Journal of Nigerian Association of Mathematical Physics, Volume 9 (November 2005)

Relative controllability and null controllability of linear delay systems with distributed delays in the state and control

Vincent A. Iheagwam and **Jerry U. Onwuatu**
Department of Mathematics and Computer Science
Federal University of Technology, Owerri, Nigeria.

Abstract

Necessary and sufficient conditions are established for the relative, absolute controllability and null controllability of the generalized linear delay system and its discrete prototype. The paper presents illuminating examples on previous controllability results by Manitius and Olbrot* [7] *and carries over the results of Onwuatu* [8] *and Klamka* [4] *to delay systems. It generalizes the results of Sebakhy and Bayoumi [11]. An algebraic approach is adopted in most of the proofs; while a manifestation of interest in the utilization of the asymptotic behaviour of solutions of differential equations provides computable criteria for relative null controllability of linear systems.

pp 221 - 238

1.0 Introduction

Quite unlike the ordinary differential control system where the action of the control is direct, a delayed control on a linear system affects the evolution of the system in an indirect manner. Consider the system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1.1}$$

The action of the control is direct in the sense that the local behaviour of the trajectory *x(t)* is affected only by the local behaviour of the control $u(t)$ at time t. However, it is known that most natural applications give rise to mechanisms of indirect action, where the decisions in the control function u are shifted, twisted or combined before affecting the evolution. An example of this delay action are the models defined by the discrete system:

$$\dot{x}(t) = \sum_{i=0}^{p} A_i(t)x(t-h_i) + \sum_{i=0}^{p} B_i(t)u(t-h_i); \ h_i \geq 0 \tag{1.2}$$

and the generalized linear delay system defined by

$$\dot{x}(t) = L(t, x_t) + \int_{-h}^{0} [d_s H(t,s)u(t+s)] \tag{1.3}$$

Klamka [4] and Onwuatu [8] have previously investigated the relative controllability of ordinary differential systems with delay in control, settling their controllability and null controllability problem in the affirmative. Chukwu [1] considered linear delay systems but without delays in the control variable. He established his result using limited controls. Balachandran in [9, 10] has demonstrated some techniques of effectively tackling the controllability problems of both linear and non linear ordinary systems with distributed delays in the control. Manitus and Olbrot [7] have obtained some controllability results for discrete systems.

Owing to the obvious difficulty of handling the many lags in both the state and control variables, not many studies are undertaken to investigate the controllability of linear delay systems with distributed delays in control. These systems are therefore the subject of interest in this research.

The present endeavour is to obtain conditions for the relative controllability and relative null controllability of system (1.3). Computable criteria for the controllability of (1.2) are also investigated.

2.0 Basic Notations and Preliminaries

Let n and m be positive integers. E, the real line $(-\infty, \infty)$. We denote by E^n the space of real n-tuples with the Euclidean norm denoted by $|\bullet|$; j is any interval of E; the usual Lebesgue space of square integrable (equivalent classes of) functions from $J \rightarrow E^n$ be denoted by $L_2(J, E^n)$. $L_1([t_0, t_1], E^n)$denotes the space of integrable functions from $L_1[t_0, t_1]$ to E^n. N_{nm} will be used for the collection of all nxm matrices with a suitable norm.

Let $h > 0$ be given. For functions $x:[t_0 - h, t_1] \rightarrow E^n$, $t \in [t_0, t_1]$ we use x_t to denote the function on $[-h, 0]$ defined by $x_t(s) = x(t + s)$ for $s \in [-h, 0]$. $C = C([-h, 0], E^n)$ is the space of continuous functions mapping the interval $[-h, 0]$ into E^n. Similarly for functions $u:[t_0 - h, t_1] \rightarrow E^m$, $t \in [t_0, t_1]$, we use u_t to denote the function on $[-h, 0]$ defined by $u_t(s) = u(t + s)$ for $s \in [-h, 0]$. We shall consider the system

$$\dot{x}(t) = L(t, x_t) + \int_{-h}^{0} [d_s H(t, s) u(t + s)] \tag{2.1}$$

where

$$L(t, x_t) = \sum_{k=0}^{\infty} A_k x(t - w_k) + \int_{-h}^{0} A(t, \theta) x(t + \theta) \tag{2.2}$$

satisfied almost everywhere on $[t_0, t_1]$ where the integral is in Lebesgue-stieltges sense with respect to s, $x(t) \in C$, $u \in L_2([t_0, t_1]$; $L(t, \phi)$ is continuous in t, linear in ϕ. $H(t, s)$ is an n x m matrix valued function which is measurable in (t, s). We shall assume that H(, s) is of bounded variation in s on $[-h, 0]$ for each $t \in [t_0, t_1]$ with $\mathrm{var}_{[-h, 0]} H(t, s) \leq m(t)$; where $m(t) \in L_1([t_0, t_1], E^n)$ and $H(t, s)$ are absolutely continuous in s on $[-h, 0]$, $A(t) \in L_1([t_0, t_1], Mnm)$. Throughout the sequel, the control sets of interest are $B = L_2([t_0, t_1], E^m)$, $U \subseteq L_2([t_0, t_1], E^m)$ a closed and bounded subset of B with zero in the interior relative to B.

If X and Y are linear spaces and $T: X \to Y$ is a mapping, we shall use the symbol $D(t)$, $R(t)$ and $N(T)$ to denote the domain, range and Null spaces of T respectively.

***Definition* 2.1 (Complete State)**

The complete state of system (2.1) at *time t is given by*

$$z(t) = \{x(t), x_t, u_t\} \tag{2.3}$$

***Definition* 2.2 (Relative controllability)**

System (2.1) is relatively controllable on $[t_0, t_1]$ if for every initial complete state $z(t_0)$ and every $x_1 \in E^n$ there exists a control $u \in B$ such that the corresponding trajectory of system (2.1) satisfies $x(t_1)$, $= x_1$.

If (2.1) is relatively controllable on each interval $[t_0, t_1]$ $t_1 > t_0$, we say (2.1) is relatively controllable.

***Definition* 2.3 (Relative Null Controllability)**

System (2.1) is said to be relatively null controllable at $t = t_1$, if for any initial complete state $z(t_0) = \{x(t_0), x_{t_0}, u_{t_0}\}$ on $[t_0 - h, t_0]$ there exists an admissible control $u(t) \in B$ defined on $[t_0, t_1]$ such that the response $x(t)$ of the system satisfies $x(t_1) = 0$.

It is relatively null controllable with constraint at $t = t_1$, if for any initial state $[x(t_0), x_{t_0}, u_{t_0}]$ on $[t_0 - h, t_0]$ there exists an admissible control $u \in U$ defined on $[t_0, t_1]$ such that the response x(t) of (2.1) satisfies $x(t_1) = 0$.

***Definition* 2.4 (Absolute Null Controllability)**

System (2.1) *is said to be absolutely null controllable at* $t = t_1$, *if for any initial complete state* $z(t_0) = [x(t_0),\ x_{t_0},\ u_{t_0}]$ *on* $[t_0 - h, t_1]$ *there exists an admissible control* $u(t) \in B$ *defined on* $[t_0, t_1 - h]$ *such that the response x(t) of the system satisfies* $x(t_1) = 0$ *using the control effort*

$$\bar{u}(t) = \begin{cases} u(t) & on \quad [t_0, t_1] \\ 0 & on \quad [t_1 - h, t_1] \end{cases} \tag{2.4}$$

2.1 **Variation of Constant Formular for the Solution of System (2.1)**

The above conditions on L(t, ϕ) and H ensure the existence of a unique absolutely continuous solution x(t) of (2.1) with initial complete state $z(t_0)$.

The solution of system (2.1) is of the form

$$x(t, t_0, \phi, u) = X(t, t_0)\,\phi(0) + \int_{t_0}^{t} X(t,\tau)\left[\int_{h}^{0} d_s H(\tau, s) u(\tau + s)\right] d\tau \tag{2.5}$$

where

$$X_t(\bullet, s)\theta = X(t + \theta, s);\ -h < \theta < 0 \tag{2.6}$$

and X(t, s) is the fundamental solution of

$$\dot{x}(t) = L(t, x_t) \tag{2.7}$$

satisfying

$$\frac{\partial X(t, s)}{\partial t} = L(t, X_t(\bullet, s));\ t \geq s \text{ a. e. in } (t, s) \tag{2.8}$$

$$X(t, s) = \begin{cases} 0 & s - h \leq t < s \\ 1 & t = s\ ;\ 1 \ \text{ identity matrix} \end{cases} \tag{2.9}$$

Using the unsymmetric Fubini's theorem for $t = t_1$ the solution (2.5) of system (2.1) becomes.

$$x(t_1\ t_0, \phi, u) = X(t_1, t_0)\phi(0) + \int_{-h}^{0} d_H \left[\int_{t_0}^{t_1} X(t_1, \tau) H(\tau, s) u(s + \tau) d\tau\right]$$

$$= X(t_1, t_0)\phi(0) + \int_{-h}^{0} d_H \int_{t_0 + s}^{t_1 + s} X(t_1, \tau - s) H(\tau - s, s) u(\tau) d\tau$$

This implies

$$x(t_1, t_0, \phi, u) = X(t_1, t_0)\phi(0) + \int_{-h}^{0} d_H \left[\int_{t_0 + s}^{t_0} X(t_1, \tau - s) H(\tau - s, s) u_{t_0}(\tau)\right]$$
$$+ \int_{-h}^{0} d_H \left[\int_{t_0}^{t_1 + s} X(t_1, \tau - s) H(\tau - s, s) u(\tau) d\tau\right] \tag{2.10}$$

By letting

$$\overline{H}(t, s) = \begin{cases} H(t, S) & for\ t \leq t_1,\ s \in E \\ 0 & for\ t > t_1,\ s \in E \end{cases} \tag{2.11}$$

we obtain

$$x(t_1, t_0, \phi, u) = X(t_1, t_0)\phi(0) + \int_{-h}^{0} d_H \left[\int_{t_0+s}^{t_0} X(t_1, \tau - s) H(\tau - s, s) u_{t_0} d\tau \right]$$

$$+ \int_{-h}^{0} d_H \left[\int_{t_0}^{t_1} X(t_1, \tau - s) \overline{H}(\tau - s, s) u(\tau) d\tau \right] \qquad (2.12)$$

Using again the unsymmetric Fubini's theorem, we have

$$x(t_1, t_0, \phi, u) = X(t, t_0)\,\phi(0) + \int_{-h}^{0} d_H \left[\int_{t_{0,s}}^{t_0} X(t_{1,}\tau - s) H(\tau - s, s) u_{t_0} d\tau \right]$$

$$+ \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t_{1,}\tau - s) d\overline{H}(\tau - s, s) \right] u(\tau) d\tau \qquad (2.13)$$

we now define the n x n controllability matrix of (2.1) by

$$W(t_0, t_1) = \int_{t_0}^{t_i} \left[\int_{-h}^{0} X(t_{1,}\tau - s) d\overline{H}(\tau - s, s) \right] \left[\int_{-h}^{0} X(t_{1,}\tau - s) d\overline{H}(\tau - s, s) \right]^T d\tau \qquad (2.14)$$

Where the symbol T denotes the matrix transpose

***Definition* 2.5**

The Reachable set $P(t_1, t_0)$ of (2.1) *at time t_1 is the subset of E^n given by*

$$P(t_1, t_0) = \int_{t_0}^{t_1} \left\{ \left[\int_{-h}^{0} X(t_{1,}\tau - s) d\overline{H}(\tau - s, s) \right] u(\tau) d\tau : u \in U \right\} \qquad (2.16)$$

The constraint reachable set with unspecified end time is given by $R(t_1, t_0) = \underset{t_1 \geq t_0}{U} R(t_{1,}t_0)$

***Definition* 2.6 (Proper systems)**

System (2.1) *is said to be proper in E^n on* $[t_0, t_1]$ *if*

$$C^T \left[\int_{-h}^{0} X(t_{1,}\tau - s) d\overline{H}(\tau - s, s) \right] = 0 \qquad (2.17)$$

a.e., $t \in [t_0, t_1]$, $c \in E^n$ *implies that $c = 0$. If* (2.1) *is proper on each interval $[t_0, t_1]$, $t_1 > t_0 > 0$, we say that the system is proper in* E^n

3.0 Relative controllability results

3.1 Proposition

The following statements are equivalent:

- *(i)* *$W(t_0, t_1)$ is non – singular for each $t_1 > t_0$*
- *(ii)* *System* (2.1) *is proper in E^n for each interval $[t_0, t_1]$.*
- *(iii)* *System* (2.1) *is relatively controllable on each interval $[t_0, t_1]$*

Proof (i) ⇒ (ii)

Let

$$W(t_0,t_1)=\int_{t_0}^{t_1}\left[\int_{-h}^{0}X(t_{1,}\tau-s)d\overline{H}(\tau-s,s)\right]\left[\int_{-h}^{0}X(t_{1,}\tau-s)d\overline{H}(\tau-s,s)\right]^T d\tau$$

Define the operator K: $L_2([t_0, t_1],E^m) \rightarrow E^n$

$$K(u)=\int_{t_0}^{t_1}\left[\int_{h}^{0}X(t_1,\tau-s)d\overline{H}(\tau-s,s)\right]u(\tau)d\tau \qquad (3.1)$$

K is a continuous linear operator from a Hilbert space to another. Thus, $R(K) \subset E^n$ is a linear subspace and its Orthogonal complement satisfies the relation.

$$(R(K))^{\perp} = N(K^*) \qquad (3.2)$$

where K^* is the adjoint of K

$$K^* : E^n \text{ --- > } U \subset L_2$$

By the non-singularity of $W\langle t_0, t_1\rangle$ the symmetric operator $KK^T = W(t_0, t_1)$ is positive definite and hence

$$\{R(K)\}^{\perp} = \{0\} \qquad (3.3)$$

By (3.2)

$$N(K^*) = \{0\} \qquad (3.4)$$

For any $c \in E^n$, $u \in L_2$

$\langle c, Ku\rangle = \langle K^*c,u\rangle$

$$\langle c, Ku\rangle = \left\langle c, \int_{t_0}^{t_1}\left[\int_{h}^{0}X(t_{1,}\tau-s)d\overline{H}(\tau-s,s)\right]u(\tau)d\tau\right\rangle \qquad (3.5)$$

$$=\int_{t_0}^{t_1}c^T\left[\int_{-h}^{0}X(t_{1,}\tau-s)d\overline{H}(\tau-s,s)\right]u(\tau)d\tau \qquad (3.6)$$

Thus K^* is given by

$$c \rightarrow c^T\left[\int_{-h}^{0}X(t_{1,}\tau-s)d\overline{H}(\tau-s,s)\right];\tau\in[t_0,t_1]$$

$N(K^*)$ is therefore the set of all such $c \in E^n$ such that

$$c^T\left[\int_{-h}^{0}X(t_{1,}\tau-s)d\overline{H}(\tau-s,s)\right]=0 \qquad (3.7)$$

a.e in $[t_0, t_1]$.

Since $N\langle K^*\rangle = \{0\}$, all such *c* are equal to zero i.e $c = 0$

This establishes the properness of system (2.1)

(ii)⇒ (iii).

We now show that if system (2.1) is proper then it is relatively controllable on each interval $[t_0, t_1]$. Let $c \in E^n$, if system (2.1) is proper then

$$c^T\left[\int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right] = 0; \text{ a.e } \; s \in [t_0, t_1] \text{ for each } t_1 \text{ implies } c = 0$$

Thus

$$\int_{t_0}^{t_1} c^T\left[\int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right]u(\tau)d\tau = 0$$

for $u \in L_2$. It follows that the only vector orthogonal to the set

$$R(t_1, t_0) = \left\{\int_{t_0}^{t_1}\left[\int_{t_0}^{0} X(t_{1,}\tau - s)dH(\tau - s, s)\right]u(\tau)d\tau : u \in L_2\right\}$$

is the zero vector, hence $\{R(t_1, t_0)\}^{\perp} = \{0\}$. i.e $R(t_1, t_0) = E^n$. This means that the system is Euclidean controllability and hence this establishes relative controllability on $[t_0, t_1]$ of system (2.1) see ref [8]

(iii) $\Rightarrow$ (i)

We now show that if the system is relatively controllable then controllability grammian $W = W(t_0, t_1)$, is non-singular. Let us assume for contradiction that W is singular. Then, there exists an n vector $v \neq 0$ such that

$$vWv^T = 0 \tag{3.8}$$

Then

$$\int_{t_0}^{t_1}\left\|v\left[\int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right]\right\|^2 d\tau = 0 \tag{3.9}$$

This implies that

$$\left\|v\left[\int_{-h}^{0} X(t_1, \tau - s)d\overline{H}(\tau - s, s)\right]\right\|^2 = 0 \text{. a.e}$$

hence

$$v\left[\int_{-h}^{0} X(t_1, \tau - s)d\overline{H}(\tau - s, s)\right] = 0 \text{. a.e} \tag{3.10}$$

for $t \in [t_0, t_1]$. This contradicts the assumption of properness of the system since $v \neq 0$. This completes the proof:

We now give another condition for relative controllability of the system (2.1)

***Theorem* 3.1**

System (2.1) *is relatively controllable if and only if* $0 \in$ *interior* $R(t_1, t_0)$ *for each* $t_1 > t_0$

Proof

$R(t_1, t_0)$ is a closed and convex subset of E^n. Therefore, a point y_1 on the boundary of $R(t_1,t_0)$ implies there is a support plane Π of $R(t_1, t_0)$ through y_1. That is $c^T(y - y_1) \leq 0$ for each $y \in R(t_1, t_0)$ where $c \neq 0$ is an outward normal to Π. If u_I is the control corresponding to y_1 we have,

$$c^T\int_{t_0}^{t_1}\left[\int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right]u(\tau)d\tau \;\leq\; C^T\int_{t_0}^{t_1}\left[\int_{-h}^{0} X(t_1, \tau - s)d\overline{H}(\tau - s, s)\right]u_1(\tau)d\tau \tag{3.11}$$

For each u ∈ U, since U is a unit sphere this last inequality holds for u ∈ U if and only if

$$c^T \int_{t_0}^{t_1}\left[\int_{-h}^{0} X(\tau - s, s)\,d\overline{H}(\tau - s, s)u(\tau)d\tau\right] \le \int_{t_0}^{t_1}\left[c^T \int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right]u_1 d\tau$$

$$= \int_{t_0}^{t_1}\left|c^T \int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right|d\tau \qquad (3.12)$$

and

$$u_1(t) = \operatorname{sgn}\ c^T \int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s) \qquad (3.13)$$

as y_1 is on the boundary. since we always have $0 \in R(t_1,t_0)$, if 0 were not in the interior of $R(t_1, t_0)$ then it is on the boundary. Hence from preceding argument this implies that

$$0 = \int_{t_0}^{t_1}\left|c^T X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right|d\tau \qquad (3.14)$$

so that

$$c^T \int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s) = 0;\ \text{a.e., } t \in [t_0, t_1]$$

This by definition of properness of systems implies that the system is not proper, since $c^T \neq 0$. Hence, if $0 \in \operatorname{int} R(t_1, t_0)$.

$$c^T \int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s) = 0;$$

a.e., $t \in [t_0, t_1]$ would imply c = 0 proving properness; and by proposition 3.1, we conclude relative controllability of system (2.1) on each interval.

4.0 Discrete System

Consider the discrete system

$$\dot{x}(t) = \sum_{i=0}^{p} A_i(t)x(t - h_i) + \sum_{i=0}^{p} B_i(t)u(t - h_i) \qquad (4.1)$$

where $x(t) \in E^n$, $u(t) \in E^n$, $A_1(t)$, $B_I(t)$ are time varying matrices of dimensions n x n and n x m respectively defined and continuous on $[t_0, t_1)$, hi are delays with $0 = h_0 < h_I < h_2 < \ldots < hp = h$.
The solution of the equation is given by

$$x(t) = X(t_1, t_0)\phi(t_0) + \sum_{1=0}^{p} \int_{t_0 - h_i}^{t_0} X(t, s + h_i)A_i(s + h_i)\phi(s)ds$$

$$+ \int_{t_0}^{t_1} X(t, s)\sum_{i=0}^{p} B_i(s)u(s - h_i)ds \qquad (4.2)$$

This formula can be rewritten as

$$x(t) = X(t_1, t_0)\phi(t_0) + \sum_{1=0}^{p} \int_{t_0\_h_i}^{t_0} X(t, s+h_i)A_i(s+h_i)\phi(s)ds$$

$$+ \sum_{i=0}^{p} \int_{t_0-h}^{t_0} X(t, s+hi)B(s+h)u_{t0}(s)ds + \sum_{1=0}^{p}\int_{t_0}^{t_1} X(t, s+h_i)B_i(s+h_i)u(s)ds \qquad (4.3)$$

If we now set

$$Z(t,s) = \sum_{1=0}^{p} X(t, s+h_i)B_i(s+h_i) \qquad (4.4)$$

Then (4.3) becomes

$$x(t) = X(t, t_0)\phi(t_0) + \sum_{1=0}^{p}\int_{t_0}^{t_1} X(t, s+h_i)A_i(s+h_i)\phi(s)ds$$

$$+ \int_{t_0-h_i}^{t_0} X(t, s+h_i)B_i(s+h_i)u_{t_0}(s)ds + \int_{t_0}^{t} Z(t,s)u(s)ds \qquad (4.5)$$

Assume piecewise continuity of Z(t,s) with respect to t; and making use of arguments as in proposition (3.1) above, we get the following conditions for relative controllability of system (4.1)

***Lemma* 4.1**

The system (4.1) *is relatively controllable on* $[t_1, t_0]$ *if and only if for* $y \in E^n$ *the relation* $y^T Z(t_1, s) \equiv 0$ *on* $[t_0, t_1]$ *implies* $y = 0$ *(T denotes transposition).*

Proof

Immediate from proposition 3.1

We provide another condition for the controllability of system (4.1)

***Lemma* 4.2**

The system is relatively controllable on $[t_0, t_1]$ *if*

$$rank\left\{\int_{t_0}^{t} Z(t,s)Z^T(t,s)ds\right\} = n \qquad (4.6)$$

Proof

Relative controllability from prop. 3.1 implies that the non-singularity of the grammian $W(t_0, t_1)$; implies that the symmetric operator

$$\int_{t_0}^{t_1} Z(t,s)Z^T(t,s)ds$$

is positive definite. But this holds if and only if

$$rank\left\{\int_{t_0}^{t_1} Z(t,s)Z^T(t,s)ds\right\} = n$$

Due to obvious difficulty in the use of these criteria when the matrix valued function Z(t,s) cannot be analytically obtained, an attempt was made in [7] to provide computable criteria for relative controllability of system (4.1) using determining equation and properties of Z(t,s).
Let $J = (J_0, J_1, \ldots, J_p)$, where J_i , i = 0, 1, 2,…, p are integers (not necessary positive) with

$$|J| = \sum_{i=0}^{p} J_i \qquad (4.7)$$

and Let E_i be a similarly defined multi index with $E_i = \delta_{ik}$ (Kronecker delta) for k = 0, 1, 2, …; p.
Clearly

$$|E_I| = 1$$

Assume $A_1(t)$, $B_I(t)$ are $(q-2)$ $(q-1)$ continuously differentiable on $[t_0, t_1]$ respectively. If $H = (h_0, h_1, \ldots, h_p)$, we set

$$< J, H > = \sum_{i=0}^{p} J_i h_i \qquad (4.8)$$

we define the determining equation as reported in [7] as follows

$$Q_k(J, t) = \sum_{i=0}^{p} A_i(t)\, Q_{k-1}(J - E_i,\ t - <E_i, H>) - \frac{d}{dt} Q_{k-I}(J - E_0, t) \qquad (4.9)$$

for $k = 1, 2, \ldots, q-1$, $t \in [t_0, t_1]$ such that

$$Q_0(J, t) = \begin{cases} B_i(t) & t \in [t_0, t_1] \quad \textit{for } J = E_i \\ 0 & \textit{otherwise} \end{cases} \qquad (4.10)$$

We deduce from (4.9) and (4.10) that
(i) $Q_k(j, t) = 0$ for $|J| \neq k+1$ or for $J \geq 0$
(ii) If some $A_I(t)$ or $B_I(t)$ are undefined for $t > t_0$ then $Q_k(j, t)$; with $J \geq 0$, $|J| = k+1$, k = 0, 1, 2,…, $q-1$ is undefined for $t < t_0$.
(iii) $Q_k(j, t)$: $|J| \neq k+1$ is undefined also for $t - (E_I, H) < t_0$ and by induction for $t - (J, H) < t_0$.
Set

$$\overline{Q_k}(J, t_1) = \sum_{k=0}^{q-1} Q_k(R, t) \text{ for } |R| = |J| = k+1, \text{ and } < R, H> = < J, H > \qquad (4.11)$$

With $R = (r_0, r_1, \ldots, r_p)$, r_i are integers . We have the following result
***Theorem* 4.1**
Assume $A_i(t)$, $B_I(t)$ are $(n-2)$, $(n-1)$ *continuously differentiable on $[t_0, t_1]$ respectively then system* (4.1) *is relatively controllable on $[t_0, t_1]$ if*

$$\text{rank } \hat{Q}_n(t_1) = n \qquad (4.12)$$

$\hat{Q}_n(t_1) = \{ \hat{Q}_k(J, t_1), k = 0, 1, \ldots, n-1; j : t_1 - (J, H) \geq t_0 \}$
The proof is contained in Manitius and Olbrot [7], here an example is given to illustrate the theorem.
Example 4.1
Consider the discrete autonomous system with lag in both state and control

$$\dot{x}(t) = A_0x(t) + A_1x(t-1) + a_2x(t-2) + B_0u(t) + B_1u(t-1) + B_2u(t-2) \qquad (4.13)$$

where

$$A_0 = \begin{pmatrix} 2 & 2 \\ -2 & -5 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Here $n = k = 2$

Take $H = (h_0, h_1, h_2) = (0_1, 1, 2)$ $J = (J_0, J_1, J_2) = (-1, 2, 1)$, $R = (r_0, r_1, r_2) = (1, 4, 0)$

From the definition, $E_i = \delta_{ik}$, i = 0, 1, 2 so, $E_0 = 0$, $E_1 = 0$, $E_2 = 1$ and $< J, H > = \sum_{i=0}^{2} j_i h_i = 0 + 2 + 2 + 4$

$$< R, H > = \sum_{i=0}^{2} r_i h_i = 0 + 4 + 0 = 4$$

$$|j| = \sum_{i=0}^{2} j_i = -1 + 2 + 1 = 2 \qquad |R| = \sum_{i=0}^{2} r_i = 1 + 4 + 0 = 5$$

$$\hat{Q}_n = \{ \hat{Q}_k (J);\ k = 0, 1, \ldots, n-1,\ t_1 - \langle J, H\rangle \geq t_0\}$$

$$\hat{Q}_2 = \{ \hat{Q}_k (J);\ k = 0, 1;\ t_1 - \langle J, H\rangle \geq t_0\}$$

$$= \{ \hat{Q}_0 (J), Q_1 (J);\ t_1 - 4 \geq t_0\}$$

By definition

$$\hat{Q}_n (J) = \sum_{n=0}^{2} Q(R) \text{ for } |R| = |J| = k + 1 \text{ and } \langle R, H\rangle = \langle J, H\rangle$$

Therefore $\hat{Q}_0 (J) = Q_0 (R)$ and $\hat{Q}_1(J) = Q_1(R)$

But

$$Q_k(J) = \sum_{i=0}^{2} A_i Q_{k-1} (J - E)\ ;\ k = 1,2, \ldots .$$

$$Q_0(J) = \begin{cases} B_i & \text{for } J = E_i \\ 0 & \text{for other } J \end{cases}$$

Hence

$$Q_0(R) = B_0 =. \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

$Q_1(R) = A_0B_0 + A_1B_1 + A_2B$

$$= \begin{pmatrix} 2 & 2 \\ -2 & -5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}$$

$$rank\,\hat{Q}_2 = rank\{ \hat{Q}_2 (J), \hat{Q}_1 (J): t_1 - 4 \geq t_0\} = \text{rank}\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 \end{pmatrix} \right\} = 2 = n$$

Invoking theorem 4.1, we conclude that system (4.13) is relatively controllable for $t_1 \geq t_0 + 4$

5.0 Relative Null Controllability Results

***Definition* 5.1**

Domain D of relative null controllability of system (2.1) is the set of all initial functions $\phi \in C$ for which the solution of (2.1) *with $x_t = \phi$ satisfies $x(t_1) = 0$ at some t_1 using $u \in U$*

***Theorem* (5.1)**

If system (2.1) *is relatively controllable on $[t_0, t_1 > t_0$ then the domain of null controllability of (2.1) contains zero in its interior.*

Proof

Assume system (2.1) is relatively controllable on $[t_0, t_1]$, $t_1 > t_0$ then by theorem (3.1), $0 \in$ Int $R(t_0, t_1)$, for each $t_1 \geq t_0$. Since $x = 0$ is a solution of (2.10) with u = 0, we have $0 \in D$. Hence, if $0 \notin$ int D then there exists a sequence $\{\phi_m\} \subseteq D$ such that $\phi_m \to 0$ as $m \to \infty$ and no ϕ_m is in D (so $\phi_m \neq 0$). From the variation of constant formula, we have.

$$0 \neq x(t_1, t_0, \phi_m, u) = X(t_1, t_0)\phi_m(t_0) + \int_{-h}^{0} dH\left[\int_{t_{0+s}}^{t_0} X(t_1, \tau - s)H(\tau - s, s)u_{t_0} d\tau\right]$$

$$+\int_{t_0}^{t_1}\left[\int_{-h}^{0} X(t_1, \tau - s)d\overline{H}(\tau - s, s)\right]u(\tau)d\tau; \quad \text{for}, \ t_1 > t_0, \qquad u \,\varepsilon\, U$$

Hence z_m defined by

$$z_m = -x(t_1, t_0, \phi_m, 0) - \int_{-h}^{0} d\left[\int_{t_{0+s}}^{t_0} X(t_1, \tau - s)H(\tau - s, s)u_{t_0} d\tau\right]$$

is not in $R(t_1, t_0)$ for any $t_1 > t_0$. Therefore, the sequence $z_m \subseteq E^n$ is such that $Z_m \notin R(t_1, t_0)$, $z_m \neq 0$ but $z_m \to 0$ as $m \to \infty$. Therefore $0 \notin$ Int $R(t_1, t_0)$. This is a contradiction and hence proves that $0 \in$ Int D.

***Theorem* 5.2**

Assume (i) system (2.1) *is relatively controllable on* $[t_0,t_1]$ *for each $t_1 > t_0$* (ii) *the zero solution of the system:*

$$\dot{x}(t) = L(t, x_t) \tag{5.1}$$

is uniformly asymptotically stable, so that the solution of (5.1) satisfies $||x(t)|| \leq k||\phi||\, e^{-\alpha(t-t_0)}$ where $\alpha > 0$, $k > 0$ are constants.

Then system (2.10) is relatively null controllable with constraint.

Proof

By condition (*i*) and theorem (5.1) the domain D of relative null controllability of (1) contains zero in its interior. Therefore there exists a ball B_1 such that $0 \in B_1 \subseteq D$. By (ii) every solution of (2.1) with u = 0) satisfies $x(t, t_0, \phi, 0) \to 0$ as $t \to \infty$. Hence at some $t_1 < \infty$ $x(t, t_0, \phi, 0) \in B_1 \subseteq D$ for $t_1 > t_0$. Therefore using t_1 and $x_1 = x(t_1, t_0, \phi, 0)$ as initial data, there exists a $u \in U$ and some $t_2 > t_1$ such that the solution $x(t, t_1, x_1, u)$ of (2.1) satisfies $x(t_2, t_1, x_1, u) = 0$.

This completes the theorem

5.1 Realisations from Theorem 5.2

Recall that system (2.1) is relatively controllable on $|t_0, t_1|$ if and only if rank $W(t_0, t_1) = n$. System (2.1) is relatively null controllable with constraint if (i) rank $W(t_0, t_1) = n$ for $t_1 > t_0$ (ii) the zero solution of system (5.1) is uniformly asymptotically stable.

***Definition* 5.2 (Closed Operator)**

An operator T: X $\rightarrow$ Y where X and Y are linear spaces is said to be closed if for any sequence $u_n \in D(T)$ such that $u_n \rightarrow u$ and $Tu_n \rightarrow v$, u belongs to D(T) and Tu =v.

***Theorem* 5.3**

Assume system (2.1) is null controllable on $[t_0,t_1]$ then for each $\phi \in C$, there exists a bounded linear mapping f: $C \rightarrow L_2$ such that the control $u = f\phi$ has the property that the solution of (2.1) *satisfies*

$$x_{t_0}(\cdot, t_0, \phi, u) = \phi \text{ and } x(t_1, t_0, \phi, f\phi) = 0$$

Proof

From the variation of parameter equation (2.13), if we set

$$T(t, t_0)\phi = X(t_1, t_0)\phi \tag{5.2}$$

then T is a continuous linear operator from $C \rightarrow C$. Now define S(t): $L_2 \rightarrow E^n$ by

$$S(t)u = \int_{-h}^{0} dH\left[\int_{t_0+s}^{t_0} X(t_1, \tau - s)H(\tau - s, s)u_{t_0}\,d\tau\right] + \int_{t_0}^{t}\left[\int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right]u(\tau)d\tau \text{ for } \tau \in [t_0, t_1] \tag{5.3a}$$

clearly $S(t)$ is a linear map. The boundedness of $S(t)$ follows from the assumptions on H(t,s). From (5.2) and (5.3) the equations (2.13) can be written as $x(t_1, t_0, \phi, u) = T(t, t_0)\phi(t_0) + S(t)u$, $t \in |t_0, t_1|$. That system (2.1) is null controllable is equivalent to the statement that for every $\phi \in C$ there exists a $u \in L_2$ such that

$$T(t_1, t_0)\phi(t_0) + S(t_1)u = 0 \tag{5.3b}$$

This implies that

$$T(t_1, t_0)\,E^n \subseteq S(B) \tag{5.4}$$

(5.4) holds by hypothesis

Let N be the null space of S and denote the orthogonal complement of N in B by $N^{\perp}$. Let S_0: $N^{\perp} \rightarrow S(t_1)B$ be the restriction of $S(t_1)$ to $N^{\perp}$. Then S_0^{-1} exists and is linear. Since $S(t_1)B$ is not necessarily closed in E^n, S_0^{-1} is not necessarily bounded. In (5.3b) we define, f: $C \rightarrow B$ by $f\phi = -S_0^{-1} T(t_1, t_0)\phi$; then,

$$x(t_1, t_0, \phi, f\phi) = T(t_1, t_0)\phi(\Theta) + S(t_1)(-S_0^{-1})T(t_1, t_0)\phi(\Theta) = 0$$

Thus, $x(t_1, t_0, \phi, f\phi = 0$.

It now remains to prove that f is bounded. Let ϕ_n be a convergent sequence in C such that $f\phi_n$ converges in B and let $\phi = \lim_{n\rightarrow\infty}\phi_n$, $u = \lim f\phi_n$

Since $N^{\perp}$ is closed in B, $u \in N^{\perp}$ and $T(t_1,.t_0)\,\phi(t_0) + S(t_1)u = \lim_{n\rightarrow\infty} (T(t_1,t_0)\phi_n(t_0) + S(t_1)u_n) = 0$

Thus $u = S_0^{-1}T(t_1,t_0)\phi = f\phi$. And therefore by the closed graph theorem f is bounded

***Theorem* (5.4)**

The system (2.1) with the control $u_{t_0}(t)$ on $[t_0 - h, t_0]$ is null controllable at $t = t_1$ if and only if

$$y(z(t_0)) = -\phi_i\, t_0) - \int_{-h}^{0} d_H \int_{t_0}^{t_0}\left[x(t_{1,}\tau - s)H(\tau - s, s)u_{t_0}\,d\tau\right]$$

belongs to the range space of the null controllability grammian

$$\Gamma(t_{0,}t_1) = \int_{t_0}^{t_1}\left[\int_{h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right]\left[\int_{-h}^{0} X(t_{1,}\tau - s)d\overline{H}(\tau - s, s)\right]^{T} d\tau$$

Proof (**Sufficiency**)

Let $y(z(t_0)) \in R(\Gamma(t_0, t_1))$ then for $z_0 \in E^n$ we have $y(z(t_0)) = \Gamma(t_0, t_1)z_0$, choose

$$u(s)=\begin{cases}\int_{-h}^{0}\left[x(t_{1,}\tau-s)dH(T-s,s)\right]^T z_0 & s\in[t_0-h,t_1]\\ 0 & s\in[t_1-h,t_1]\end{cases} \tag{5.5}$$

then substituting (5.5) into the variation of parameter equation for (2.1) in 2.12, we obtain

$$x(t_1,t_0,\phi,u)=X(t_1,t_0)\left[\phi(t_0)+\int_{-h}^{0}d_H\int_{t_{0+s}}^{t_0}X(t_{1,}\tau-s)H(\tau-s,s)u_{t_0}d\tau\right]$$

$$+\int_{t_0}^{t}\int_{-h}^{0}X(t_{1,}\tau-s)d\overline{H}(\tau-s,s)u(\tau)d\tau = X(t_1,t_0)[-y(z(t_0))]+X(t_1,t_0)\Gamma(t_1,t_0)z_0$$

$$= -X(t_1, t_0)\Gamma(t_0, t_1)z_0 + X(t_1, t_0)\Gamma(t_1,t_0)z_0 = 0$$

establishing null controllability.

Necessity

Suppose for a contradiction, that is $y(z(t_0)) \notin R(\Gamma(t_0, t_1))$ then there exists $z_1, z_2 \in E^n$ such that

$$y(z(t_0)) = z_1 + z_2; \qquad z_2 \neq 0$$

where

$$z_1 \in R(\Gamma(t_0, t_1)),\ z_2 \in N(\Gamma(t_0, t_1))$$

Thus

$$\langle z_2, \Gamma(t_0, t_1)z_2\rangle = \int_{t_0}^{t_1-h}\|\Gamma(t_0,t_1)z_2\|^2\, d\tau = \int_{t_0}^{t_1-h}\|\int_{-h}^{0}X(t_0,\tau-s)dH(\tau-s,s)^T z_2\|\, d\tau$$

since the integrand is non-negative, we obtain

$$\left[\int_{-h}^{0}X(t_{0,}\tau-s)d\overline{H}(\tau-s,s)\right]^T z_2=0 \quad \tau\in[t_0,t_1-h] \tag{5.6}$$

By hypothesis, however,

$(x(t_0),\ x_{t_0},\ u_{t_0})$ can be brought to the origin by some control effort on $[t_0, t_1]$ i.e

$$\int_{t_0}^{t_1 h}\left[\int_{-h}^{0}X(t_{0,}\tau-s)d\overline{H}(\tau-s,s)\right]u(\tau)d\tau=0 \tag{5.8}$$

$$z_2^T\left[\int_{-h}^{0}X(t_{0,}\tau-s)d\overline{H}(\tau-s,s)\right]u(\tau)d\tau=0 \tag{5.9}$$

By combining (5.6) and (5.9) we obtain the contradiction

$$\| z_2 \|^2 = 0 \Rightarrow z_2 = 0$$

but by assumption $z_2 \neq 0$

Hence,

$$y(z(t_0)) \in R(\Gamma(t_0, t_1)).$$

5.2. **Null controllability with constraint**

***Theorem* 5.5**

Assume (2.1) *is relatively null controllable with* L_2 *controls then it is locally relatively null controllable with constraint*.

Proof

Since (2.1) is relatively null controllable by theorem 5.5, there is a bounded linear operator f: C → B such that for each $\phi \in C$ the solution $x(t, t_0, \phi, f\phi)$ of (2.1) satisfies $x_{t_0}(\bullet, t_0, \phi, f\phi) = \phi$, $x(t_1, t_0, \phi, f\phi) = 0$. Because f is continuous, it is continuous at the origin E^n. Hence, for each neighhourhood N_1 of the origin L_2 there exists a neighbourhood N_2 in C such that $f(N_2) \subseteq N_1$. In particular N_1 can be chosen to be any open set in B containing zero which is contained in U, since U is a closed and bounded subset of B which has zero in its interior. Hence, there exists an open set N_2 around the origin in C such that $f(N_2)$ N_1 $\subseteq$ U.. Every $\phi \in N_2$ can be steered to zero by the control $u = f\phi \in U$. Hence (2.1) is locally relative null controllable with constraint.

***Theorem* 5.6**

Assume (*i*) *system* (2.1*) is relatively null controllable on [t_0, t_1} for each $t_1 > t_0$ (ii) the zero solution of the system* $\dot{x}(t) = L(t, x_t)$ *is uniformly asymptotically stable then system* (2.1*) is null controllable with constraint.*

Proof

Condition (i) and theorem 5.3 guarantee an open ball $N_1 \subseteq C$ such that every $\phi \in N_1$ can be steered to zero point of E^n with controls from U in time $t_1 < \infty$. Condition (ii) ensures that every solution of system (5.1) which is a solution of (2.1) with u = 0 satisfies $x(t, t_0, \phi, 0) \to 0$ as $t \to \infty$. Thus using u = 0, there exists a $t_2 < \infty$ such that $x_2 = x(t_2, t_0, \phi, 0) \in N_1$ with x_2 and t_2 as initial data there exists $t_3 > t_2$ such that for some $u \in U$ $x(t_2, t_2, x_2, u) = x_2$ and $x(t_3, t_2, x_2, u) = 0$. Thus, the control

$$w = \begin{cases} 0 & in \quad [t_0, t_1] \\ u & in \quad [t_2, t_3] \end{cases}$$

transfers ϕ to the origin in time $t_1 < \infty$. This completes the theorem.

The above results on the null controllability of system (2.1) can easily be shown to apply to the discrete system (4.1). To this end we state the following theorems: without proofs.

***Theorem* 5.7**

Assume system (4.1) is relatively controllable on [t_0, t_1], $t_1 > t_0$ (ii) the zero solution of the system

$$\dot{x}(t) = \sum_{i=0}^{p} A_i(t)\, x(t - h_i) \tag{5.10}$$

is uniformly asymptotically stable, so that the solution of (5.10) *satisfies*

$$\|x(t)\| \le k \, \|\phi\| \, e^{-\alpha(t-t_0)} \quad \alpha > 0,\ k > 0 \text{ are constants}$$

Then system (4.1) is relatively null controllable with constraint

***Theorem* 5.8**

The system (4.1) *with the control* $u_{t_0}(t)$ *on [$t_0 - h$, t_0] is null controllable at $t = t_1$ if and only if.*

$$y(z(t_0)) = -\phi(t_0) - \sum_{i=0}^{k} \int_{t_0 - h_1}^{t_0} X(t_{1,s+hi}) B_i(s + h_i) u_{t_0}(s) ds$$

belongs to the range space of the null controllability grammian.

The null controllability grammian is given by

$$\Gamma(t_0, t_1) = \sum_{i=1}^{k} \int_{t-h_{i+1}}^{t-h_i} \left[\sum_{p=0}^{i} X(t_0, s + h_p) B_p(s + h_p) \right] \left[\sum_{p}^{i} X(t_0, s + h_p) B_p(s + h_p) \right]^T ds.$$

***Theorem* 5.9**

Assume (*i*) *system* (4.1) is relatively controllable on $[t_0, t_1]$ *for each* $t_1 > t_0$ (*ii*) *the zero solution of system* (5.10) *if uniformly asymptotically stable, then system* (4.1) *if relatively null controllable with constraint.*

6.0 Computable Criteria for Relative Null Controllability of Discrete Systems.

We can now state an algebraic condition for the relative null controllability of system (4.1). Here we exploit the results previously obtained on the relative controllability of the system (4.1) by Manitius and Olbrot [7], together with the fact that if the zero solution of system (5.10) is uniformly asymptotically stable that is det.$\Delta(\lambda) = 0$ has roots with negative real parts, where

$$\Lambda(\lambda) = \lambda_1 - \sum_{i=0}^{n} A_i e^{-h_i \lambda} \tag{6.1}$$

since null controllability could be proved by assuming controllability of the system and the asymptotic stability of the free system, we have the following

***Theorem* 6.1**

System (4.1) *is relatively null controllable with constraint on* $[t_0, t_1]$, $t_1 > t_0$ *if*

(*i*) *rank* $\hat{Q}_n(t_1) = n$

where

$\hat{Q}_n(t_1) = \{\hat{Q}(J, t_1), k = 0, 1, \ldots, n-1 : t_1 - \langle JH \rangle \geq t_0\}$

(ii) The roots of $\det\Delta(\lambda) = 0$ of (5.10) have negative real parts.

We now give a concrete touch to the sequel by considering the particular system which has A, B, C, B as constant matrices of appropriate dimensions.

Consider the system

$$\dot{x}(t) = Ax(t) + Bx(t-h) + Cu(t) + Du(t-h) \tag{6.3}$$

This system shares much in common with the system studied by Gahi [4]

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-w(t)) + C(t)u(t) + D(t)u(t-h(t)) + g(t) \tag{6.4}$$

for $t \in J$ and $x(t) = q(t)$ for $t \in I$-w(0), 0], Gahl [3] obtained for system (6.4) that if $C(t) = C$ a constant matrix, $h(t_1) > 0$ and $w(t_1) > 0$ then the system will be completely controllable on J if rank

$$[C, A(t_1)C] = n \tag{6.5}$$

The application of this result to system (6.3) is immediate. For the system to be relatively controllable on the interval $[t_0, t_1]$; $t_1 > t_0$, rank$[C, AC] = n$ we now obtain the form of the unique absolutely continuous solution of the system (6.3) satisfying the condition $x_{t_0} = \phi$

This is given by

$$\begin{aligned} x(t) = X(t_1,t_0)\phi &+ \int_{t_0-h}^{t_0} X(t, s+h)B(s+h)\phi(s)\,ds + \int_{t_0-h}^{t_0} \big(X(t,s)C(s) \\ &+ X(t, s+h)D(t+s)\big)u_{t_0}(s)ds + \int_{t_0}^{t-h} \big(X(t,s)C(s) \\ &+ X(t,s+h)D(s+h)\big)u(s)ds + \int_{t-h}^{t} X(t,s)C(s)u(s)ds \end{aligned} \tag{6.6}$$

The controllability grammian is given by

$$W(t_0,t_1) = \int_{t_0}^{t-h} \big[x(t_1,s)C(s) + X(t_1, s+h)D(s+h)\big]\big[x(t_1,s)C(s) + X(t_1, s+h)D(s+h)\big]^T,$$

$$+\int_{t_h}^{t} X\ (t_1, s)c(s)c(s)^T X(t_1, s)^T ds \qquad (6.7)$$

and the null controllability grammian is given by

$$\Gamma(t_0, t_1)\int_{t_0}^{t_1-h} [x(t,s)\ C(s) + X(t_1, s + h)D(s + h)][x(t, s)\ C(s) + X(t_1,s + h)D(s + h)]^T ds. \qquad (6.8)$$

Theorems 5.7, 5.8, 5.9 are applicable to system (6.3).

6.1 Examples

In order to shed more light on our articulations, let us consider the following examples

***Example* 1**

$$\dot{x}(t) = Ax(t) + Bx(t-h) + Cu(t) + du(t-h) \qquad (6.9)$$

where

$$A = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We first obtain

$$AC = \begin{pmatrix} 2 & 2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix}$$

$$rank[C, AC] = rank\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} = 2 = n$$

This establishes the relative controllability of the system.

We now show that the zero solution of the free part of the system is uniformly asymptotically stable. Let us first obtain

$$\Delta(\lambda) = \left[\lambda_1 - \left(A + Be^{-h\lambda}\right)\right] = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \left[\begin{pmatrix} 2 & 2 \\ 2 & -5 \end{pmatrix} + \begin{pmatrix} 0 & e^{-h\lambda} \\ 0 & e^{h\lambda} \end{pmatrix}\right]$$

$$= \begin{bmatrix} \lambda - 2 & -\left(2 + e^{-h\lambda}\right) \\ 2 & \lambda + 5 + e^{-h\lambda} \end{bmatrix} \qquad (6.10)$$

$$\det \Delta(\lambda) = \lambda^2 + 3\lambda + \lambda e^{-h\lambda} - 6 = 0$$

Comparing this result with the equation $\lambda^2 + b\lambda + q\lambda e^{-h\lambda} + k = 0$ which according to Driver, [2] in pg. 32 example 3 will have negative real parts for its roots if $b > q$, $b > 0$, $q > 0$.

In the light of this, we conclude that the roots of the characteristic equation for the system (6.9 have negative real parts since

$$b = 3, \quad q = 1, 3 > 1 \qquad 3 > 0, \quad 1 > 0.$$

Hence the zero solution of the free part of system (6.9) is uniformly asymptotically stable. Hence we conclude that the system (6.9) is relatively null controllable with constraint.

***Example* 2**

Consider the system

$$\dot{x}(t) = A_0x(t) + A_1x(t-1) + A_2x(t-2) + B_0u(t) + B_1u(t-2) + B_2u(t-2) \qquad (6.11)$$

where

$$A_0 = \begin{pmatrix} 2 & 3 \\ -2 & -5 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 1 & 0 \\ -2 & 0 \end{pmatrix} \qquad B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

This is the discrete autonomous system (4.13) with lag in both the state and control whose controllability has been settled previously we now show that the free part of the system is uniformly asymptotically sable.

The free part of the system is obtained when u = 0 i.e.

$$x(t) = A_0x(t) + A_1x(t-1) + A_2x(t-2)$$

The solution of the system is uniformly asymptotically stable if $x(t) \to 0$ as $t \to \infty$.
This can happen if the roots of the characteristic equation of the free part have negative real parts.

$$\text{Det}[\lambda] - (A_0 + A_1e^{-\lambda} + A_2e^{-2\lambda}] = 0$$

$$Det\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} - \left\{\begin{pmatrix} 2 & 2 \\ -2 & -5 \end{pmatrix} + \begin{pmatrix} 0 & e^{-\lambda} \\ 0 & e^{-\lambda} \end{pmatrix} + 0\right\}\right] = 0$$

$$Det\begin{bmatrix} \lambda^{-2} & -(2+e-\lambda) \\ 2 & \lambda + 5 + e^{-\lambda} \end{bmatrix} = 0$$

$$\lambda^2 + 5\lambda + \lambda e^{-\lambda} - 2\lambda - 10 - 2e^{-\lambda} + 4 + 2e^{-\lambda} = 0$$

$$\lambda^2 + 3\lambda + \lambda e^{-\lambda} - 6 = 0$$

Comparing this result, with the equation $\lambda^2 + b\lambda + q\lambda e^{-\lambda h} + k = 0$ whose roots will have negative real parts if $b > q$, $b > 0$, $q > 0$. (See Driver [2] pp 32 example 3)

We conclude that the roots of the characteristic equation of the system 7.3 has negative real part – b = 1 q = 1 3 > 1, 3 > 0, q > 0. Hence the zero solution of the free part is therefore uniformly asymptotically stable.

Taking together, the relative controllability of the system and the uniform asymptotic stability of its free part, we conclude in the light of theorem (5.9) that system (6.10) is relatively null controllable with constraint.

References

[1] Chukwu E. N. Euclidean Controllability of Linear Delay Systems with Limited controls. IEEE Transaction on Automatic Control 24 No 5 (1979) P. 798 – 800.

[2] Driver R. D. Ordinary and Delay Differential Equations Springer Verlag New York 1977.

[3] Gahl R. D. Local Controllability of Non-linear Systems. Entrata in Redazione il 12 Luglia 1978.

[4] Hale J. K. Theory of Functional Differential Equations. Springer Verlag New York 1977

[5] Klamka Jerzy. Relative Controllability and Minimum Energy Control of Linear Controls with Distributed Delays in Control IEEE Transactions on Automatic Control 1976.

[6] Manitius A. Optimal Control of Hereditary Systems IAEA – SM 17/95 1975.

[7] Manitius A. and Olbrot W. Controllability Conditions for Linear Systems with Delay Delay State and Control arch auto. Tel. 1972.

[8] Onwuatu J. U On Controllability and Null Controllability of Linear Systems with Distributed Delays in the Control Int. J. Diff and Integral Equations Theory and Applications 1989.

[9] Balachandran K. Relative Controllability of Non-Linear Systems with Distributed Delays in Control. Journal a Vol 28 No. 11987.

[10] Balachandran K. Global Relative Controllability of Non-Linear Systems with Time Varying Multiple Delays in Control Int. J. control Vol. 46 No. 1 (1987) P. 193 – 200.

[11] Sebakhy O. and Bayoumi M. M. Controllability of Linear Time Varying Systems with Delay in Control, Int. Journal of Control. Vol. 17 No. 1 (1973) P. 127- 135.

[12] Onwuatu J. U. Optimal Control for Discrete System. J. N. M.S. Vol 192000