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Total controllability for nonlinear perturbed discrete systems

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Abstract

We assume G to be a target set in E^n, the n-dimensional Euclidean space. A control system is said to be totally G-controllable if each point of the state space E^n can be steered to G in finite time t_1 ($0< t_1< \infty$) and thereafter stabilized in G for all time $t_2 \geq t_1$ ($t_2< \infty$). In this paper we adopt Leray-Schauder fixed point theorem to develop sufficient conditions, which guarantee that whenever an unperturbed nonlinear discrete system is totally G-controllable, then so is its perturbation.

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1.0 Introduction

In the control of most systems, focus has been on the continuity of functions making up the systems. This is because the variation of constant formula for the solutions of such systems is readily deduced. In [1] and [3], Chukwu and Eke respectively handled systems (1.1) and (1.2) as follows:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.1a}$$

and its perturbation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + F(t,x(t),u(t)) \tag{1.1b}$$

and

$$\dot{x}(t) = f(t,x) + B(t)u(t),\ \ x(t_o) = x_o \tag{1.2a}$$

and its perturbation

$$\dot{y}(t) = f(t,y) + B(t)u(t) + F(t,y(t),u(t)),\ y(t_o) = y_o \tag{1.2b}$$

Both systems are continuous. While Chukwu [1] stressed on affined manifold control systems, Eke [3] worked on a purely nonlinear perturbed system. Both authors were able to deduce totally G-controllability of their systems. This paper goes beyond the sentiments of continuity to treat the nonlinear discrete control system

$$\dot{x}(t) = f(t,x) + \sum_{i=0}^{k} B_i(t)u(t),\ \ x(t_0) = x_0 \tag{1.3}$$

and its perturbation

$$\dot{y}(t) = f(t,y) + \sum_{i=0}^{k} B_i(t)u(t),\ \ y(t_0) = y_0 \tag{1.4}$$

in the *n*-dimensional Euclidean space E^n whose target set is given in E^n by

$$G = \{x:\ Tx = Tb\} \tag{1.5}$$

where T is a linear operator, b is a point in E^n , B is a continuous nxm real matrix defined on $[t_o, \infty) = I$ say; $f : I \times E^n \rightarrow E^n$; $F : I \times E^n \times E^m \rightarrow E^n$ are nonlinear real n-vector functions which are continuous ; f is Lipschitzian in x ; F is Lipschitzian in both y and u and u is an admissible control which is a bounded m-valued measurable function $u: I \rightarrow E^n$. Total controllability is of interest to researchers of control theory. A special case of controllability where cores of the target set G are characterized can be found in Iheagwam and Nse [5], Nse [7].

2.0 Preliminaries

Suppose $x(t,t_o,x_o,0) = x(t,t_o,x_o)$ is the unique solution of the uncontrolled form of (1.1) for $u = 0$. Assume that $f_x(t,x)$,the derivative of $f(t,x)$ with respect to the solution $x(t,t_o,x_o)$ of (1.1) with $u = 0$ exists and is continuous for $(t,x) \in I \times E^n$. Let $\phi(t,t_o,x_o)$ be the continuous nonsingular fundamental matrix solution of the variational equation x = $f_x(t,x(t,t_o,x_o))x$ with $\phi(t,t_0,x_0)$ = I ,the identity matrix. Then the solution of the free system of (1.1) is unique, differentiable with respect to initial data and it is such that

$$\frac{\partial x}{\partial x_0}(t,t_0,x_0)=\phi(t,t_0,x_0) \qquad (2.1)$$

and
$$\frac{\partial x}{\partial t_0}(t,t_0,x_0)=-\phi(t,t_0,x_0)f(t_0,x_0) \qquad (2.2)$$

Under these conditions and since it is known ([6], p9) that $x(t,t_o,x_o)$ and $\phi(t,t_o,x_o)$are further related by

$$x(t,t_0,x_0)=\left(\int_{t_0}^{t}\phi(t,t_0,sx_0)ds\right)x_0 \qquad (2.3)$$

The solution of (1.3) and (1.4) by the nonlinear variation of parameter become respectively

$$y(tt,t_0,x_0,u)=H(t)x_o+\int_{t_0}^{t}\left\{\phi(t,s,x(s))\sum_{i=0}^{k}B_i(s)u(s)ds\right\} \qquad (2.4)$$

and
$$x(tt,t_0,x_0,u)=H(t)x_o+\int_{t_0}^{t}\phi(t,s,y(s))\left[\sum_{i=0}^{k}B_i(s)u(s)+F(s,y(s),u(s))ds\right] \qquad (2.5)$$

where we have defined the nxm nonsingular matrix $H(t,t_0,x_0)=\int_{0}^{t}\phi(t,t_0,sx_0)ds=H(t)(\text{say})$

It follows that we shall denote the controllability gramian of (1.1) by

$$W(t)=\int_{t_0}^{t}\phi(T)\sum_{i=0}^{k}B_i(T)B_i^{\bullet}(T)\phi^{\bullet}(T)dT \qquad (2.6)$$

where we have set $W(t,t_o)=W(t) : E^n\to E^n$ and $\phi(t,\bullet,\bullet)=\phi(t)$ and B^* denotes the transpose of B. We now give characterization for the total G-controllability of the system (1.3) and (1.4). Firstly we have proved (Chukwu [1], for the linear case) the following lemma:

***Lemma* 2.1**

Suppose $W^+(t)$, the generalized inverse of the controllability gramian of (1.3) exists, if the system is totally G-controllable, then for $x_o \in E^n$, there exists some $g \in G$ such that whenever $t \in$ $[t_o,\infty]$, then $[I - W(t)][g - H(t)x_o]$ *= 0 where I is the identity matrix in E^n . Now for $t \in [t_o,\infty]$ define the matrix T_o by $T_o = TH(t)$ so that $T_o=\{Th_1(t),Th_2(t),\ldots,Th_n(t)\}$where $h_1,h_2,\ldots,h_n$ are entries of H. As in* [4] *we assume that T_o is invertible*.

Also we can easily prove the following lemma.

***Lemma* 2.2**

If the system (1.2) is totally G-controllable, then for $t_o \geq 0$ we have

$$x = y(t_o) = T_o(Tb - Tp(\bullet,y,u)) \qquad (2.7)$$

and the solution *y(t)* of (1.2) is given by

$$y(t) = H(t)T_o^{-1}(Tb-Tp(t,y,u)) + p(t,y,u) \qquad (2.8)$$

where $p(t,y,u) = \int_{t_0}^{t} \phi(t,s,y(s)) \left[\sum_{i=0}^{k} B_i(s) + F(s,y(s),u(s)) ds \right]$. Now it is easy to see from lemmas (2.1) and (2.2) that a characterization for the total G-controllability of system (1.4) in terms of the current notation is

$$[I-W(t)W^{+}(t)][y(t)-H(t)\{x_o-q(\cdot,y,u)\}] = 0 \qquad (2.9)$$

where $q(t,y,u) = \int_{t_o}^{t} \phi(t,s,y(s))F(s,y(s),u(s))ds$. To solve our problem, we make use of the Leray-Schauder fixed point Theorem by Karsatos [6].

3.0 Main Results:

In what follows, we shall use the notation $\|\bullet\|$ to denote the norm of real matrices as appropriate. We now have the following theorem:

***Theorem* 3.1**

Consider the system (1.3) *and its perturbation* (1.4) *with their target set* (1.5). *Assume that:*

(i) *System* (1.3) *is totally G-controllable*

(ii) $\Phi(t), H(t)$ *are such that* $\|H(t)\| \le m_o$, $\|\phi(t)\| \le n_o$, $\|\phi^*(t)\| \le \rho_o$ *for all* $t \ge t_o$ *where* m_o, n_o, ρ_o *are small positive constants.*

(iii) $\max_{t \ge t_0} \|B(t)\| \le \beta$, $\max_{t\ge t_0} \|B^*(t)\| \le \beta^*$ where $\beta, \beta^* > 0$ *are constants*

(iv) *There exists functions h, r, s* $\in \zeta(I, I)$ *such that* $\|\varphi(t)B_i(t)\| \le h(s)$, $\|\phi(t)F(t,y(t),u(t))\| \le r(t)[\|y\| + \|u\|] + s(t)$

(v) $W^{+}(t)$ *exists and for some* $\lambda \ge 0$ *we have* $\|W^{+}(t)\| \le \lambda$, $t \ge t_o$.

(vi) *If* $H = \int_{to}^{\infty} h(t)dt < \infty$, $R = \int_{to}^{\infty} r(t)dt < \infty$ *and* $S = \int_{to}^{\infty} s(t)dt$. T_o *is invertible and it is such that* $\lambda_2 \exp(R+H) < 1$ *provided* $\lambda_2 = m_o \|T_o^{-1}\|\{\|T\|(R+H)\} + \beta^{+}\rho_o\lambda\{1 + n_o\|T_o^{-1}\|\|T\|(R+H)+S\}$; *then the nonlinear perturbed system* (1.2) *is totally G-controllable.*

Proof

Let us assume that (1.3) is totally G-controllable, then any solution of (1.4) which satisfies Ty(t) = Tb is a fixed point of the operator S defined by $S(y,u)(t) = (e(t),v(t))$, $t \ge t_o$ where $e(t) = H(t)T_o^{-1}[Tb - Tp(\bullet,y,u)] + p(t,y,u)$ and

$$v(t) = B^*(t)\phi^*(t)W+(t)(y-H(t)[T_o^{-1}(Tb-Tp(\cdot,y,u)]-q(\cdot,y,u) \qquad (3.1)$$

which are relatively derived from the solution of (1.4) and the admissible control. We wish to determine a fixed point of $S(y,u)$ such that (1.5) is satisfied. To do this, we introduce a parameter $\mu \in [0, 1]$ into the boundary value problem (1.4) thus;

$$y(t) = f(t,y) + \mu[B(t)u(t) + F(t,y(t),u(t))] \qquad (3.2)$$
$$Ty=\mu Tb$$

and consider the operator $\bar{S}$ defined by $\bar{S}(\bar{y},\bar{u},\mu) = (\bar{e}(t),\bar{v}(t))$ where $\bar{e}(t) = \mu e(t)$ and $v(t) = \mu\ (t)$.

To get a fixed point of $\bar{S}$ it is enough to show that in a suitable function space D there exists a function pair $(y,u) \in$ D such that $S(y,u,1) = (y,u)$ and this is done by means of Leray-Schauder Theorem [] (See appendix). Now let $J_1=[t_o,t_1]$, $t_1 > t_o$, such that $x(t_1) \in$ G. Define $w = (\bar{y},u) \in \xi[J_1, E^{n+m}]$ with norm $\|w\| = w(t)$ and consider the set $C_1=D_1 \times H_1$ where $D_1 = \{e(t): e \in \zeta(J_1,E^n)\}$ and $H_1 = H_1 = \{\bar{v}(t) : v\xi(J_1,E^n)\}$.

Let the norm in C_1 be $\|w\|_1$ so that the product space C_1 with norm $\|\bullet\|_1$ becomes a Banach space Let $S_1:C_1\to C_1$ be an operator defined by $S_i(\bar{y},\bar{u},\mu) = (e,v)$. Then for $\mu = 1$ this operator has a fixed point in C_1. By Lebesgue Dominated Convergence Theorem in addition to the estimates in the hypotheses of the theorem, S_1 is easily shown to be uniformly continuous.

Now let K be a bounded subset of C_1 with bound *bk*. Consider the family of functions $\{\overline{a}_n\}$ where $\overline{a}_n = S_i(\overline{w},\mu), \overline{w} \in K$ and μ ∈ [0,1]. Set $w = (\overline{y},\overline{u})$, $w_n = (\overline{y}_n,\overline{u}_n)$. where for *n* =1,2, $w_n, w \in C_1$. Let $\overline{a} = (e,v) = S_1(y,u)$. Clearly, for various *t*'s, $\{a_n\}$ is equicontinuous. Also $\{\overline{a}_n\}$ is uniformly bounded for by the hypotheses (ii) and (iii) of Theorem 3.1 in addition to the estimates which define the equicontinuity of $\{a_n\}$, we get

$$\|y\| \leq m_o\|T_o^{-1}\|(\|Tb\|+\|T\|[(H+R)b_k+S])+(H+R)b_k+S;$$

$$\|v\| \leq \lambda\beta^+\rho_o [b_k+m_o (\|T_o^{-1}\|\{\|Tb\|+\|T\|[(H+R)b_k+S\}]+(H+R)b_k+S$$

and these show that $\{a_n\}$are uniformly bounded. Hence, the equicontinuity and the uniform boundedness of {an} imply (by the Ascoli-Arzela Theorem [9] p.85) that $S(K,\mu)$ is relatively compact in C_1 for each μ∈[0,1].

Now consider the operator equation $$S_i(\overline{w},\mu) - \overline{w} = 0. \qquad (3.3)$$

Assume that (3.3) has a solution w for a fixed μ ∈ [0,1] in C_1. We have

$$|w(t)| \leq |y(t)| + |v(t)| \leq mo\|To-1\|\{\|Tb\|+\|T\|(H\|u\|+R\|w\|+S) + \int_{t_0}^{t} |h(\tau)|d\tau$$

$$+ \int_{t_0}^{t} r(\tau)|w(\tau)d\tau+S+\beta^+\rho_o\lambda\{\|y\|+m_o[\|T_o^{-1}\|(\|Tb\|+\|T\|[H\|u\|+R\|w\|+S)+R\|w\|+S])$$

so that we get $$|w(t)| = \lambda_1+\lambda_2\|w\| + \int_{to}^{t}[(h(\tau))|w(t)|]d\tau \qquad (3.4)$$

where $\lambda_1 = m_o\|T_o^{-1}\|\{\|Tb\|+\|T\|+S\}+S+\beta^+\rho_o\lambda\{m_o\|T_o^{-1}\|(Tb\|+\|T\|+S)+S\}$

and $\lambda_2 = m_o\|T_o^{-1}\|\|T\|(H+R)+\beta^+\rho_o\lambda\{1+m_o\|T_o^{-1}\|\|T\|(R+H)+S\}$

Now applying Gronwall"s inequality, (Hale [4], Lemma 3.1 p.115) to (3.4) we get $\|w(t)\| \leq (\lambda_1+\lambda_2\|w\|\exp(R+H)$. From hypothesis (vi) of the theorem we have $\lambda_2\exp(R+H) < 1$; it then follows that the solutions of the equation (3.3) are uniformly bounded with respect to μ ∈ [0, 1]. Hence, the conditions of the Leray-Schauder Theorem (see appendix A) are satisfied for the interval $J_1= [t_o,t_1]$ so that $S_i(\overline{w},1)$ has a fixed point $\overline{w}$ in C_1, that is $S_i(\overline{w},1) = \overline{w} \in C_1$ and so (1.2) is totally *G*-controllable when we consider J_1.

Appendix A

Here we give a complete statement of the Leray-Schauder Fixed Point Theorem as used in Kartsatos [6]

***Lemma* A.1 (Leray-Schauder Fixed Point Theorem)**

Let B be a Banach space. For the equation

$$S(y,u,\mu) = 0 \text{ , for some } \mu \in [0,1] \qquad (A1)$$

A*ssume that*

(i) *S(y,u,μ) is defined on B x* [0,1] *with values in B and is completely continuous in (y,u): that is, for each μ ∈* [0,1], *S(y,u,μ) is continuous in (y,u) and maps every bounded subset of B into a relatively compact set Moreover, if K is a bounded subset of B, S(y,u,μ).is continuous in μ uniformly with respect to (y,u) ∈ K*

(ii) *S(y,u,μ)=0 for some μ_o ∈* [0,1] *and for every (y,u) ∈B.*

(iii) *If there are any solutions of A1, then they belong to some close ball ⊥Bof B independently of* μ

Then, there exists a continuum of solutions of (A1) corresponding to all values of μ ∈ [0,1]. All solutions lie in ⊥B

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