4\overline{\beta_2} - 2\delta_3\delta_4\overline{\delta_2} - \frac{1}{24}\delta_3\delta_4\overline{\beta_1} - \frac{1}{2}\delta_3\delta_4\overline{\delta_1} \Bigg) h^{14} y^{(14)}(x_n) + o\left(h^{15}\right)$$

and this is simplified to give

$$LTE = \frac{h^{14}}{2876883609600} y^{(14)}(x_n) + \frac{h^{15}}{2876883609600} y^{(15)}(x_n) + \frac{73h^{16}}{316457197056000} y^{(16)}(x_n)$$

$$+ \frac{109h^{17}}{949371591168000} y^{(17)}(x_n) + o(h^{18})$$

by considering further terms from (3.13). The order of the method (3.12) is $p = 12$.

4.0 Numerical implementation and applications

For numerical results consider the problem in Lambert and Watson (1975) [17], Cash (1981) [4], Fatunla (1985) [7], and Fatunla, Ikhile and Otunta (1997) [8]:

$$y'' = -y + 0.001e^{ix},$$
$$y(0) = 1, \quad y'(0) = 0.9995\,i \qquad i^2 = -1 \tag{4.1}$$

which theoretical solution can be obtained as

$$y(x) = U(x) + iV(x)$$
$$U(x) = Cos\ x + 0.0005\ x\ Sin\ x \tag{4.2}$$
$$V(x) = Sin\ x - 0.0005\ x\ Cos\ x$$

The solution of the differential system represents motion on a perturbed circular orbit that spirals slowly outward with the distance $T(x) = \sqrt{U^2(x) + V^2(x)}$ (4.3)

from the origin. The system (4.1) was solved numerically with the scheme (3.12) in the interval [0, 40π]. This corresponds to 20 orbits of the point y(x). The integration was effected with uniform meshsizes

$$h = \pi 2^{-q},\ q = 3(1)11 \tag{4.4}$$

using the predictor $y_{n+2}^{[p]} - 2y_{n+1} + y_n = h^2 f_{n+1}$ (4.5)

in our corrector method (3.13) and (3.22). Thus

$$y_{n+2} = 2y_{n+1} - y_n + h^2 f(x_{n+1}, \frac{1}{44} y^{[p]}{}_{n+2} + \frac{42}{44} y_{n+1} + \frac{1}{44} y_n + h^2((\frac{1617}{1829520}$$
$$- \frac{2142r}{1829520}) f^{[p]}{}_{n+2} + (\frac{107644}{1829520} + \frac{2142r}{1829520}) f_{n+1} + (\frac{-5}{17424}) f_n)$$
$$+ h^2(-\frac{36}{6168960} - \frac{84k}{6168960}) F_{n+r}^{[p]})$$

$$Y_{n+r} = -(2-r)y_{n+1} - (1-r)y_{n+2} + h^2 f(x_{n+1}, \frac{5}{2142} y^{[p]}{}_{n+2} + \frac{2132}{2142} y_{n+1} + \frac{5}{2142} y_n$$
$$+ h^2 f((-\frac{36}{6168960} - \frac{84k}{6168960}) f^{[p]}{}_{n+2} + (\frac{40392}{6168960} + \frac{84k}{6168960}) f_{n+1}$$
$$+ (\frac{-1}{51408}) f_n) + h^2(\frac{84}{6168960}) F_{n+k}^{[p]})$$

$$Y_{n+k} = -(2-k)y_{n+1} - (1-k)y_{n+2} + h^2 f(x_{n+1}, \frac{1}{84} y^{[p]}{}_{n+2} + \frac{82}{84} y_{n+1} + \frac{1}{84} y_n$$
$$+ h^2(-\frac{1}{7056} f^{[p]}{}_{n+2} + \frac{1}{3528} f_{n+1} + \left(-\frac{1}{7056}\right) f_n$$

setting v = 1, r = 1, k = 1 makes the methods symmetric as required for P-stability. Observe that all the solutions generated by the new scheme spiral outward in agreement with theoretical solution showing orbital stability, while it was shown in Lambert and Watson (1976) [17], that the solutions generated with the orbitally instable Stormer -Cowell five step scheme

$$y_{n+5} - 2y_{n+4} + y_{n+3} = \frac{h^2}{240}(18\,f_{n+5} + 209\,f_{n+4} + 4\,f_{n+3} + 14\,f_{n+2} - 6\,f_{n+1} + f_n)$$

spiral inward. See numerical solutions given in the tables below, these are compared with existing results as shown.

Table 1: Lambert and Watson (1976) [17]

q	P = 4; Stormer-Cowell: τ	P = 6; Symmetric: τ
4	0.965645	1.003067
5	0.993734	1.002217
6	0.999596	1.002047
9	1.001829	1.001978
12	1.001953	1.001973

Table 2. Cash (1981) [4]

q	P = 4; Estimate of (x_f): τ	P = 6; Estimate of (x_f): τ
4	1.004118	1.001984
5	1.002856	1.001975
6	1.002400	1.001973
9	1.002057	1.001972
12	1.002000	1.001972

Table 3. Fatunla (1984):

q	Estimate of (x_f):τ	10^6x Error in τ $\lvert 10^6 \cdot (\tau-\tau(x_f)) \rvert$
3	1.05365	51682
4	1.01523	132600
5	1.00515	11803
6	1.00276	791
7	1.00217	198
8	1.00202	49
9	1.00198	12
10	1.00197	3

Table 4. Fatunla (1985) [7]

q	Estimate of $\tau(x_f)$: τ	10^6x Error in τ $\lvert 10^6 . (\tau-\tau(x_f)) \rvert$
3	1.010853	8881
4	1.004106	2134
5	1.002502	530
6	1.002104	132
7	1.002005	33
8	1.001980	8
9	1.001974	2
10	1.001972	0

Table 5. Fatunla (1985)

q	Estimate of $\tau(x_f)$:τ	10^6x Error in τ $\lvert 10^6 . (\tau-\tau(x_f)) \rvert$
3	1.010383	8410
4	1.004079	2106
5	1.002501	529
6	1.002104	132
7	1.002005	33
8	1.001980	8
9	1.001974	2
10	1.001972	0

Table 6. Fatunla, Ikhile and Otunta (1997) [8]

q	Estimate of $\tau(x_f)$: τ	10^6x Error in τ $\lvert 10^6 . (\tau-\tau(x_f)) \rvert$
3	1.009580	7620.33
4	1.0096427	7676.84
5	1.006985	5016.20
6	1.0048087	2838.25
7	1.0035590	1587.986
8	1.0025975	625.729
9	1.0020842	112.4143
10	1.0019327	39.2198

Table 7. Method (3.12): $k = 1.0$, $r = 1.0$

q	Estimate of $\tau(x_f)$:τ	10^6x Error in τ $\lvert 10^6 . (\tau-\tau(x_f)) \rvert$
3	1.00174407079197	2.279057425225162e+002
4	1.00189859983109	73.37670340623603
5	1.00195590285486	16.07367962819239
6	1.00196713761456	4.83891993230401
7	1.00197076367587	1.21285861687248
8	1.00197205792043	0.08138593710072
9	1.00197190067059	0.07586390182723
10	1.00197205375990	0.07722540495969
11	1.00197201989199	0.04335750114492

5.0 Conclusion

The numerical solution values generated by the new schemes (3.13, 3.22) can be observed to be consistent with the results of Cash (1981) [4], Lambert and Watson (1976) [17], Fatunla (1984,1985) [6,7] and Fatunla, Ikhile and Otunta (1997) [8], and show some improvements over these methods.

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