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Null controllability criterion for discrete nonlinear systems with distributed delays in the control: A fixed point approach

Celestin A. Nse
Department of Mathematics and Computer Science.
Federal University of Technology
Owerri, Imo State.
e-mail:canseus@yahoo.com

Abstract

In this work, we consider null controllability results of the base system

$$\dot{x}(t) = \mathrm{A(t)}\,\chi(\mathrm{t}) + \sum_{i=o}^{\mathrm{k}} B_i(t)\mu(t-h_i), \quad k = 0,1,2,\cdots$$

and its perturbed equivalence

$$\dot{x}(t) = \mathrm{A\ (t)\ x(t)} + \sum_{\mathrm{i=o}}^{\mathrm{k}} \mathrm{B_i\ (t)}\ \mu(\mathrm{t - h_i}) + f(t,x,\mu)$$

The non-singularity of the controllability gramian, the properness of the differential system and the Schauder's Fixed Point Theorem are veritable tools used to obtain results.

Keywords: Null Controllability, Schauder's Fixed Point Theorem, Properness

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1.0 Introduction:

In his paper [2], A.N. Eke posed an open problem; the problem of extending his results from a Euclidean space to a function space. This work is not unconnected with his work though with a little bias to discrete systems of the form.

$$\dot{x}(t) = \mathrm{A(t)\ x(t)} + \sum_{i=1}^{\mathrm{k}} B_i(t)\mu(t-h_i) \tag{1.1}$$

and

$$\dot{x}(t) = \mathrm{A(t)}\ x(\mathrm{t}) + \sum_{i=1}^{\mathrm{k}} B_i(t)\beta\mu(t-h_i) + \mathrm{f}(x,\mathrm{t},\mu) \tag{1.2}$$

Investigations on null controllability of control system is not new. These have been done extensively by various authors. For instance, Shintendorf and Barmish [4], Chukwu [1] among others. What makes this paper unique is the application of a version of the Schauder's Fixed Point Theorem to prove null controllability of the perturbed linear system. By the variation of parameter the solution of (1.1) is given by

$$x(t_1, \mathrm{t_o}, x_o, \mu) = F(t_1, t_o)\left\{x_\mathrm{o} + \sum_{i=o}^{k}\left[\int_{t_1-hi+1}^{t_1-hi} \sum_{p=o}^{i} F(t_o, s+h_p)\right] \times \left[B_p\ (\mathrm{s+h_p})\ \mu(\mathrm{s})\ \mathrm{ds}\right]\right\} \tag{1.3}$$

where $F(t, t_o)$ is the fundamental matrix solution of the system (1.1) for B=o with F(0) = I, the identity matrix. The null controllability is achieved by imposing on (1.1) the boundary condition.

$$Tx = 0 \qquad (1.4)$$

Here T is a bounded linear operator defined on $C[E^+, E^n]$ the space of all bounded and continuous operators from E^+ to E^n.

2.0 Preliminaries

Let E^n denote the n-dimensional Euclidean space with norm denoted by $\|\bullet\|$. If J is any interval of E, the usual Lebesque space of square integrable function's from J to E^n will be denoted by $L_2[J, E^n]$. $N_{n,m}$ will denote the collection of all real n x m matrices with a suitable norm. Let h > 0, $h_i(t) > 0$ be given. For functions

$$U = L_2([t_o, t_1], E^n), \ t \in [t_o, t_1],$$

we use μ_t to denote the function on $[-h, o]$ defined by

$$\mu_t(s) = \mu(t+s), s \in [-h,0] \qquad (2.1)$$

We shall consider the system (1.1) satisfied almost everywhere on (t_o, t_1) where the integral is in Lebesque- Stielties sense with respect to s. $x(t) \in E^n$, $\mu \in L_2([t_o\, t_1], E^n)$, A(t) is an n x n matrix valued function which is measurable in t. We shall assume that B(t), $B_j(t)$ are of bounded variation in s on [-h, 0] for each $t \in [t_o, t_1]$ and are absolute continuous in s on [-h, 0].

In the sequel, the control of interest is

$$\mu = L_2([t_o, t_1], C^m) \text{and } \mu \in L_2([t_o, t_1], E^m)$$

where

$$C^m = \{\mu \in E^n : /\mu/ \leq / \}$$

That is, the unit ball with zero in its interior relative to μ . If X and Y are linear spaces and T; X → Y is a mapping we shall use the symbols D (T), R (T) and N (T) to denote respectively the domain, range and null spaces of T.

***Definition* 2.1**

Let X and Y be normed linear spaces. An operator T: X→ Y is said to be completely continuous if T maps bounded sets in X into relatively compact sets in Y. A completely continuous operator is compact.

***Definition* 2.2**

An operator T: X →Y where X and Y are linear spaces is said to be closed if for any sequence $u_n \in D(t)$ *such that* $u_n \to u$ and $Tu_n \to v$, μ *belongs to* $D(T)$ and $Tu \to v$.

***Definition* 2.3**

The complete state of system (1.1) *at time t is given by* $z(t) = \{x(t), \mu\}$

***Definition* 2.4 (Proper system)**

The system (1.1) is proper in E^n for t∈ (t_o, t_1) if for $C \in E^n$,

$$C^T \left[\int_{t_o}^{t_1} F(t_1, s) B(s) \mu(s)\, ds = 0 \right. \qquad (2.2)$$

almost everywhere implies $C = 0$ for all $\mu \in U$

***Definition* 2.5 (Complete controllability)**

System (1.1) *is completely controllable if for every* $x_o\, x_1, \in E^n$, *there exists a continuous function* $\mu: I \to E^n$ *such that the solution of* (1.1) *satisfies* $x(t_o) = x_o$ and $x(t_1) = x_1$. *It is null controllable with constraints at* $t = t_1$ *for any initial state* $\{x_o, \mu_{to}\}$ *on* $[t_o - h, t_o]$ *if there exists an admissible control* $\mu \in u$, *defined* on $[t_o, t_1 - h]$ *such that the response x (t) of system* (1.1) *satisfies* $x(t_o) = 0$ *using the control effort.*

$$\mu(t) = \begin{cases} \mu(t) & \text{on } [t_o - h, t_o] \\ 0 & \text{on } [t_o, t_1 - h] \end{cases}$$

***Definition* 2.6 (Domain of null controllability).**

The Domain D of null controllability of the system (1.1) *is the set of all initial points* $x_o \in E^n$ *for which the solution* $x(t_o) = x_o$ *satisfies* $\mu(t_1) = 0 \in E^n$ *at some* t_1 *using* $\mu \in u$.

***Definition* 2.7**

Given system (1.2), *set* $\hat{A} = A + \frac{\partial f}{\partial x}(t,0,0,),\ \hat{B} = B_o + \frac{\partial f}{\partial x}(t,0,0)$ where A and B_o are from system (1.1). *If the linear part satisfies Rank condition then it is completely controllable which it turn implies that the system* (1.2) *is also completely controllable*.

3.0 Main Results

***Theorem* 3.1:**

The following are equivalent

(i) $W(t_o,t_1)$, *the control grammian, is non singular (positive definite)*

(ii) *The system* (1.1) *is completely controllable on* $[t_o, \mathbf{t}_1]$

(iii) *The system* (1.1) *is proper on* $[t_o, t_1]$

Proof

To show that (i)$\Rightarrow$(ii)

Let W^{-1} exists and define μ by

$$\mu(t) = \sum_{i=o}^{k} \left\{ \sum_{p=o}^{i} B^*_p(s+h_p) F(t_o, s+h_p)\, W^{-1}\left[F^{-1}(t)x_1 - x_o\right. \right\} \qquad (3.1)$$

Let $x_o, x_1 \in E^n$

$$x(t_1, t_o, x_o\ \mu) = F(t_1,t_o)\left\{ x_o + \sum_{i=o}^{k}\left[\int_{t_1-hi+1}^{t_1-hi} \sum_{p=o}^{i} F(t_o, s+h_p)\ B_p(s+h_p)\mu(s)ds \right\}\right. \qquad (3.2)$$

$x(t_o) = x_o, \quad x(t_1) = I$

$$x(t_1) = F(t_1)x. + F(t_1)\left[\sum_{i=o}^{k} \int_{t_1-hi+1}^{t_1-hi\ o} \sum_{p=o}^{i} F(t_o, s+h_p). \right. \qquad (3.3)$$

$$B_p(s+h_p)\left\{ B^*{}_p \wedge\ (s+h_p)\, F^*(t_o, s+h_p) \right\}$$

$$W^{-1}[F^{-1}(t_1)]\,(s+h_p)\{B^*{}_p(s+h_p)\, F^{\alpha}(t_o, s+h_p)\}$$

$$= F(t_1)x_o + F(t_1)WW^{-1}\left[F^{-1}(t_1)x_1 - x_o\right]$$

$$= F(t_1)x_o + F(t_1)\left[F^{-1}(t_1)x_1 - x_o\right]$$

$$= F(t_1)x_o + x_1 - F(t_1)x_o = x_1$$

where * denotes transpose.

Thus we have found μ such that $x(t_1, t_o, x_o) = x_1$, hence the system is completely controllable.

To show (i) $\Rightarrow$ (iii). If $W(t_o,t_1)$ is nonsingular then it is positive definite and all the eigenvalues are positive. Equivalently $< \eta W, \eta > \ldots > 0$ for all $\eta \neq 0$

$$\eta^T\left[\sum_{i=o}^{k}\left[\sum_{p=o}^{i}F(t_o,s+h_p)B_p(s+h_p)\right]\right]\equiv 0 \qquad (3.4)$$

$\Rightarrow \eta = 0$ hence the system (1.1) is proper on $[0, t_1]$

To show (ii) $\Rightarrow$ (iii). Suppose the system is not proper then there exists $\eta \neq o$ such that

$$\eta^T y(t) = \eta^T \int_o^T y(s)\mu(s)ds \qquad (3.5)$$

where $y(t) = \sum_{i=o}^{k}\left\{\sum_{p=o}^{i}F(t_o,s+h_p)B_p(s+h_p)\right\}$, *for* all $\mu \in L_2$

Consider $\eta^T\left(F^{-1}(t_1)x_1 - x_o\right) = \eta^T \int_o^t y(s)\mu(s)ds = 0$

Since $x(t_1) = F(t_1)x_o + F(t_1)\int_o^t y(s)\mu(s)ds$

Then $\eta^T\left[F^{-1}(t_1)F(t_1)\right] - X_o = \eta^T \int_o^t y(s)\mu(s)ds = 0$, for all $\mu \in L_2$

x_i are all the points that can be attained using all the admissible controls

$$\eta^T F^{-1}(t_1)x_1 = \eta^T x_o = c(say)$$

Therefore all the points that can be attained using all the admissible controls is a translation of a subspace of co dimension 1 and not R^n. This contradicts our supposition; hence complete controllability implies properness.

***Theorem* 3.2:**

The system (1.1) *is proper on* $[t_o,t_1]$ *if and only if* $o \in Int\ R(t_o,t_1)$

Proof

If $y^* \in \Re(t_o,t_1)$, then there exists a $y \in R(t_o,t_1)$ and $\eta \neq o, \eta \in E^n$ such $\eta^T(y - y\,\hat{}) \leq 0$

That is $$\eta^T y \leq \eta^T y^*$$

$$\Rightarrow \eta^T\left[\sum_{i=o}^{k}\int_{t_1-h_{i+1}}^{t_1-h_i}\sum_{p=o}^{i}F(t_o,s+h_p)\right]B_p(s+h_p)\mu(s)ds$$

$$\leq \eta^T\left[\sum_{i=o}^{k}\int_{t_1-h_{i+1}}^{t_1-h_i}\sum_{p=o}^{i}F(t_o,s+h_p)B^*p(s+h_p)\mu(s)ds\right] \qquad (3.6)$$

Since U is a unit sphere the last inequality holds for each $\mu \in U$ if and only if

$$\eta^T\left[\sum_{i=o}^{k}\int_{t_1-h_{i+1}}^{t_1-h_i}\sum_{p=o}^{i}F(t_o,s+h_p)B_p(s+h_p)\mu(s)ds\right.$$

$$\leq \eta^T\left[\sum_{i=o}^{k}\int_{t_1-h_{i+1}}^{t_1-h_i}\sum_{p=o}^{i}F(t_o,s+h_p)B_p(s+h_p)\mu^*(s)ds\right.$$

$$= \int_{t_1-h_{i+1}}^{t_1-h_i} \left[\eta^T \left[\sum_{i=o}^{k} \left[\sum_{p=o}^{i} F(t_o, s+h_p) B_p(s+h_p) \right]\right]\right]$$

If $0 \in$ Int R (t_o, t_1) *then* $o \in \Re(t_o, t_1)$. If o$\in$ $\Re(t_o, t_1)$ then

$$o = \int_{t_1-h_{i+1}}^{t_1-h_i} \left[\eta^T \sum_{i=o}^{k} \left[\sum_{p=o}^{i} F(t_o, s+h_p) B_p(s+h_p) \right] ds\right.$$

so that
$$\eta^T \left\{ \sum_{i=o}^{k} \left[\sum_{p=o}^{i} F(t_o, s+h_p) B_p(s+h_p) \right]\right\} = 0$$

almost everywhere and there exist $\eta \neq o$ for s$\in$ $[t_o, t_1]$

***Theorem* 3.3**

The system (1.1) *is proper if and only if*

$$Rank\left[B_0, AB_0, \cdots A^{n-1}B_0, B_1A, B_1, \cdots A^{n-1}B_1, \cdots B_k, AB_k, \cdots A^{n-1}B_k\right] = n$$

Proof

System (1.1) is proper if an only if

$$\eta^T \sum_{i=0}^{k} \left[\sum_{p=0}^{i} F(t_o, s+h_p) B_p(s+h_p) \right] = 0 \qquad a.e \Rightarrow \eta = 0 \tag{3.7}$$

$$\sum_{i=0}^{k} \left\{ \sum_{p=0}^{i} F(t_0, s+h_p) B_p(s+h_p) \right\} = e^{-At} \eta^T \sum_{i=0}^{k} \sum_{p=0}^{i} F(t_0, s+h_p)\, B_p(s+h_p)$$

$$= \eta^T e^{-At} \sum_{i=o}^{k} \left[\sum_{p=o}^{i} F(t_0, s+h_p) B_p(s+h_p) \right] = 0 \qquad a.e \Rightarrow \eta = 0$$

since an analytic function can have almost a finite number of zeros.

By differentiating, we have

$$\eta^T (-A^k) e^{-At} \sum_{i=o}^{k} \left\{ \sum_{p=o}^{i} B(s+h_p) \right\} = 0 \Rightarrow \eta = 0, \text{ for } k = 0,1,2,\cdots \tag{3.8}$$

Setting $t = 0$ we have

$$\eta^T A^k \sum_{i=0}^{k} \left\{ \sum_{p=0}^{i} B_p(s+h_p) \right\} = 0 \quad \Rightarrow \eta = 0,\ k = 0,1,2,\cdots, n-1$$

That is η is orthogonal to

$$\left[B_0, AB_0, \cdots, A^{n-1}B_0, B_1A, B_1, \cdots, A^{n-1}B_1, \cdots, B_k, AB_k, \cdots, A^{n-1}B_k\right] \quad \Rightarrow \eta = 0$$

Since $\eta \in E^n$ it means the vector

$$\left[B_0, AB_0, \cdots, A^{n-1}B_0, \cdots, B_k, AB_k \cdots A^{n-1}B_k\right]$$

has Rank *n*.

Conversely suppose $\left[B_0, AB_0, \cdots, A^{n-1}B_0, \cdots, B_k, AB_k \cdots, A^{n-1}B_k\right]$ has Rank less than n, then there exists $\eta \in E^n$, $\eta \neq o$ such that

$$\eta^T B_0 = \eta^T AB_0 = \cdots = \eta^T A^{n-1}B_0 = \cdots = \eta^T B_k = \cdots = \eta^T A^{n-1}B_k = 0$$

By the Hamilton Clayey theorem, A^n is a linear combination of A^{n-1},..., A,I

$$\det(A-\lambda I) = \lambda^n + a_n, \lambda^{n-1} + \cdots + a_1, \lambda + a_0 = I$$

and $A^n + a_n A^{n-1} + \cdots + a_1 A + a_0 I = 0 \Rightarrow A^n = a_n A^{n-1} = \cdots = a_o I$

$$\eta^T B_0 + \eta^T AB_0 + \eta A^{n-1}B_0 + \cdots + \eta^T A^{n-1}B_k = 0$$

or

$$A^n B_0 = a_n A^{n-1}B_0 = \cdots. = A^n B_k = a_0 A^{n-1}B_k = \cdots = a_0 B_0 = 0$$

hence

$$\eta^T A^k \sum_{i=1}^{k} \left\{ \sum_{p=0}^{i} B_p(s+h_p) \right\} = 0, \text{ for all } k$$

***Theorem* 3.4:**

The system (1.1) with the control μ (t) on (t_o-h, t_o) *is null controllable with constraints at* $t = t_1$ *if and only if*

$$y(z(t_o)) = x(t_o) - \sum_{i=o}^{k} \left[\int_{t_1-h_{i+1}}^{t_1-h_i} \sum_{p=o}^{i} F(t_o, s+h_p) B_p(s+h_p) \mu(s) ds \right. \qquad (3.9)$$

belongs to the range space of null controllability Gramian.

Proof

Let $y(z(t_o,t_1)) \in R(\Gamma(t_o,t_1))$ then $z_o \in E^n$. We have $y(z(t_o)) = \Gamma(t_o,t_1)z_o$

Choose

$$\mu(s) = \left\{ \int_{t_1-h_{i+1}}^{t_1-h_i} \left[\sum_{p=o}^{i} F(t_o, s+h_p) B_p(s+h_p) \right]^T z_o \right. \qquad (3.10)$$

$$s \in [t_o, t_1 - hi]$$

Substituting (3.10) into the variation of parameter equation for system (1.1), we obtain

$$x(t_1, t_o, x_o\ \mu) = F(t_1,t_o) \left\{ x_o + \sum_{i=o}^{k} \left[\int_{t_1-h_{i+1}}^{t_1-h_i} \sum_{p=o}^{i} F(t_o, s+h_p) B_p(s+h_p) ds \right] \right.$$

$$+ \sum_{i=o}^{i} \left[\int_{t_1-h_{i+1}}^{t_1-h_i} \sum_{p=o}^{i} F(t_o, s+h_p)\, B_p(s+h_p) \mu(s) ds \right.$$

$$= F(t_1,t_o)[-y(z(t_o))] + F(t_1,t_o)\, \Gamma(t_o,t_1,)z_o$$

$$= -F(t_1,t_o)[(t_o,t_1)z_o + F(t_1,t_o)]\Gamma(t_o,t_1)z_o$$

Conversely suppose for a contradiction that $y(z(t_o)) \in R\Gamma(t_o,t_1)$) then there exists z, $z_2 \in E^n$ such that

$$y(z(t_o)) = z_1 + z_2 \neq 0$$

where $z_1 \in \Gamma((t_o,t_1,), z_2 \in N((t_o,t_1))$

then $(z_2\ \Gamma(t_o,t_1)z_2) = \int_{t_o}^{t_1-hi} \|\ \Gamma(t_o,t_1)z_2\ \|^2 ds$

Since the integrand is non-negative, we obtain.

$$\left[\sum_{i=0}^{k}\sum_{p=0}^{i} F(t_0, s+h_p)B_p(s+h_p)\right]^T z_2 = 0 \quad (3.11)$$

By hypothesis however, $x(t_o)$ can be brought to the origin by some control effort on $[t_o, t_1]$. That is

$$\sum_{i=0}^{k}\int_{t_1-h_{i+1}}^{t_1-h_i}\sum_{p=0}^{i} F(t_0, s+h_p)B_p(s+h_p)\mu(s)ds = 0 \quad (3.12)$$

$$z_2^T\left[\sum_{i=0}^{k}\left[\sum_{p=0}^{i} F(t_0, s+h_p)B_p(s+h_p)\mu(s)ds\right]\right] = 0 \quad (3.13)$$

By combining (3.11) and (3.13) we obtain the contradiction // z //=o when z ≠ o. Hence y (z (t_o)) $\in R\Gamma(t_o,t_1)$)

***Theorem* 3.5:**

Consider f(t,x,μ) *of the system* (1.2). *Assume that*

$$\lim_{/x,\mu/\to\infty} \left|\frac{f(t_0 x,\mu)}{(x,\mu)}\right| = 0 \quad (3.14)$$

uniformly for t∈I. If system (1.1) *is completely controllable. Then system* (1.2) *is also controllable*

Proof

Assume that system (1.1) is completely controllable, choose $x_o, x_1 \in E^n$ and let

$$\bar{x} = F^{-1}(t_1)x_1 - x_0$$

Let ℓ be the Banach space of continuous functions $(x,\mu): I \to E^n x E^m$ with the usual supremum norm $\|(x,\mu)\| = \sup\{|x(t),\mu(t)|, t\in I\}$.

Define a continuous operator T on ℓ as follows

$$T(x,\mu) = (z,v)$$

where *v(t)* and *z(t)* are respectively given by

$$v(t) = \sum_{i=0}^{k}\left[\sum_{p=0}^{i} B_p^*(s+h_p)F^*(t_0, s+h_p)w^{-1}\left[\bar{\chi}\int_{t_0}^{t} F^{-1}(s)\ F(s, x(s), \mu(s))ds\right]\right] \quad (3.15a)$$

$$z(t) = F(t_1,t_0)\left\{\chi_0 + \sum_{i=0}^{k}\left[\int_{t_1-hi+1}^{t_1-hi}\sum_{p=0}^{i} F(t_0, s+h_p)\ B_p(s+h_p)\mu(s)ds\right]\right\} \quad (3.15b)$$

μ(t) is as defined. We now take

$$k = Max\left\{\|x\|\ \|x^{-1}B\|(t_1 - t_0\right\}$$

$$c_1 = 4k\left\{\left\|\sum_{i=0}^{k}\int_{t_1-hi+1}^{t_1-hi}\sum_{p=0}^{i} F(t_o, s+h_p) + \right\|\ w^{-1}\right\|\ F^{-1}\|(t_1 - t_0)\right\}$$

$$d_1 = 4k\left\|\sum_{i=0}^{k}\sum_{p=0}^{i} B_p^*(s+h_p)F^*(t_0, s+h_p)\right\|\ \|\ w^{-1}\ \|\bar{x}\|$$

$$c_2 = 4\| F \| \| F^{-1}\| (t_1, t_0)$$

$$d_2 = 4\| F\| \|x_0 \|$$

$$c = \text{Max}\{c_1, c_2\}$$

$$d = Max\{d_1, d_2\}$$

***Proposition* 3.1**

There exists r such that if

$$|(x,\mu) \le r \text{ and } s \in \text{I then } c \mid f\ (s,x,\mu)| + d \le r$$

$Let\ \ell_r = \{(x,\mu) \in t_o : \| (x,\mu) \| \le r\} if\ (x,\mu) \in \ell_r$, we have

$$\|v\| \le (4k)^{-1}(d_1 + c_1) \underset{S\in \text{I}}{Sup} | f(s,x(s),\mu(s))|) \le (4k)^{-1}[d_o + c_o \underset{S\in \text{I}}{Sup} | f(s,x(s),\mu(s))|]$$

$$\le (4\text{k})^{-1} r \le \frac{r}{4}$$

$$\|z\| = \frac{d_2}{4} + k\ \|r\| + (\frac{c_2}{4} \underset{S\in \text{I}}{Sup} | f(s,x(s),\mu(s)| \le \frac{d}{4} + \frac{r}{4.} + \frac{c}{4} \underset{S\in \text{I}}{Sup} | f(s,x(s),\mu(s) \le \frac{r}{2}$$

Hence $\|z\| + \| r\| \le \frac{3r}{4} < r$ and T maps the convex closure of ℓ_r into itself.

Since f is bounded in ℓ_r, T (ℓ_r) is equicontinuous and hence relatively compact. By schauder Tyconov's fixed point theorem, T has a fixed point

$$T(x,\mu) = (x,\mu)$$

Thus the integral equations (3.15a) and (3.15b) have solutions since $x_{o,} x_1$, $\in E^n$ are arbitrary.Thus the system (1.2) is completely controllable.

4.0 Conclusion

Criteria for the null controllability of discrete nonlinear systems have been presented. It has been shown that, systems that are proper are null controllable and by direct application of the fixed point theorem of Schander, we established controllability for the perturbed system.

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