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Relative controllability of nonlinear systems with delays in state and control

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Abstract

In this work, sufficient conditions are developed for the relative controllability of perturbed nonlinear systems with time varying multiple delays in control with the perturbation function having implicit derivative with delays depending on both state and control variable, using Darbo's fixed points theorem.

Keywords: Controllability, Relative controllability, Non-linear systems, Perturbation function, multiple delays.

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1.0 Introduction

Controllability of systems presents a challenging but fascinating area of study in control theory. Several studies have been conducted on the controllability of linear systems (see [6, 9, 12]) and independent results obtained.

It is now known that true life dynamical systems are often nonlinear and non-deterministic since they are often affected by friction, noise, etc. Controllability of such systems has attracted lots of literature from several authors (see [1, 2, 3, 4, 5, 7, 8, 10, 11]).

There is however, no standard technique for the controllability of nonlinear systems, but the linearization and fixed point approach have proved useful and have been greatly in use. Various fixed point theorem have been applied in the controllability of nonlinear systems. In [7] and [8], the notion of linearization and Schauder's fixed point theorem are used for the study of the relative controllability of nonlinear systems with distributed delays in the control. In [2], the measure of non-compactness of set and Darbo's fixed point theorem are used to investigate the global relative controllability of nonlinear systems with time-varying multiple delays in control and having implicit derivatives.

From these studies one readily see the difficulty involved in converting a given nonlinear control system and making it satisfy the set of conditions for the application of a fixed point theorem of interest. For nonlinear systems having implicit derivative with multiple delays depending on both state and control variables, the problem is more complex, with the difficulty of making choice of appropriate state space and conditions to suit the fixed point of interest. The investigation of the relative controllability of such systems given by

$$\dot{x}(t) = L(t, x(t), x_t) + \sum_{i=0}^{N} B_i(t, x(t), x_1)\, u(h_i(t)) +$$
$$f\,(u(t), x(t - h(x(t), u(t), t)), x_t, \dot{x}(t), t)$$

is the main objective of this research work.

2.0 Basic notations and preliminaries

Let $E=(-\infty,\infty)$ and E^n be the n-dimensional Euclidean space with norm $|\cdot|$. The symbol $C=C([-h,0],E^n)$ denotes the space of continuous functions mapping the interval $[-h,0], h>0, h\in E^n$ into E^n with the suprenum norm $||.||$ defined by $\|\phi\|=\underset{-h\le\theta\le 0}{Sup}|\phi(\theta)|,\ \phi\in C$ while $C'=C'([-h,0],\ E^n)$ denotes the space of differentiable functions mapping the interval $[-h,0]$ into E^n.

Let $(X,\|\cdot\|)$ be the Banach space. The measure of non-compactness of β is given by $\mu(\beta)=\inf\{r>0:\beta$ can be covered by a finite number of balls of radii less than r }. For the space of continuous functions $C'([-h,0],E^n)$, the measure of non-compactness of β is given by

$$\mu(\beta)=\frac{1}{2}\varphi_0(\beta)=\frac{1}{2}\lim_{h\to 0}\varphi(\beta,h)$$

where $\varphi(\beta,h)$ is the common modulus of continuity of the functions which belong to the set β, that is

$$\varphi(\beta,\ h)\ =\ \sup_{x\in\beta}\{\sup|x(t)-x(s)|:\ |t-s|\le h\}$$

for the space of differentiable functions $C'([-h,0],E^n)$ we have

$$\mu(\beta)=\frac{1}{2}\varphi(P\beta)$$

where $P\beta=\{\dot{x}:x\in\beta\}$

If $t\in[t_0,t_1]$ we let $x_t\in C'$ be defined by $x_t(t)=x(t+s),\ s\in[-h,0]$. Also, for functions $u:[t_0-h,t_1]\to E^m,\ h>0$ and $t\in[t_0,t_1]$, then u_t denotes the functions on $[-h,0]$ defined by $u_t(s)=u(t+s)$ for $s\in[-h,0]$. The integrals are the Lebesgue Stieltjes sense.

We consider the system

$$\dot{x}(t)=L(t,x(t),x_t)+\sum_{i=0}^{N}B_i(t,x(t),x_1)u(h_i(t)) \qquad (2.1)$$
$$+\ f(u(t),x(t-h(x(t),u(t),t)),x_t,\dot{x}(t),t)$$

with the following basic assumptions.

$$L(t,x(t),x_t)\ =\ \int_{-h}^{0}d\eta\ (t,s,x(t),x(t+s)) \qquad (2.2)$$

satisfied almost everywhere such that the $n\times n$ matrix valued function η is measurable in $(t,s)\in E\times E$ and of bounded variation in s on $[-h,0]$. There is a locally integrable function *m* on E such that $|L(t,x(t),x_t)|\le m(t)\|x(t)\|$ so that $L(t,x(t),x_t)$ is continuous, f is an n-vector continuous function with $h(x(t),u(t),t) \geq 0$. Let the initial function $\phi(t)$ and the delay $h(x(t),u(t),t)$ be continuous and set

$$a(t) = (t - h(x(t),u(t),t)x_t)$$
$$a = \inf a(t) \quad and \quad -\infty < a < t_0$$

We assume that the function $h_i : [t_0, t_1] \to E, \quad i = 0,12,\dots N$ are twice continuously differentiable and strictly increasing in $[t_0, t_1]$. Further $h_i(t) \leq t$ for $t \in [t_0, t_1], \ i = 0,1,2,\dots N$.
We introduce the time lead function r_i , (see [6]) with $r_i(t) : [h_i(t_0), h_i(t_1)] \to [t_0, t_1]$ Such that $r_i(h_i(t)) = t \quad for \quad i = 0,1,2,\dots,N, \ \ t \in [t_0, t_1]$. Without loss of generality, we assume that $h_0(t) = t$ and the following inequalities hold for $t = t_1$

$$\begin{aligned} h_N(t_1) &\leq h_{N-1}(t_1) \leq \dots \leq h_{n+1}(t_1) \leq t_0 \leq h_n(t_1) \\ &< h_{n-1}(t_1) = \dots = h_i(t) = h_0(t_1) = t_1 \end{aligned} \qquad (2.3)$$

substituting the arguments, $x(t), \ x_t$, and $\dot{x}(t), \ (t_0 \leq t \leq t_1)$ by arbitrary functions $z, v, \dot{z} \in C_n[t_0, t_1]$ respectively. The state $x(t)$ is an n-vector and the control $u(t)$ is an m-vector. $L(t, x(t), x_t)$ is an $n \times n$ matrix, $B_i(t, x, x_t)$ for *i = 0,1,2,…, N* are $n \times m$ matrices and $f(u(t), x(t - h(x(t),u(t),t)), x_t, \dot{x}(t), t)$ is an n-vector continuous function. We assume that the elements n_{jk} of $L(j,k = 1,2,\dots,n)$ and b_{ijk} of $B_i(j = 1,2,\dots,n$ and $k = 1,2,\dots,m)$ for $i = 0,1,2,\cdots,N$ are continuous functions satisfying:

$$\left.\begin{array}{lll} (a) & |n_{ik}(t, v)| \leq M & for\ each \\ (b) & |b_{ijk}(t, v, \dot{z})| \leq L_i & t \in [t_0, t_1] \\ (c) & |f(u, z, v, \dot{z}, t)| \leq K & z, v, \dot{z} \in E^n, u \in E^m \end{array}\right\} \qquad (2.4)$$

where $M, \ L_i \ (i = 0,1,2,\cdots,N)$ and K are some positive constants. Also for every $x, x_t, \dot{x} \in E^n, \ u \in E^m$ and $t \in [t_0, t_1]$ we have

$$|n_{jk}(t,z,v) - n_{jk}(t,\bar{z},v)| \leq \frac{k_1}{n^2}|z - \bar{z}|$$

$$|b_{ijk}(t,z,v) - b_{ijk}(t,\bar{z},v)| \leq \frac{\alpha_i}{nm}|z - \bar{z}|$$

$$|f(u,z,v,\dot{z},t) - f(u,\bar{z},v,\dot{z},t)| \leq k_2|z - \bar{z}|$$

where $k_1, k_2 \; and \alpha_i (i = 0,1,2,..., N)$ are positive constants and $0 \le k_2 \le \frac{1}{3}$. Define the norm of a continuous $n \times m$ matrix valued function $\|P\|$ by

$$\|P(t)\| = \max \sum_{jk=1}^{m} |P_{jk}(t)|$$

where P_{jk} are elements of P, for system (2.1), Let X satisfy the equation

$$\frac{\partial}{\partial t} X(t,s) = L(t, z, X_t(\cdot, s)), \quad t \ge s$$

where

$$X(t,s) = \begin{cases} 0 & s-h \le t < s \\ I & t = s \quad (I \equiv identity) \end{cases}$$

Then the solution of (2.1) is given by

$$x(t) = X(t,t_0)\phi(t_0) + \int_{t_0}^{t} X(t,s) \sum_{i=0}^{N} B_i(s,z,v) u(h_i(s)) ds + \int_{t_0}^{t} X(t,s) \, f(u(s), z(s - h(z,u(s),s), v, \dot{z}, s) ds \qquad (2.5)$$

for $t_0 \le t \le t_1, \quad x(t) = \phi(t)$ for $t \in [-\infty, t_0]$ with initial state

$$y(t_0) = (x(t_0), \phi, \psi)$$

where $u(s) = \psi(s)$ for $s \in [t_0, -h, t_0]$. Then $X(t,t_0)\phi(t_0)$ is the solution of

$$\dot{x}(t) = L(t, x(t), x_t).$$

Using time lead function and the inequalities (2.3), the solution (2.5) can be expressed for $t = t_1$ as

$$\left.\begin{aligned} x(t_1) = X(t,t_0)\phi(t_0) &+ \sum_{i=0}^{n} \int_{h_i(t_0)}^{t_0} X(t, r_i(s)) \, B_i(r_i(s), z, v) \dot{r}_i(s) \psi(s) \, ds \\ &+ \sum_{i=n+1}^{N} \int_{h_i(t_0)}^{h_i(t_1)} X(t_1, r_i(s)) \, B_i(r_i(s), z, v) \, \dot{r}_i(s) \psi(s) ds \\ &+ \sum_{i=0}^{n} \int_{t_0}^{t_1} X(t_1, r_i(s)) \, B_i(r_i(s), z, v) \, \dot{r}_i(s) \, u(s) ds \\ &+ \int_{t_0}^{t_1} X(t_1, s) f(u(s), z(s - h(z, u(s), s)), v, \dot{z}, s) ds \end{aligned}\right\} \qquad (2.6)$$

Let us introduce the following notations for brevity.

$$H(t,s,u)=\sum_{i=0}^{n}\int_{h_i(t_0)}^{t_0} X(t,r_i(s))\,B_i(r_i(s),z,v)\,\dot{r}_i(s)\psi(s)ds$$
$$+\sum_{i=n+1}^{N}\int_{h_i(t_0)}^{h_i(t)} X(t,r_i(s))\,B_i(r_i(s),z,v)\,\dot{r}_i(s)\,\psi(s)ds \qquad (2.7)$$

$$q(t_1,s,u)=X(t,t_0)\phi(t_1)-\phi(t_0)-H(t,s,u)$$
$$-\int_{t_0}^{t_1} X(t_1,s)f(u(s),z(s-h(z,u(s),s),v,\dot{z},s)ds \qquad (2.8)$$

$$V(t,s,z)=\sum_{i=0}^{n}\int_{t_0}^{t} X(t,r_i(s))\,B_i(r_i(s),z,v)\,\dot{r}_i(s)$$
$$+\int_{t_0}^{t} X(t,s)\,f(u(s),x(s-h(z,u,s)),v,\dot{z},s)ds \qquad (2.9)$$

$$G_i(t_0,s)=\sum_{j=0}^{i} X(t_0,r_j(s))\,B_j(r_j(t),z,v)\,\dot{r}_j(t) \qquad (2.10)$$

Define the controllability matrix of (2.6) at time t as

$$W(t_0,t,z)=\int_{t_0}^{t} G_n(t_0,s)\,G_n^T(t_0,s)\,ds \qquad (2.11)$$

where T denote the matrix transpose

***Definition* 2.1**

The set $y(t)=\{x(t),\phi,\psi\}$ is to be the complete state of system (2.1).

***Definition* 2.2**

System (2.1) is said to be relatively controllable on $[t_0,t_1]$, if for every initial complete state $y(t_0)$ and every $x_1\in E^n$, there exists a control $u(t)$ defined on $[t_0,t_1]$ such that the corresponding trajectory of system (2.1) satisfies $x(t_1)=x_1$.

3.0 Controllability Result

***Theorem* 3.1**

Assume that

$$\text{inf def } W(t_0,t,z)>0,\quad x\in C_n' \qquad (3.1)$$

Then system (2.1) with conditions as in (2.4) is relatively controllable on $[t_0,t_1]$.

Proof

Define the control $u(t)$ for $t\in[t_0,t_1]$ as follows:

$$u(t) = G^T(t_0, s, z)W^{-1}(t_0, t, z)q(t_1, s, u) \qquad (3.2)$$

where $y(t_0)$ and x_1 are chosen arbitrarily. The inverse of W is made possible by condition (3.1). Substituting (3.2) into (2.5) to replace $u(t)$ and using (2.10), (2.11) and (3.2), it is clear that the control $u(t)$ defined by (3.2) steers the initial complete state $y(t_0)$ to the final state $x(t_1) = x_1 \in E^n$. The actual substitution of (3.2) into (2.5) yields

$$\begin{aligned} x(t) = X(t,t_0)[\phi(t_0) + H(t,s,u) \\ + V(t,s,z)G_i^T(t_0,s,z)W^{-1}(t_0,t,z)q(t_1,s,u)ds] \end{aligned} \qquad (3.3)$$

Consider the right hand side of (3.2) and (3.3) as a pair of operators $T_2([u,z])(t)$ and $T_1([u,z])(t)$ respectively. Define the continuous nonlinear operator T which maps the space β into itself by

$$T([u,z])(t) = \left[T_1([u,z])(t); T_2([u,z])(t)\right] \qquad (3.4)$$

Let us consider the closed convex subset of β

$$H = \{[u,z] \in \beta : \|u\| \le K_1, \|z\| \le K_2, \|Pz\| \le K_3\}$$

We introduce the following notations for brevity

$$\begin{aligned} J_1 = |x_0| + \sum_{i=0}^{n}(t_0 - h_i(t_0)nmL_i c_i b \exp(nM(r_i(t_0) - t_0)) \\ + \sum_{i=n+1}^{N}(h_i(t_1) - h_i(t_0))nmL_i c_i b \exp(nM(t_1 - t_0) \\ + k(t_1 - t_0)\exp(nM(t_1 - t_0)) \end{aligned}$$

$$J_2 = \sup_{t\in[t_0,t_1]} \left\|W^{-1}(t_0,t,z)\right\| \sum_{i=0}^{n} nmL_i c_i \exp(nM(t_1 - t_0))$$

$$J_3 = \sup_{z\in C_n'[t_0,t_1]} \left\|W^{-1}(t_o,t,z)\right\| \sum_{i=0}^{n} nmL_i c_i \exp(nM(r_i(t_1) - r_i(t_0)))$$

$$c_i = \|\dot{r}_i(s)\|, \quad b = \|u(s)\|$$

and define the positive constants K_1, K_2, K_3 as follows

$$K_1 = \left[|X_1| \exp(nM(t_1 - t_0)) + J_1 \right] J_2$$

$$K_2 = \exp(nM\,(t_1 - t_0)[J_1 + (t_1 - t_0)\,J_3 \left(|\,x_1| \exp(nM(t_1 - t_0) + J_1 \right) \times \sum_{i=0}^{n} nmL_i c_i \exp(nM(r_i(t_1) - r_i(t_0))] \tag{3.5}$$

$$K_3 = nMK_2 + \sum_{i=0}^{N} L_i nmK_1 + K$$

$$0 \le k_1 K_2 < \frac{1}{3}, \; 0 \le (\propto_0 + \propto_1 + .. + \propto_N) K_1 < \frac{1}{3}$$

It is easily seen that T transforms H into itself and for each pair $[u,z] \in H$, we have

$$\varphi(T_2([u,z]),h) \le \varphi(G_n^T,h)c$$

where

$$c = \sup_{[u,z] \in H} \left[\left\| W^{-1}(t_0,t,z) \right\| q(t_1,s,u) \right]$$

Thus all the functions $T_2([u,z])(t)$ have a uniformly bounded modulus of continuity, and are therefore equicontinuous. Note that all the functions $T_1([u,z])(t)$ are equicontinuous. We now consider the modulus of continuity of $PT_1([u,z])(t)$ for $t, s(t_0,t_1)$ as

$$\begin{aligned}
&|PT_1([u,z])(t) - PT_1([u,z])(s)| \le \\
&\left| m(t) \| z(t) \| T_1([u,z])(t) - m(s) \| z(s) \| T_1([u,z])(s) \right| \\
&\quad + \left| m(t) \| z(s) \| T_1([u,z])(t) - m(s) \| z(s) \| T_1([u,z])(s) \right| \\
&\quad + \sum_{i=0}^{N} \left| B_i(t, z(t), v)\, u(h_i(t)) - B_i(t, z(s), v)\, u(h_i(t)) \right| \\
&\quad + \sum_{i=0}^{N} \left| B_i([t, z(s), v) u(h_i(t)) - B_i(s, z(s), v) u(h_i(t)) \right| \\
&+ \left| f(T_2([u,z])(t), z(t - h(z(t), T_2([u,z](t),t))), v, \dot{z}(t), t) \right. \\
&\quad \left. - f(T_2([u,z])(s), z(t - h(z(s), T_2([u,z](s),s),s)), v, \dot{z}(s), s) \right| \\
&\quad + \left| f(T_2[(u,z)](s), z(s - h(z(s), T_2([u,z])(s),s)), v, \dot{z}(s), s) \right. \\
&\quad \left. - f(T_2([u,z])(s), z(s - h(z(s), T_1([u,z])(s),s)), v, \dot{z}(s), s) \right|
\end{aligned} \tag{3.6}$$

Taking upper estimates of the second, fourth and sixth terms of the right hand side of inequality (3.6) as $\psi_0(|t-s|)$, $\psi_1(|t-s|)$, $\psi_2(|t-s|)$ respectively where ψ_i are non-negative functions, such that $\underset{h->o}{him}\psi_i(h)=0$. We find that the first, third and fifth terms of inequality (3.6) can be written as $k_1 K_2 |z(t)-z(s)|$, $\sum_{i=0}^{N} \propto_1 K_1 |z(t)-z(s)|$ and $k_2|z(t)-z(s)|$ respectively. Setting $k = k_1 K_2 + (\propto_0, \propto_1 + \cdots + \propto_N) K_1 + k_2$ and $\psi = \psi_0 + \psi_1 + \psi_2$, we finally obtained

$$|PT_1([u,z],h)(t) - PT_1([u,z](s))| \le k|z(t)-z(s)| + \psi(|t-s|) \tag{3.7}$$

hence

$$\varphi(PT_1([u,z],h)) \le k\varphi(Pz,h) + \psi(h)$$

Thus we have for any set

$$E \subset H, \ \varphi_0(PT_1 E) \le k\varphi_0(PE_1) \text{ and } \varphi_0(T_2 E) = 0$$

where E_1 is the natural projection of the set E on $C_n^{'}[t_0, t_1]$. Hence, it follows that

$$\mu(TE) \le k\mu(E)$$

By the Darbo's fixed point theorem, the mapping T has at least one fixed point; therefore there exist function $u^* \in C_m[t_0, t_1]$ and $z^* \in C_n[t_0, t_1]$ such that

$$u^*(t) = T_1([u^*, z^*])(t) \tag{3.8}$$

$$z^*(t) = T_2([z^*, u^*])(t) \tag{3.9}$$

Differentiating with respect to t we see that $x(t)$ given by (3.9) is a solution to the system (2.1) for the control $u(t)$ given by (3.8), we find that any control $u(t) = u^*(t)$ steers the system (2.1) from the initial complete state $y(t_0)$ to the desired vector $x \in E_n$ on the interval $[t_0, t_1]$ and, since $y(t_0)$ and x_1 have been chosen arbitrarily, then by definition (2.2) the system (2.1) is relatively controllable on $[t_0, t_1]$.

3.1 **Remark**

If we assume that the nonlinear function satisfies a Lipschitz condition with respect to the implicit variable, the response is uniquely determined by the control $u(t)$

4.0 Conclusion

Using the Darbo's fixed point theorem, sufficient conditions for relative controllability of the perturbed nonlinear systems with time varying multiple delays in control with the perturbation function having implicit derivative with delays depending on both state and control variable have been derived.

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