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Stability of collinear equilibrium points in the generalized photo-gravitational restricted three-body problem

Jagadish Singh
Department of Mathematics
Ahmadu Bello University (ABU), Zaria Nigeria
e-mail: jgds2004@yahoo.com

Abstract

We study the effect of oblateness and radiation pressure forces of the primaries on the locations and the stability of the collinear equilibrium points. We find that the locations of these points are affected by the radiation pressure forces and oblateness of the primaries but their stability is not influenced by them, and they remain unstable.

Keywords: stability, collinear points, generalised, photogravitational RTBP.

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1.0 Introduction

The classical restricted three-body problem is unable to discuss the motion of the third infinitesimal body when at least one of the participating bodies is a source of radiation or an oblate spheroid. In recent times, many perturbing forces have been included in the study of the restricted three-body problem.

Radzievskii [1] formulated the photogravitational restricted three-body problem and discussed it for three specific bodies: the sun, a planet and a dust particle. Chernikov [2] extended his work by including aberational deceleration (the Poynting Robertson effect). He demonstrated the instability of the solutions. Sharma [3] studied the linear stability of triangular libration point of the restricted problem when the more massive primary is a source of radiation and an oblate spheroid as well. Simmons et. al. [4] gave a classical treatment of the more general problem with radiation emanating from both primaries. Dankowicz [5] described the motion of grains in orbit around asteroids under the influence of radiation pressure originating in the flux of solar photons. His recent paper [6] accounts for gravitational interactions with the asteroid and the sun and the radiation pressure from the sun. Knitsyn [7,8] investigated the stability of triangular and collinear libration points in the photogravitational three-body problem. Vidyakin [9] studied the effect of oblateness of both primaries on the existence of five stationary solutions. Subbarao and Sharma [10] investigated the restricted problem with one of the primaries as an oblate spheroid and proved that there was an increase in the coriolis force and the centrifugal force due to oblateness. Singh and Ishwar [11] examined the stability of triangular points when both primaries are sources of radiation and oblate spheroids as well.

In this paper, we wish to study the "stability of collinear equilibrium points in the generalised photogravitational restricted three-body problem". The problem is generalised in the sense that both primaries are taken as oblate spheroids. It is photogravitational as they are sources of radiation. The participating bodies in the classical restricted three-body problem are strictly spherical in shape. But in actual situations we find that several heavenly bodies like Saturn and Jupiter are sufficiently oblate. The minor planets and the meteoroids are of irregular shape. On account of the small dimensions of these bodies in comparison with their distances from the primaries, they are considered as point masses. But in many cases the dimensions of these bodies are larger than the distances from their respective primaries. Thus, the above assumption is not justified and the results obtained are far from the realistic approach. The lack of the sphericity or the oblateness of the planet causes large perturbations from the two-body orbit. The motions of artificial earth satellites are examples. This enabled many researchers (e.g. [3],[9], [10], [11]) to study the restricted problems by taking into account the shapes of the bodies. In stellar systems numerous examples are available where a body is moving under the gravitational field of one or two radiating bodies, for example, binary star systems (where both primaries radiate) or a sun-planet system (only one of the primaries radiate).

In studying the motion of a material point in the sun-planet system, the classical model of the restricted circular three-body problem is not valid. In this connection it is reasonable to modify the model by superposing a light repulsion field. As the solar radiation pressure force changes with the distance by the same law as the gravitational attraction force and acts opposite to it, it is possible to consider that the result of the action of this force will lead to reducing the effective mass of the massive particle. A series of papers (e.g. [1], [8], [11]) investigated the effect of radiation pressure in the restricted problem. Itraised a curiosity in our mind to study the present problem. This paper considers the stability of the collinear points under the effects of both oblateness and radiation of both primaries. So it differs from the other's problem**.**

2.0 Equations of motion

Using dimensionless variables and a synodic coordinate system (x, y) as Szebehely [12] and, Singh and Ishwar [11], the equations of motion of the third body of infinitesimal mass when both primaries are sources of radiation and oblate spheroids, are

$$\ddot{x} - 2n\dot{y} = U_x,\ \ \ddot{y} + 2n\dot{x} = U_y, \tag{2.1}$$

where
$$U = \frac{n^2}{2}\left(x^2 + y^2\right) + q_1\frac{(1-\mu)}{r_1} + q_2\frac{\mu}{r_2} + A_1 q_1\frac{(1-\mu)}{2r_1^3} + A_2 q_2\frac{\mu}{2r_2^3}, \tag{2.2}$$

$$r_1^2 = (x-\mu)^2 + y^2,\ r_2^2 = (x+1-\mu)^2 + y^2, \tag{2.3}$$

Here dots indicate differentiation with respect to time t. The parameter μ is the ratio of the mass of the smaller primary to the total mass of the primaries and $0<\mu \le ½$. n is the perturbed mean motion of the primaries given by $n^2 = 1 + \frac{3}{2}(A_1 + A_2)$, $\qquad 0 < A_i << 1 \qquad$ (2.3)

where A_i (i = 1,2) being oblateness coefficients of the bigger and the smaller primaries respectively.

$$q_i = \frac{\left(F_i^g - F_i^p\right)}{F_i^g} = 1 - \beta_i \le 1,\ \ |\beta_i| << 1,\ \ i = 1,2 \tag{2.4}$$

are the mass reduction coefficients due to radiation, where F_i^p and F_i^g being the forces of gravity and light pressure respectively.

3.0 Locations of Collinear Points

Equilibrium points are solutions of equations $\quad U_x = 0$ and $U_y = 0 \qquad$ (3.1)

U_x and U_y can be written as
$$U_x = n^2 x - \frac{q_1(1-\mu)(x-\mu)}{r_1^3} - \frac{q_2\mu(x+1-\mu)}{r_2^3} - \frac{3}{2}\frac{A_1 q_1(1-\mu)(x-\mu)}{r_1^5} - \frac{3}{2}\frac{A_2 q_2\mu(x+1-\mu)}{r_2^5} = g(x,y) \tag{3.2}$$

$$U_y = y\left[n^2 - \frac{q_1(1-\mu)}{r_1^3} - \frac{q_2\mu}{r_2^3} - \frac{3}{2}\frac{A_1 q_1(1-\mu)}{r_1^5} - \frac{3}{2}\frac{A_2 q_2\mu}{r_2^5}\right] = yh(x,y) \tag{3.3}$$

The solution of Equation. (3.1) results in five points, three on the line joining the primaries, called the collinear points, and two forming triangles with the primaries, called the triangular points.

The collinear points are denoted by L_1, L_2, L_3 whereas triangular points by L_4, L_5. To find the collinear points we write y = 0 in Equations. (3.1). Their abscissae are the roots of the equation

$$f(x) = n^2 x - \frac{q_1(1-\mu)(x-\mu)}{|x-\mu|^3} - \frac{q_2\mu(x+1-\mu)}{|x+1-\mu|^3} - \frac{3}{2}A_1 q_1 \frac{(1-\mu)(x-\mu)}{|x-\mu|^5} - \frac{3}{2}A_2 q_2 \frac{\mu(x+1-\mu)}{|x+1-\mu|^5} = 0 \tag{3.4}$$

Now since $\frac{df(x)}{dx} \succ 0$ in each of the open intervals $(-\infty, \mu-1), (\mu-1, \mu)$ and (μ, ∞), the function f is strictly increasing in each of them. Also f(x) approaches $-\infty$ as x approaches $-\infty$ or $(\mu-1)+0$ or $\mu+0$ and *f(x)* approaches ∞ as x approaches $(\mu-1)-0$ or $\mu-0$ or ∞.

Therefore there exists one and only one value of x in each of the above intervals such that *f(x)* = 0. Further using Equations (2.3) and (2.4) in Equation (3.4) and restricting only linear terms in $A_i, \beta_i (i = 1,2)$ we have

$$f(\mu-2) = (\mu-1)\left[\frac{7}{4} + \frac{3}{2}(A_1 + 2A_2) + \frac{\beta_1}{4}\right] - \frac{3}{32}A_1(1+\mu) - \beta_2\mu < 0,$$

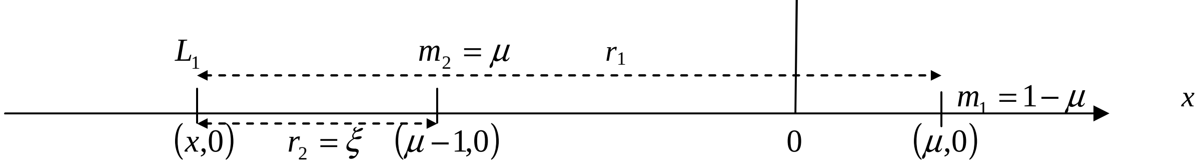
$$f(0) = \frac{\left[(1-2\mu)\left\{(1-\mu)^2 + (1-\mu)\mu + \mu^2\right\} + \mu^3(\beta_1+\beta_2) + 3\mu(1-\mu)\beta_1 - \beta_1\right]}{\left[\mu^2(1-\mu)^2\right]} + \frac{3}{2}\frac{\left[A_1(1-\mu)^5 - A_2\mu^5\right]}{\mu^4(1-\mu)^4} > 0$$

and
$$f(\mu+1) = \mu\left[\frac{7}{4} + 3A_1 + \frac{45}{32}A_2 + \frac{\beta_2}{4} - \beta_1\right] + \frac{3}{2}A_2 + \beta_1 > 0$$

Therefore there are only three real roots of Equation (3.4), one lying in each of the open intervals $(\mu-2, \mu-1)$, $(\mu-1, \mu)$ and $(\mu, \mu+1)$. This shows the locations of the three collinear points L_1, L_2, L_3. The first collinear point is located left of the second primary, the second is between the primaries, and the third collinear point is to the right of the first primary. Three cases arise:

***Case* 1:**

We consider the model depicted in the following figure:



Here $r_1 = \mu - x$, $r_2 = \mu - x - 1$, Let $r_2 = \xi$ so that $r_1 = 1+\xi$, $x = \mu - 1 - \xi$. Applying these in Equation (3.4), we get

$$(\xi+1-\mu)n^2+\frac{q_1(\mu-1)}{(1+\xi)^2}-\frac{\mu q_2}{\xi^2}+\frac{3}{2}A_1q_1\frac{(\mu-1)}{(1+\xi)^4}-\frac{3}{2}A_2q_2\frac{\mu}{\xi^4}=0 \tag{3.5}$$

In the classical case i.e. when $q_1=1,\ \ q_2=1,\ \ A_1=0,\ \ A_2=0,$ Equation (3.5) becomes

$$\xi^5+(3-\mu)\xi^4+(3-2\mu)\xi^3-\mu\xi^2-2\mu\xi-\mu=0, \tag{3.6}$$

Its series solution in powers of the quantity $\upsilon=\left[\frac{\mu(1-\mu)}{3}\right]^{\frac{1}{3}}$ is, Szebehely [12]

$$\xi=\upsilon\left[1+\frac{1}{3}\upsilon-\frac{1}{9}\upsilon^2-\frac{31}{81}\upsilon^3-\frac{119}{243}\upsilon^4-\frac{1}{9}\upsilon^5+0\left(\upsilon^7\right)\right] \tag{3.8}$$

An expansion with respect to $[\mu/3]^{1/3}$ is $\xi=\left(\frac{\mu}{3}\right)^{\frac{1}{3}}\left[1+\frac{1}{3}\left(\frac{\mu}{3}\right)^{\frac{1}{3}}-\frac{1}{9}\left(\frac{\mu}{3}\right)^{\frac{2}{3}}+\ldots\right]$ (3.9)

Simplifying Equation (3.5), we have

$$a_1\xi^9+a_2\xi^8+a_3\xi^7+a_4\xi^6+a_5\xi^5+a_6\xi^4+a_7\xi^3+a_8\xi^2+a_9\xi+a_{10}=0 \tag{3.10}$$

where $a_1=n^2,\ a_6=n^2-\left(n^2-q_1+6q_2-\frac{3}{2}A_1q_1+\frac{3}{2}A_2q_2\right)\mu-q_1-\frac{3}{2}A_1q_1,$

$a_2=n^2(5-\mu),\ a_7=-(4q_2+6A_2q_2)\mu,\ \ a_3=n^2(10-4\mu),\ \ a_8=-(q_2+9A_2q_2)\mu,$

$a_4=10n^2-(6n^2-q_1+q_2)\mu-q_1,\ a_9=-6A_2q_2\mu,$

$$a_5=5n^2-(4n^2-2q_1+4q_2)\mu-2q_1,\ a_{10}=-\frac{3}{2}A_2q_2\mu, \tag{3.11}$$

By Descartes' rule of signs there is only one positive root of Equation (3.10), since the coefficients of the powers of ξ change sign only once.

Let γ be the value of ξ in the classical case. This implies that γ is a real root of Equation (3.6).

Let the value of ξ be slightly changed due to radiation and oblateness of the primaries. Let the new value of ξ be defined by $\xi=\gamma+\delta,\qquad \delta<<1$ (3.12)

Substituting this new value of ξ in Equation (3.10) and making use of Equations (2.3), (2.4) in Equation (3.11) and considering only linear terms in $\delta, A_1, A_2, \beta_1, \beta_2,$ we obtain

$$\delta(Q_0+Q_1\beta_1+Q_2\beta_2+Q_3A_1+Q_4A_2)=P_0+P_1\beta_1+P_2\beta_2+P_3A_1+P_4A_2$$

$$\Rightarrow\delta=X_0+X_1\beta_1+X_2\beta_2+X_3A_1+X_4A_2 \tag{3.13}$$

where $X_0=\frac{P_0}{Q_0},\ X_1=\frac{(P_1Q_0-P_0Q_1)}{Q_0^2},\ X_2=\frac{(P_2Q_0-P_0Q_2)}{Q_0^2},$

$X_3=\frac{(P_3Q_0-P_0Q_3)}{Q_0^2},\ X_4=\frac{(P_4Q_0-P_0Q_4)}{Q_0^2},$

$P_0=-\left[a+(9-6\mu)\gamma^6+(3-6\mu)\gamma^5-6\mu\gamma^4-4\mu\gamma^3-\mu\gamma^2\right],$

$P_1=-(1-\mu)\left[\gamma^6+2\gamma^5+\gamma^4\right],\ P_2=-\mu\left[\gamma^6+4\gamma^5+6\gamma^4+4\gamma^3+\gamma^2\right],$

$$P_3 = -\frac{3}{2}\left[a + (5-3\mu)\gamma^6 + (5-4\mu)\gamma^5\right]$$
$$P_4 = P_3 - \frac{3}{2}\left[(1-2\mu)\gamma^4 - 4\mu\gamma^3 - 6\mu\gamma^2 - 4\mu\gamma - \mu\right],$$
$$Q_0 = b + 18(3-2\mu)\gamma^5 + 15(1-2\mu)\gamma^4 - 24\mu\gamma^3 - 12\mu\gamma^2 - 2\mu\gamma,$$
$$Q_1 = 2(1-\mu)\left(3\gamma^5 + 5\gamma^4 + 2\gamma^3\right), \qquad Q_2 = 2\mu\left[3\gamma^5 + 10\gamma^4 + 12\gamma^3 + 6\gamma^2 + \gamma\right],$$
$$Q_3 = \frac{3}{2}b + 18(5-3\mu)\gamma^5 + \frac{15}{2}(5-4\mu)\gamma^4,$$
$$Q_4 = Q_3 + 6(1-2\mu)\gamma^3 - 18\mu\gamma^2 - 18\mu\gamma - 6\mu,$$
$$a = \gamma^9 + (5-\mu)\gamma^8 + (10-4\mu)\gamma^7,$$
$$b = 9\gamma^8 + 8(5-\mu)\gamma^7 + (10-4\mu)\gamma^6 \tag{3.14}$$

Using Equations (3.8), (3.13), (3.14) in Equation (3.12), we get the required solution of Equation (3.10). Thus, we find the position of collinear point L_1. Similarly, we can obtain the positions of L_2 and L_3.

Since, the solutions contain $A_i, \beta_i (i = 1,2)$. Therefore, the locations of the collinear points are affected by radiation pressure and oblateness of the primaries.

4.0 Stability of Collinear Points

The motion of the infinitesimal body will be stable near the equilibrium point when given a very small displacement and small velocity, the body oscillates for a considerable time around the said point.

Let u,v denote small displacements of the infinitesimal body from the equilibrium point (x_0, y_0). Putting $x = x_0 + u, \quad y = y_0 + v,$ in Equation (2.1) and then expanding its R.H.S. in a Taylor's series and considering only first order terms, we have

$$\ddot{u} - 2n\dot{v} = u\left(U^0_{xx}\right) + v\left(U^0_{xy}\right),\ \ddot{v} + 2n\dot{u} = u\left(U^0_{yx}\right) + v\left(U^0_{yy}\right) \tag{4.1}$$

Here only linear terms in u and v have been taken. The second partial derivatives of U are denoted by subscripts. The superscripts 0 indicate that the derivative is to be evaluated at the equilibrium point (x_0, y_0). The characteristic equation corresponding to Equation (4.1) is

$$\lambda^4 - \left(U^0_{xx} + U^0_{yy} - 4n^2\right)\lambda^2 + \left(U^0_{xx}\right)\left(U^0_{yy}\right) - \left(U^0_{xy}\right)^2 = 0 \tag{4.2}$$

If all the λ_i obtained from Equation (4.2) and pure imaginary numbers, then u and v are periodic and thus give stable periodic solutions in the vicinity (x_0,y_0). If, however, any of the λ_i are real or complex numbers, then u and v increase with time so that the solution is unstable.

To examine the stability of collinear points we pay an additional attention to the evaluation of the second derivatives from Equations (3.2) and (3.3) on the line joining the primaries.

$$U_{xx} = g_x(x,0) = n^2 + 2q_1(1-\mu)\frac{(x-\mu)^2}{r_1^5} + 2\mu q_2\frac{(x+1-\mu)}{r_2^5}$$
$$+ 6A_1q_1(1-\mu)\frac{(x-\mu)^2}{r_1^7} + 6A_2q_2\mu\frac{(x+1-\mu)^2}{r_2^7}, \tag{4.3}$$

$$U_{yy} = h(x,0) = n^2 - q_1\frac{(1-\mu)}{r_1^3} - \frac{q_2\mu}{r_2^3} - \frac{3}{2}A_1q_1\frac{(1-\mu)}{r_1^5} - \frac{3}{2}A_2q_2\frac{\mu}{r_2^5}, \tag{4.4}$$

$$U_{xy} = yh_x(x,0) = 0, \tag{4.5}$$

First we consider the point corresponding to L_1. At this point $x-\mu=-r_1$ and $x+1-\mu=-r_2$ and so

$$g_x(x_1,0) = n^2 + 2q_1\frac{(1-\mu)}{r_1^3} + 2q_2\frac{\mu}{r_2^3} + 6A_1q_1\frac{(1-\mu)}{r_1^5} + 6A_2q_2\frac{\mu}{r_2^5} > 0, \tag{4.6}$$

Also, of course, at all collinear points $g(x_i,0)=0,$ so

$$n^2(\mu - r_1) + q_1\frac{(1-\mu)}{r_1^2} + q_2\frac{\mu}{r_2^2} + \frac{3}{2}A_1q_1\frac{(1-\mu)}{r_1^4} + \frac{3}{2}A_2q_2\frac{\mu}{r_2^4} = 0, \tag{4.7}$$

Equation (4.4) can be written in the form

$$h(x,0) = n^2 - q_1\frac{(1-\mu)}{r_1^2}\cdot\frac{1}{r_1} - q_2\frac{\mu}{r_2^3} - \frac{3}{2}A_1q_1\frac{(1-\mu)}{r_1^5} - \frac{3}{2}A_2q_2\frac{\mu}{r_2^5},$$

and using Equation (4.7) to express $q_1(1-\mu)r_1^{-2}$, we have

$$h(x_1,0) = \frac{\mu}{r_1}\left[n^2 + \frac{q_2}{r_2^2} + \frac{3}{2}\frac{A_2q_2}{r_2^4} - \frac{q_2r_1}{r_2^3} - \frac{3}{2}A_2q_2\frac{r_1}{r_2^5}\right],$$

Using Equations (2.3) and (2.4) $h(x_1,0) = \frac{\mu}{r_1}\left[\left(1-\frac{1}{r_2^3}\right) + \frac{3}{2}A_2\left(1-\frac{1}{r_2^5}\right) + \frac{3}{2}A_1 + \frac{\beta_1}{r_2^3}\right]$, since $0<\mu<\frac{1}{2}$, $r_2<1,$ $0<A_i<<1,$ $\left|\frac{\beta_1}{r_2^5}\right| < |\beta_1|$ and $|\beta_1|$ is very small, therefore $h(x_1,0)<0.$

The analysis for L_2 and L_3 is quite similar and it does not present new problems.

Thus, at collinear points, we have $U_{xy}^0=0,$ $U_{xx}^0>0,$ and $U_{yy}^0<0.$ Now the characteristic Equation (4.2) becomes $\lambda^4 - \left(U_{xx}^0 + U_{yy}^0 - 4n^2\right)\lambda^2 + \left(U_{xx}^0\right)\left(U_{yy}^0\right) = 0.$ Because $\left(U_{xx}^0\right)\left(U_{yy}^0\right)<0,$ the discriminant is positive and the four roots of the characteristic equation can be written as $\lambda_1 = s,$ $\lambda_2=-s,$ $\lambda_3 = is',$ $\lambda_4=-is',$ where s and s' are real. So, the solution is unstable. Thus collinear points are unstable.

5.0 Conclusion

We conclude that the locations of collinear points are affected by radiation pressure forces and oblateness of the primaries but their stability is not influenced by them, and they remain unstable.

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References

[1] V.V. Radzievskii, A. J. (USSR) 27, 5 (1950), 250.
[2] Yu. A. Chernikov, Sov. Astr. J. 14 (1970), 176.
[3] R. K. Sharma, Sun and Planetary System, 435 (1982), W. Fricke & G. Teleki (eds.), by D. Reidel, 435 – 436.
[4] J. F. L. Simmons, A. J. C. Mc Donald, J. C. Brown, Cel. Mech. 35(1985), 145 – 187.
[5] H. Dankowicz, Cel. Mech. & Dyn. Astron. 67(1997), 63-85.
[6] H. Dankowicz, Cel. Mech. & Dyn. Astron. 84(2002), 1 – 25.
[7] A. L. Kunitsyn, J. Appl. Maths Mechs. 64,5(2000), 757 – 763. [Prikl. Mat. Mekh. 64, 5(2000), 788-794]
[8] A. L. Kunitsyn, J. Appl. Maths Mechs, 65, 4(2001), 703 – 706. [Prikl. Mat. Mekh.65,4(2001), 720 – 724].
[9] V. V. Vidyakin, Astron. Zh. 51,5 (1974), 1087 – 1094.
[10] P. V. Subbarao and R. K. Sharma, Astron. Astrophys. 43(1975), 381.
[11] J. Singh and B. Ishwar, Bull. Astr. India 27(1999), 415 – 424.
[12] V. Szebehely, Theory of Orbits (1967), Academic Press, Inc. New York.