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Initial post dynamic buckling of a quadratic-cubic column pressurized by a sinusoidally slowly varying dynamic load

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Abstract

In this investigation, we determine the dynamic buckling load of an imperfect finite column resting on a mixed quadratic-cubic nonlinear elastic foundation trapped by an explicitly time dependent sinusoidally slowly varying dynamic load .The resultant coefficients are dynamically slowly varying and the formulation contains two small but Mathematically unrelated parameters upon which regular perturbation procedure are used in asymptotic expansions of the variables. The imperfection is assumed in the shape of the mth term in the Fourier sine series expansion with small (in absolute value) Fourier coefficients. A generalization of Lindsted-Poincare procedure is used and the structure under investigation is, on the main, a nonlinear oscillatory system with small perturbations. The results obtained are strictly asymptotic and are valid for small absolute values of the perturbation parameters.

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1.0 Introduction

This analysis seeks to find an entirely analytical approach in determining the dynamic buckling load of an imperfect finite column resting on a mixed quadratic –cubic non-linear elastic foundation modulated by a loading history that is harmonically and dynamically slowly varying .We remark that the first investigation into slowly varying parameters in a dynamical system was enunciated by Kuzmak [1] .Later, Luke [2] extended Kuzmak's analysis to higher levels of approximation. Notable later contributions on slowly varying parameters in dynamical systems include Cole [3, chapter 3], Kevorkian [3, 5], Kevorkian and Cole [6], Li [7] and Li and Kevorkian [8] among others .Perhaps the main substance of this inquiry is the application of the idea of slowly varying parameters in a dynamical systems to a purely non-autonomous dynamic buckling problem where the time parameter is explicit.

We recall that an earlier investigation into the static buckling of the same structure was undertaken by Elishakoff [9] who used a slightly different nondimensionalization from that used here. The corresponding step loading investigation was done by Ette [10]. Other similar inquiries that have specifically addressed dynamic buckling problems include Zhu et al [11], Heinen and Bullesbach [12] and Schenk and Schueller [13] among others.

2.0 Differential Equations

The differential equation [9,10] satisfied by the displacement $W(X,T)$ of an imperfect but finite column resting on a mixed quadratic–cubic nonlinear elastic foundation trapped by an arbitrary loading history $P(T)$ (where T is the time variable) is

$$m_0 W,_{TT} + EIW,_{XXXX} + P(T)W,_{XX} + k_1 W - \alpha_2 k_2 W^2 - \alpha_3 k_3 W^3 = -P(T)\frac{d^2\overline{W}}{dX^2}, \mathrm{T} > 0 \quad (2.1)$$

where m_0 is the mass per unit length of the column and *EI* is the bending stiffness (where *E* is the Young's modulus and *I* is the moment of inertia) and $\overline{W}$ is the twice differentiable imperfection function . X is the spatial variable and $k_1>0$, $k_3>0$ where $-\alpha_2 k_2 W^2 - \alpha_3 k_3 W^3$ is the quadratic– cubic nonlinear response of the elastic foundation. We have neglected axial inertia as well as all non-linearties higher than the cubic .We shall assume homogeneous initial conditions and now introduce the following non-dimensional quantities:

$$x = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X \ , \ w = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} W \ , \ \lambda f(\delta \bar{t}) = \frac{P(T)}{2(EIk_1)^{\frac{1}{2}}} \qquad (2.2a)$$

$$\bar{t} = \left(\frac{k_1}{m_0}\right)^{\frac{1}{2}} T \ , \ \in \overline{w} = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} \overline{W} \ , \ \alpha = \frac{\alpha_2 k_2}{\sqrt{k_1 k_3}}$$

The non-dimensional form of (2.1) becomes

$$w_{,\bar{t}\bar{t}} + w_{,xxxx} + 2\lambda f(\delta \bar{t}) w_{,xx} + w - \alpha w^2 - \alpha_3 w^3 = 2\lambda \in f(\delta \bar{t}) \frac{d^2 \overline{w}}{dx^2} \ , \bar{t} > 0, \qquad (2.2b)$$

$$0 < x < \pi \qquad (2.2c)$$

$$w(x,0) = w_{,\bar{t}}(x,0) = 0 \ , 0 < x < \pi \qquad (2.2d)$$

$$w = w_{,xx} = 0 \ \text{at} \ x = 0, \pi \ \ \text{for} \ \bar{t} \geq 0 \qquad (2.2e)$$

Here α and α_3 are imperfection –sensitivity parameters and for our case, we demand that $\alpha > 0$, $\alpha_3 > 0$. Partial differentiation is indicated by using a subscript following a comma and we assume $0 < \delta < 1$, $0 < \in << 1$ where δ and $\in$ are small but not Mathematically related parameters .However, the analysis is generally true for $|\delta| < 1$ and $|\in| << 1$. λ is a non-dimensional load parameter satisfying the inequality $0 < \lambda < 1$ and our aim is to determine a particular value of λ, say λ_D , for which the structure buckles dynamically . The dynamic buckling load λ_D is here defined as the largest load parameter for which the problem has a bounded solution. The procedure shall involve the following: We shall first determine a uniformly valid asymptotic expression of the displacement w(x, $\bar{t}$) in the small parameters δ and $\in$. We shall next determine the maximum displacement as a function of space and time variables and finally determine the dynamic buckling load using a suitable maximization process .For our case we shall take

$$f(\delta \bar{t}) = \cos \delta \bar{t} \qquad (2.3a)$$

and note that $|f(\delta \bar{t})| \leq 1 \ for \ \bar{t} \geq 0$ .Based on the boundary conditions (2.2e) we shall take

$$\overline{w} = \bar{a}_m \sin mx \ , \ |\bar{a}_m| << 1 \ , m = 1,2,3,\cdots \qquad (2.3b)$$

We shall now let

$$\tau = \delta \bar{t} \ ; \ \omega = \left(m^4 - 2m^2 \lambda \cos \delta \bar{t} + 1\right) = \left(m^4 - 2m^2 \lambda \cos \tau + 1\right); \frac{d\tilde{t}}{d\bar{t}} = \omega^{\frac{1}{2}} \qquad (2.4a)$$

for m as in (2.3b). We similarly let

$$t = \tilde{t} + \frac{(\in \mu_1 + \in^2 \mu_2 + \in^3 \mu_3 + \cdots)}{\delta}, \ \mu_i = \mu_i(\tau); \ \mu_i(0) = 0 \ \text{ for } \ i = 1,2,3,\cdots \quad (2.4b)$$

We shall now let $w(x,\bar{t}) = U(x,t,\tau,\in,\delta)$ and now obtain

$$w,_{\bar{t}} = \omega^{\frac{1}{2}} U,_t + \left(\in \mu_1' + \in^2 \mu_2' + \in^3 \mu_3' + \cdots\right) U,_t + \delta U,_\tau \quad (2.5)$$

$$w,_{\bar{t}\bar{t}} = \omega U,_{tt} + \left(\in \mu_1' + \in^2 \mu_2' + \in^3 \mu_3' + \cdots\right)^2 U,_{tt} + \delta^2 U,_{\tau\tau}$$

$$+ 2\omega^{\frac{1}{2}}\left(\in \mu_1' + \in^2 \mu_2' + \in^3 \mu_3' + \cdots\right) U_{tt} + 2\delta\left(\in \mu_1' + \in^2 \mu_2' + \in^3 \mu_3' + \cdots\right) U_{t\tau} \quad (2.6)$$

$$+ 2\delta\omega^{\frac{1}{2}} U_{t\tau} + \delta\left(\in \mu_1'' + \in^2 \mu_2'' + \in^3 \mu_3'' + \cdots\right) U_t + \frac{\delta \omega^{-\frac{1}{2}} \omega' U,_t}{2}$$

where $\frac{d(\ )}{d\tau} \equiv (\ )'$.

We shall now let $$U(x,t,\tau,\in,\delta) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} U^{ij}(x,t,\tau) \in^i \delta^j \quad (2.7)$$

where the ij on U^{ij} are superscripts and not powers .We now substitute (2.5)-(2.7) into (2.2b-e) and simplify to get the following sequence of equations

$$M\ U^{10} \equiv U,_{tt}^{10} + \frac{1}{\omega}\left\{U,_{xxxx}^{10} + 2\lambda U,_{xx}^{10} \cos\tau + U^{10}\right\} = \frac{2\lambda m^2 \bar{a}_m \cos\tau \sin mx}{\omega} \quad (2.8a)$$

$$M\ U^{11} = -2\omega^{-\frac{1}{2}} U,_{t\tau}^{10} - \frac{\omega^{-\frac{3}{2}} \omega' U,_t^{10}}{2} \quad (2.8b)$$

$$M\ U^{20} = \frac{\alpha\left(U^{10}\right)^2}{\omega} - 2\omega^{-\frac{1}{2}} \mu_1' U,_{tt}^{10} \quad (2.9a)$$

$$M\ U^{21} = \frac{2\alpha U^{10} U^{11}}{\omega} - 2\omega^{-\frac{1}{2}} U,_{t\tau}^{20} - \frac{\omega^{-\frac{3}{2}} \omega' U,_t^{20}}{2} - 2\omega^{-\frac{1}{2}} \mu_1' U,_{tt}^{11}$$
$$- \omega^{-1}\left(2\mu_1' U,_{t\tau}^{10} + \mu_1'' U,_t^{10}\right) \quad (2.9b)$$

$$M\ U^{30} = \frac{2\alpha U^{20} U^{10}}{\omega} + \frac{\alpha_3 \left(U^{10}\right)^3}{\omega} - 2\omega^{-\frac{1}{2}}\left(\mu_2' U,_{tt}^{10} + \mu_1' U,_{tt}^{20}\right) \quad (2.10a)$$

$$M\,U^{31} = \frac{2\alpha}{\omega}\left(U^{20}U^{11} + U^{21}U^{10}\right) + \frac{3\alpha_3 U^{11}\left(U^{10}\right)^2}{\omega} - 2\,\omega^{-\frac{1}{2}}\left(\mu_2' U_{,tt}^{11} + \mu_1' U_{,tt}^{21}\right)$$

$$-2\omega^{-1}\left(\mu_1' U_{,t\tau}^{20} + \mu_2' U_{,t\tau}^{10}\right) - 2\,\omega^{-\frac{1}{2}} U_{,t\tau}^{30} \qquad (2.10b)`$$

$$-\omega^{-1}\left(\mu_1'' U_{,t}^{20} + \mu_2'' U_{,t}^{10}\right) - \frac{\omega^{-\frac{3}{2}}\omega' U_{,t}^{30}}{2}$$

The initial conditions, evaluated at $t = \tau = 0$ are

$$U^{ij} = 0\ ,\ U_{,t}^{10}(0,0) = 0 \qquad (2.11a)$$

$$U_{,t}^{1k} + \omega^{-\frac{1}{2}}(0)\,U_{,\tau}^{1p} = 0\ ,\ k = p+1,\ p = 0,1,2,3,\cdots \qquad (2.11b)$$

$$U_{,t}^{20} + \mu_1'(0)\,\omega^{-\frac{1}{2}}(0)\,U_{,t}^{10}(0,0) = 0 \qquad (2.12a)$$

$$U_{,t}^{2k} + \mu_1'(0)\,\omega^{-\frac{1}{2}}(0)\,U_{,t}^{1k} + \omega^{-\frac{1}{2}}(0)\,U_{,\tau}^{2p} = 0, \qquad (2.12b)$$

$$k = p+1,\ p = 0,1,2,3,$$

$$U_{,t}^{30} + \omega^{-\frac{1}{2}}(0)\left[\mu_1'(0)U_{,t}^{20} + \mu_2'(0)\,U_{,t}^{10}\right] = 0 \qquad (2.13a)$$

$$U_{,t}^{3k} + \omega^{-\frac{1}{2}}(0)\left[\mu_1'(0)U_{,t}^{2k} + \mu_2'(0)\,U_{,t}^{1k} + U_{,\tau}^{3p}\right] = 0, \qquad (2.13b)$$

$$k = p+1,\ p = 0,1,2,3,\cdots$$

To solve (2.8a) we let $U^{10}(x,t,\tau) = \sum_{n=1}^{\infty} U_n^{10}(t,\tau)\sin nx$ (2.14)

We substitute (2.14) into (2.8a) and simplify to get, for $n = m$

$$U_{m,tt}^{10} + U_m^{10} = B(\tau);\quad B(\tau) = \frac{2\lambda m^2 \bar{a}_m \cos\tau}{\omega(\tau)} \qquad (2.15a,b)$$

$$U_m^{10}(0,0) = U_{m,t}^{10}(0,0) = 0$$

The solution of (2.15a,b) is

$$U_m^{10}(t,\tau) = a_{10}(\tau)\cos t + b_{10}(\tau)\sin t + B(\tau) \qquad (2.16a,b)$$

$$a_{10}(0) = -B(0) \equiv -B_0;\ B_0 = \frac{2\lambda m^2 \bar{a}_m}{m^4 - 2m^2\lambda + 1};\ b_{10}(0) = 0;\ B'(0) = 0$$

We next substitute for terms into (2.8b) and to ensure a uniformly valid solution, set to zero the coefficients of cost and *sint* and get respectively

$$b_{10}' + \frac{\omega' b_{10}}{4\omega} = 0 \quad \text{and} \quad a_{10}' + \frac{\omega' a_{10}}{4\omega} = 0 \qquad (2.17)$$

The solutions of (2.17) subject to (2,16b) are

$$b_{10}(\tau)\equiv 0\ ,\ \mathrm{a}_{10}(\tau)=\mathrm{a}_{10}(0)\left(\frac{\omega(0)}{\omega(\tau)}\right)^{\frac{1}{4}} \tag{2.18a}$$

Thus we have $$U_m^{10}=a_{10}\cos t+B(\tau)\ ,\ \mathrm{U}^{10}=U_m^{10}\sin mx \tag{2.18b}$$

The remaining equation in the substitution into (2.8b) is

$$MU^{11}=0$$
$$U^{11}(0,0)=0\ ;\ \mathrm{U}_{,\mathrm{t}}^{11}(0,0)+\omega^{-\frac{1}{2}}(0)\,\mathrm{U}_{,\tau}^{10}(0,0)=0 \tag{2.19a,b}$$

We let

$$U^{11}(x,t)=\sum_{n=1}^{\infty}U_n^{11}\sin nx \tag{2.19c}$$

and substitute same into (2.19a,b) and solve and get

$$U_m^{11}(t,\tau)=a_{11}(\tau)\cos t+b_{11}(\tau)\sin t$$
$$a_{11}(0)=0\ ,\ \mathrm{b}_{11}(0)=-B_0\omega^{-\frac{1}{2}}(0). \tag{2.20a,b}$$

We note in passing that

$$a_{10}'(0)=0\ ,\omega'(0)=0\ ,\ \omega''(0)=2m^2\lambda \tag{2.20c}$$

Analysis beyond the level of accuracy retained here shows that b_{11} and a_{11} satisfy equations similar to b_{10} and a_{10} in (2.17), namely

$$b_{11}'+\frac{\omega' b_{11}}{4\omega}=0 \quad \text{and} \quad \mathrm{a}_{11}'+\frac{\omega' a_{11}}{4\omega}=0 \tag{2.21a}$$

The solutions of (2.21a) subject to (2.20b) are

$$a_{11}(\tau)\equiv 0\ ;\ \ \mathrm{b}_{11}(\tau)=b_{11}(0)\left(\frac{\omega(0)}{\omega(\tau)}\right)^{\frac{1}{4}};b_{11}'(0)=\text{-}\,\mathrm{b}_{11}(0)=\mathrm{B}_0\omega^{-\frac{1}{2}}(0) \tag{2.21b}$$

Thus, we have $U_m^{11}(t,\tau)=\mathrm{b}_{11}(\tau)\sin mx;\ \ \mathrm{U}^{11}(x,t,\tau)=U_m^{11}(t,\tau)\sin \mathrm{mx}$ (2.21c)

We shall let

$$U^{20}=\sum_{n=1}^{\infty}U_n^{20}(t,\tau)\sin mx \tag{2.22a}$$

and now substitute for terms into (2.9a). To ensure a uniformly valid solution we equate to zero the coefficient of cost and get

$$\mu_1'(\tau)=-\frac{8\alpha}{3\pi\,\mathrm{m}\omega^{\frac{1}{2}}}\ ,\ \mu_1'(0)=-\frac{8\alpha}{3\pi\,\mathrm{m}\omega^{\frac{1}{2}}(0)}\ ;\mu_1''(0)=0\ ,\mathrm{m\ odd} \tag{2.22b}$$

The remaining equation and initial conditions in the substitution into (2.9a) are

$$U^{20}_{m,tt} + U^{20}_{m} = \frac{8\alpha\left(A_0 + \frac{a_{10}^2 \cos 2t}{2}\right)}{3\pi m \omega}, \text{ m odd}$$

$$U^{20}_{m}(0,0) = 0,\ \mathrm{U}^{20}_{\mathrm{m,t}}(0,0) + \mu_1'(0)\omega^{-\frac{1}{2}}(0)U^{10}_{m,t}(0,0) = 0 \quad \text{(2.23a,b,c)}$$

$$A_0 = B^2 + \frac{a_{10}^2}{2}, A_0(0) = \frac{3B_0^2}{2},\ \mathrm{A}_0'(0) = 0$$

The solution of (2.23a-c) is

$$U^{20}_{m} = a_{20}\cos t + b_{20}\sin t + \frac{8\alpha}{3\pi \mathrm{m}\omega}\left(A_0 - \frac{a_{10}^2 \cos 2t}{6}\right) \quad \text{(2.24a)}$$

$$a_{20}(0) = -\frac{32\alpha \mathrm{B}_0^2}{3\pi \mathrm{m}\omega(0)}, b_{20}(0) = 0 \quad \text{(2.24b)}$$

We next substitute into the right hand side of (2.9b) and assume

$$U^{21} = \sum_{\mathrm{n}=1}^{\infty} U^{21}(t,\tau)\sin \mathrm{nx} \quad \textbf{(2.25a)}$$

We now equate to zero the coefficients of cost and *sint* in the substitution into (2.9b) and get

$$b_{20}' + \frac{\omega' \mathrm{b}_{20}}{4\omega} = 0\ ;\ \mathrm{a}_{20}' + \frac{\omega' \mathrm{a}_{20}}{4\omega} = -\mathrm{h}_1(\tau) \quad \text{(2.25b)}$$

$$\mathrm{h}_1(\tau) = \left[\mu_1' b_{11} + \frac{\omega^{-\frac{1}{2}}}{2}\left\{2\mu'\mathrm{a}_{10}' + \mu_1''\mathrm{a}_{10} + \frac{16\alpha B b_{11}}{3\pi m \omega}\right\}\right]; \quad \text{(2.25c)}$$

$$\mathrm{h}_1(0) = \frac{8\alpha B_0(1 - B_0)}{3\pi m \omega(0)};$$

$$\mathrm{a}_{20}'(0) = -\mathrm{h}_1(0) = -\frac{8\alpha B_0(1 - B_0)}{3\pi m \omega(0)} \quad \text{(2.25d)}$$

The solutions of (2.25b-d) are

$$b_{20}(\tau) = 0, \mathrm{a}_{20}(\tau) = \omega^{-\frac{1}{4}}(\tau)\left[\mathrm{a}_{20}(0)\omega^{\frac{1}{4}}(0) - \int_0^{\tau} h_1(s)\omega^{\frac{1}{4}}(s)\,\mathrm{ds}\right] \quad \text{(2.25e)}$$

Thus we have

$$U^{20} = \sum_{m=1,3,5,\cdots}^{\infty} U^{20}_{m} \sin mx \quad \text{(2.25f)}$$

The remaining equations in the substitution into (2.9) are

$$U^{21}_{m,tt} + U^{21}_{m} = \mathrm{A}_1(\tau)\sin 2t \quad \text{(2.26a)}$$

$$U_m^{21}(0,0)=0\,;\; U_{m,t}^{21}(0,0)+\mu_1'(0)\,\omega^{-\frac{1}{2}}(0)\,\mathrm{U}_{\mathrm{m},\,t}^{10}(0,0)$$

$$+\,\omega^{-\frac{1}{2}}(0)\,\mathrm{U}_{\mathrm{m},,\tau}^{10}(0,0)=0 \qquad (2.26b)$$

$$A_1(\tau)=\frac{16\alpha\, \mathrm{a}_{10}\, \mathrm{b}_{11}}{3\pi\,\mathrm{m}\,\omega}-\frac{16\,\alpha\,\omega^{-\frac{1}{2}}}{9\,\pi\,\mathrm{m}}\left(\frac{a_{10}^2}{\omega}\right)'-\frac{4\,\alpha\omega'\,\mathrm{a}_{10}^2}{9\pi\,\mathrm{m}\omega^{\frac{5}{2}}} \qquad (2.26c)$$

$$A_1(0)=\frac{16\,\alpha\,\mathrm{B}_0^2}{3\,\pi\,\mathrm{m}\,\omega^{\frac{3}{2}}(0)}\,,\ \mathrm{A}_1'(0)=-B_0^2\alpha\, Q_1\ ,\ \ \mathrm{Q}_1=\frac{16}{3\,\pi\,\mathrm{m}\,\omega^{\frac{1}{2}}(0)}+\frac{4\,\omega''(0)}{9\,\pi\,\mathrm{m}\,\omega^{\frac{5}{2}}(0)} \qquad (2.26d)$$

The solution of (2.26a-d) is

$$U_m^{21}=a_{21}\cos t+b_{21}\sin t-\frac{A_1\sin 2t}{3}$$

$$a_{21}(0)=0\ \ \mathrm{b}_{21}(0)=\frac{\alpha\ \left(8B_0-24\mathrm{B}_0^2\right)}{3\,\pi\,\mathrm{m}\,\omega^{\frac{3}{2}}(0)} \qquad (2.26e\text{-}f)$$

We therefore have $$U^{21}(x,t,\tau)=\sum_{m=1,3,5,\cdots}^{\infty}U_m^{21}\sin mx \qquad (2.27)$$

Henceforth summation over m , unless otherwise stated ,shall be deemed to be over all positive odd integers. This thus avoids the expressed indication of the range of summation in subsequent analysis. We next substitute into the right hand side of (2.10a) and assume

$$U^{30}=\sum_{\mathrm{n}=1}^{\infty}U_n^{30}(t,\tau)\sin nx \qquad (2.28)$$

We easily see that when $n = m$, we get

$$U^{30}_{m,tt} + U^{30}_{m} = \left(\frac{3\alpha_3 A_2}{4\omega} + \sum \frac{A_6}{m}\right)$$

$$+\left\{\frac{3\alpha_3 A_3}{4\omega} + 2\omega^{-\frac{1}{2}}\left(\mu_2' \, a_{10} + \mu_1' \sum a_{20}\right) + \sum \frac{A_7}{m}\right\}\cos t$$

$$+\left\{\frac{3\alpha_3 A_4}{4\omega} + \sum\left(\frac{A_8}{m} - \frac{32\,\alpha\,\mu_1'\,a_{10}^2}{9\,\pi\,m\,\omega^{\frac{3}{2}}}\right)\right\}\cos 2t \qquad (2.29a)$$

$$+\left(\frac{3\alpha_3 A_4}{4\omega} + \sum \frac{A_9}{m}\right)\cos 3t$$

$$U^{30}_{m}(0,0) = 0\,,\ U^{30}_{m.,t} + \omega^{-\frac{1}{2}}(0)\left[\mu_2'(0)\,U^{10}_{m,t}(0,0) + \mu_1'(0)\sum U^{20}_{m,t}(,0,0)\right] = 0 \qquad (2.29b)$$

Similarly when $n = 3m$, we get

$$U^{30}_{3m,tt} + \phi_{3m}(\tau) U^{30}_{3m} = -\frac{\alpha_3}{4\omega}\left[A_2 + A_3\cos t + A_4\cos 2t + A_5\cos 3t\right]$$

$$\phi_{3m}(\tau) = \frac{81m^4 - 18m^2\lambda\cos\tau + 1}{m^4 - 2m^2\lambda\cos\tau + 1}\,;\ U^{30}_{3m}(0,0) = U^{30}_{3m,t}(0,0) = 0 \qquad (2.30a,b)$$

where

$$A_2 = B^3 + 3Ba_{10}^2\,,\ A_2(0) = 4B_0^3,\ A_2'(0) = 0\,;\ A_3 = \frac{3}{4}a_{10}^3 + 3B^2 a_{10}$$

$$A_3(0) = -\frac{15B_0^3}{4},\ A_3'(0) = 0 \qquad (2.31a,b)$$

$$A_4 = \frac{3Ba_{10}^2}{2}\,,\ A_4(0) = \frac{3B_0^3}{2},\ A_4'(0) = 0\ ;\ \ A_5 = \frac{a_{10}^3}{4},\ A_5(0) = -\frac{B_0^3}{4},\ A_5'(0) = 0 \qquad (2.31c)$$

$$A_6 = \frac{a_{10}a_{20}}{2} + \frac{8\,\alpha\,BA_0}{3\,\pi\,m\,\omega},\ A_6(0) = \frac{52\alpha B_0^3}{9\,\pi\,m\,\omega(0)},\ A_6'(0) = \frac{4\,\alpha\,B_0^2(1-B_0)}{3\,\pi\,m\,\omega(0)} \qquad (2.31d)$$

$$A_7 = Ba_{20} + \frac{8\,\alpha\,a_{10}A_0}{3\,\pi\,m\,\omega},\ A_7(0) = -\frac{52\,\alpha\,B_0^3}{9\,\pi\,m\,\omega(0)} = -A_6(0),\ A_7'(0) = \frac{4\,\alpha\,B_0^2(1-B_0)}{3\,\pi\,m\,\omega(0)} \qquad (2.31e)$$

$$A_8 = \frac{a_{10}a_{20}}{2} - \frac{4\,\alpha\,B\,a_{10}^2}{9\,\pi\,m\,\omega},\ A_8(0) = \frac{4\,\alpha\,B_0^3}{3\,\pi\,m\,\omega(0)},\ \ A_8'(0) = \frac{8\,\alpha\,B_0^2(1-B_0)}{3\,\pi\,m\,\omega(0)} \qquad (2.31f)$$

$$A_9 = -\frac{4\,\alpha\,a_{10}^3}{9\,\pi\,m\,\omega},\ \ A_9(0) = \frac{4\,\alpha\,B_0^3}{9\,\pi\,m\,\omega(0)}\ ,\qquad A_9'(0) = 0 \qquad (2.31g)$$

To ensure a uniformly valid solution, we equate to zero in (2.29a) the coefficients of cost and get

$$\mu_2' = -\frac{\omega^{\frac{1}{2}}}{2\,a_{10}}\left\{\frac{3\,\alpha_3 A_3}{4\,\omega} + \sum \frac{A_7}{m}\right\} - \frac{\mu_1' \sum a_{20}}{a_{10}} \quad (2.32a)$$

$$\mu_2'(0) = \frac{45\,\alpha_3 B_0^2}{16\,\omega} + \frac{\omega^{\frac{1}{2}}(0)}{2}\sum \frac{52\,\alpha\, B_0^2}{9\pi\, m^2 \omega(0)} - \mu_1'(0)\sum \frac{32\,\alpha\, B_0}{9\pi\, m\omega(0)} \quad (2.32b)$$

$$\mu_2''(0) = \alpha\left\{\sum \frac{2\,B_0(1 - B_0)}{3\pi\, m^2 \omega^{\frac{1}{2}}(0)} - \mu_1'(0)\sum \frac{8(1 - B_0)}{3\,\pi\, m\omega(0)}\right\} \quad (2.32c)$$

The solution of the remaining equation In (2.29a,b) is

$$U_m^{30} = a_{30}\cos t + b_{30}\sin t + A_{10} - \frac{A_{11}\cos 2t}{3} - \frac{A_{12}\cos 3t}{8} \quad (2.33a)$$

$$A_{10} = \frac{3\,\alpha_3 A_2}{4\,\omega} + \sum \frac{A_6}{m}\ ;\ A_{10}(0) = \frac{3\,\alpha_3 B_0^3}{\omega} + \sum \frac{52\,\alpha\, B_0^3}{9\,\pi\, m^2 \omega(0)}; \quad (2.33b)$$

$$A_{10}'(0) = \sum \frac{4\alpha B_0^2}{3\,\pi\, m^2 \omega(0)} \quad (2.33c)$$

$$A_{11} = \left\{\frac{3\,\alpha_3 A_4}{4\omega} + \sum\left(\frac{A_8}{m} - \frac{32\,\alpha\, \mu_1'\, a_{10}^2}{9\,\pi\, m\omega^{\frac{3}{2}}}\right)\right\} \quad (2.34a)$$

$$A_{11}(0) = \frac{9B_0^3\alpha_3}{8\,\omega(0)} + \alpha\sum\left(\frac{4B_0^3}{3\,\pi\, m^2\omega(0)} - \frac{32\mu_1'(0)B_0^2}{9\,\pi\, m\omega^{\frac{3}{2}}(0)}\right);\ A_{11}'(0) = \sum \frac{8\,\alpha\, B_0^2(1 - B_0)}{3\,\pi\, m^2 \omega(0)} \quad (2.34b)$$

$$A_{11}'(0) = \sum \frac{8\,\alpha\, B_0^2(1 - B_0)}{3\,\pi\, m^2 \omega(0)} \quad (2.34c)$$

$$A_{12} = \frac{3\,\alpha_3 A_4}{4\,\omega} + \sum \frac{A_9}{m}\ ,\ \ A_{12}(0) = \frac{9\,\alpha_3 B_0^3}{8\,\omega(0)} + \sum \frac{4\,\alpha\, B_0^3}{9\,\pi\, m\omega^{\frac{3}{2}}(0)};\ \ A_{12}'(0) = 0 \quad (2.35)$$

We also have

$$a_{30}(0) = -A_{10}(0) + \frac{A_{11}(0)}{3} + \frac{A_{12}(0)}{8} = -\frac{113\, B_0^3 \alpha_3}{64\omega}$$

$$- \alpha \sum \left\{ \frac{95\, B_0^3}{18\pi\, m^2 \omega(0)} + \frac{32\, \mu_1'(0) B_0^2}{27\pi\, m\, \omega^{\frac{3}{2}}(0)} \right\}; \; {}_{30}(0) = 0 \qquad (2.36)$$

To solve (2.30a,b), we note that

$$\phi_{3m}(\tau) = \phi_{3m}(0) + \tau\, \phi_{3m}'(0) + \frac{\tau^2 \phi_{3m}''(0)}{2} + \cdots \qquad (2.37a)$$

We easily see that $\phi_{3m}'(0) = 0$ so that within the limit of accuracy retained in this work we can conveniently approximate $\phi_{3m}(\tau)$ by

$$\phi_{3m}(\tau) \approx \phi_{3m}(0) = \phi^2 = \left(\frac{81m^4 - 18m^2\lambda + 1}{m^4 - 2m^2\lambda + 1} \right) > 0 \quad \forall m \qquad (2.37b)$$

Thus on substituting (2.37b) into (2.30a,b) and solving we get

$$U_{3m}^{30} = a_{3m} \cos\phi\, t + b_{3m} \sin\phi\, t - \frac{\alpha_3}{4\,\omega} \left[\frac{A_2}{\phi^2} + \frac{A_3 \cos t}{\phi^2 - 1} + \frac{A_4 \cos 2t}{\phi^2 - 4} + \frac{A_5 \cos 3t}{\phi^2 - 9} \right] \qquad (2.38a)$$

$$a_{3m}(0) = \frac{B_0^3 \alpha_3 Q_0}{4\,\omega(0)},\; Q_0 = \left[\frac{3}{\phi^2} - \frac{15}{4\,(\phi^2 - 1)} + \frac{3}{2\,(\phi^2 - 4)} - \frac{1}{4(\phi^2 - 9)} \right]; \; b_{3m}(0) = 0 \quad (2.38b)$$

We next substitute on the right hand side of (2.10b) and assume

$$U^{31} = \sum_{n=1}^{\infty} U_n^{31}(t, \tau) \sin nx \qquad (2.39a)$$

When *n* = *m*, we equate to zero the coefficients of cost and *sint* and obtain respectively

$$b_{30}' + \left(\frac{\omega'}{4\omega} \right) b_{30} = 0 \;;\; a_{30}' + \left(\frac{\omega'}{4\omega} \right) a_{30} = -\frac{\omega^{\frac{1}{2}}}{2} h_2(\tau) \qquad (2.40a)$$

$$h_2 = \left[\omega^{-1} \left(\mu_2'' a_{10} + \mu_1'' \sum a_{20}' \right) + 2\omega^{-1} \left(\mu_2' a_{10}' + \mu_1' \sum a_{20}' \right) \right.$$

$$+ \frac{9\, \alpha_3 b_{11}}{4\omega} \left(A_0 - \frac{a_{10}^2}{4} \right) + 2\omega^{-\frac{1}{2}} \left(\mu_2' b_{11} - \mu_1' \sum b_{21} \right) \qquad (2.40b)$$

$$\left. + \sum \frac{16}{3\pi\, m} \left\{ \left\{ \frac{2\,\alpha\, b_{11}}{\omega} \left\{ \frac{8\,\alpha}{3\,\pi\, m\, \omega} \left(A_0 + \frac{a_{10}^2}{12} \right) \right\} + \frac{2\,\alpha}{\omega} \left(Bb_{21} - \frac{A_0 a_{10}}{6} \right) \right\} \right\} \right] \qquad (2.40c)$$

A detailed approximation of $h_2(0)$ is

$$h_2(0)=\sum\left(\frac{2\,\alpha\,B_0^3}{3\pi\;m^2\omega^{\frac{3}{2}}(0)}\right)-\frac{45\,\alpha_3 B_0^3}{8\,\omega^{\frac{3}{2}}(0)}+\sum\frac{16\alpha}{3\pi\;m}\left(\frac{B_0^3}{2\,\omega(0)}-\frac{220\mathrm{B}_0^3}{9\pi\;m\omega^{\frac{5}{2}}(0)}\right) \qquad (2.40\text{d})$$

$$+0\,(B_0^2)+0\,(B_0)\;;\qquad a'_{30}(0)=-\frac{\omega^{\frac{1}{2}}(0)}{2}h_2(0)$$

The solutions of (2.40a-c) subject to (2.36) are

$$b_{30}(\tau)\equiv 0\,,\,a_{30}(\tau)=\omega^{-\frac{1}{4}}(\tau)\left[a_{30}(0)\omega^{\frac{1}{4}}(0)-\frac{1}{2}\int_0^\tau \omega^{\frac{3}{4}}(s)\,\mathrm{h}_2(s)\,\mathrm{ds}\right] \qquad (2.41)$$

The remaining equation after substituting into the right hand side of (2.10b) and equating term that would lead to secular terms is $\quad U^{31}_{m,\mathrm{tt}}+U^{31}_m=A_{13}\sin 2t+A_{14}\sin 3t \quad$ (2.42a)

$$U^{31}_{m,\mathrm{t}}(0{,}0)=0\,,\;\mathrm{U}^{31}_{\mathrm{m.t}}(0{,}0)+\omega^{-\frac{1}{2}}(0)\left[\mu'_1(0)\sum U^{20}_{m,t}(0{,}0)+\mu'_2(0)\,U^{10}_{m,t}(0{,}0)\right]=0 \qquad (2.42\text{b})$$

$$A_{13}=\frac{16}{3\pi}\sum\frac{\alpha}{m}\left\{\frac{b_{11}a_{20}}{\omega}+\frac{1}{\omega}\left(b_{21}a_{10}-\frac{2A_1\,\mathrm{B}}{3\omega}\right)\right\}$$

$$+\frac{9\,\alpha_3 b_{11}B\,\mathrm{a}_{10}}{4\omega}-\frac{8\,\mu'_1}{3}\sum A_1\omega^{-\frac{1}{2}}-\mu'\sum\frac{16\,\alpha\,\omega^{-1}}{9\pi\,m}\left(\frac{a_{10}^2}{\omega}\right)' \qquad (2.42\text{c})$$

$$-\frac{4\,\omega^{-\frac{1}{2}}A'_{11}}{3}-\frac{\omega^{-\frac{3}{2}}\omega'\,\mathrm{A}_4}{3}-\frac{8\,\mu'_1\,\alpha}{9\pi\,\mathrm{m}\,\omega}\sum\left(\frac{\alpha_{10}^2}{\omega}\right)$$

$$A_{14}=-\frac{16\,\alpha}{3\pi}\sum\frac{1}{m}\left\{\frac{8}{3\pi\,\mathrm{m}\,\omega}\left(\frac{a_{10}^2}{12}\right)+\frac{A_1\,\mathrm{a}_{10}}{3\,\omega}\right\}+\frac{9\,\alpha_3 b_{11}a_{10}^2}{16\,\omega}-\frac{3\,\omega^{-\frac{1}{2}}A'_{12}}{4}$$

$$-\frac{3\,\omega^{-\frac{3}{2}}\omega' A_{12}}{16} \qquad (2.42\text{d})$$

Similarly when n = 3m in the substitution into (2.10b), we ensure non-secular terms (using (2.37b)) by equating the coefficients of $\cos\phi$ t and sin ϕ t and getting respectively

$$b'_{3m}+b_{3m}\left(\frac{\omega'}{4\,\omega}\right)=0\;;\;\mathrm{a}'_{3m}+a_{3m}\left(\frac{\omega'}{4\,\omega}\right)=0 \qquad (2.43\text{a})$$

The solutions of (2.43a) are $\quad b_{3m}(\tau)\equiv 0\,;\;\mathrm{a}_{3\mathrm{m}}(\tau)=\mathrm{a}_{3\mathrm{m}}(0)\left(\frac{\omega(0)}{\omega(\tau)}\right)^{\frac{1}{4}} \quad$ (2.43b)

The remaining equations for the case n=3m in (2.10b) are

$$U^{31}_{3m\,tt} + \phi^2 U^{31}_{3m} = A_{15}\sin t + A_{16}\sin 2t + A_{17}\sin 3t$$
$$U^{31}_{3m}(0,0) = 0 \;,\; U^{31}_{3m,t}(0,0) = 0 \qquad (2.44a,b)$$

$$A_{15} = \frac{3\alpha_3 b_{11}}{4\omega}\left(\frac{a_{10}^2}{4} - A_0\right) - \frac{\omega^{-\frac{1}{2}}\alpha_3}{4(\phi^2-1)}\left(\frac{A_3}{\omega}\right)' - \frac{\omega'\alpha_3 A_3}{8\omega^{\frac{5}{2}}(\phi^2-1)} \qquad (2.44c)$$

$$A_{16} = -\frac{3\alpha_3 b_{11} B a_{10}}{4\omega} + \frac{\omega^{-\frac{1}{2}}\alpha_3}{(\phi^2-4)}\left(\frac{A_4}{\omega}\right)' - \frac{\omega'\alpha_3 A_4}{4\omega^{\frac{5}{2}}(\phi^2-4)} \qquad (2.44d)$$

$$A_{17} = \frac{3\alpha_3 b_{11} a_{10}^2}{16\omega} - \frac{3\omega^{-\frac{1}{2}}\alpha_3}{2(\phi^2-9)}\left(\frac{A_5}{\omega}\right)' - \frac{3\omega'\alpha_3 A_5}{8\omega^{\frac{5}{2}}(\phi^2-9)} \qquad (2.44e)$$

The solution of (2.42a-d) is

$$U^{31}_m = a_{31}\cos t + b_{31}\sin t - \frac{A_{13}\sin 2t}{3} - \frac{A_{14}\sin 3t}{8}$$
$$a_{31}(0) = 0 \;,\; b_{31}(0) \neq 0 \qquad (2.45,a,b)$$

Similarly, the solution of (2.44a-e) is

$$U^{31}_{3m} = a_{41}\cos\phi\, t + b_{41}\sin\phi\, t + \frac{A_{15}\sin t}{\phi^2-1} + \frac{A_{16}\sin 2t}{\phi^2-4} + \frac{A_{17}\sin 3t}{\phi^2-9}$$
$$a_{41}(0) = 0 \;,\; b_{41}(0) \neq 0 \qquad (2.46a,b)$$

Analysis that will soon follow shows that in the final evaluation, U^{31}_m and U^{31}_{3m} are going to be evaluated at $t = 0, \tau = 0$ in which case U^{31}_m will vanish and U^{31}_{3m} will also vanish except for the insignificant term $b_{41}(0)\sin\pi$. Thus at this level of approximation, we can ignore this insignificant non-vanishing term since it contributes an infinitesimal addition to the solution. Thus for further analysis, the terms U^{31}_m and U^{31}_{3m} are ignored. However for the sake of completeness, we still write

$$U^{30} = U^{30}_m \sin mx + U^{30}_{3m}\sin 3mx \;;\; U^{31} = U^{31}_m \sin mx + U^{31}_{3m}\sin 3mx \qquad (2.47)$$

where *m* is odd. So far the expression for displacement $U(x,t,\tau,\in,\delta)$ is

$$U(x,t,\tau,\in,\delta) = \in\left(U^{10}_m + \delta U^{11}_m\right)\sin mx + \in^2 \sum\left(U^{20}_m + \delta U^{21}_m\right)\sin mx$$
$$+ \in^3 \left\{ U^{30}_m \sin mx + U^{30}_{3m}\sin 3mx + \delta\left(U^{31}_m \sin mx + U^{31}_{3m}\sin 3mx\right)\right\} \qquad (2.48)$$
$$+ 0\left(\in\delta^2\right) + 0\left(\in^2\delta^2\right) + 0\left(\in^3\delta^2\right)$$

3.0 Maximum Displacement

The maximum displacement is Ua = $U(x_a, t_a, \tau_a, \in, \delta)$ where x_a, t_a and τ_a are the critical values of the associated variables at maximum displacement. Similarly we shall let $\bar{t}_a$ and $\tilde{t}_a$ be the critical values of $\bar{t}$ and $\tilde{t}$ respectively. The conditions for maximum displacement are

$$U_{,x}=0\ ;\ U_{,t}+\omega^{-\frac{1}{2}}\left\{\left(\in\mu_1'+\in^2\mu_2'+\cdots\right)U_{,t}+\delta\, U_{,\tau}\right\} \qquad (3.1)$$

We shall now assume the following series

$$\begin{aligned}
t_a &= t_0+\delta\, t_{01}+\cdots+\in\left(t_{10}+\delta\, t_{11}\cdots\right)+\in^2\left(t_{20}+\delta\, t_{21}\cdots\right)+\cdots\\
\tilde{t}_a &= \tilde{t}_0+\delta\, \tilde{t}_{01}+\cdots+\in\left(\tilde{t}_{10}+\delta\, \tilde{t}_{11}\cdots\right)+\in^2\left(\tilde{t}_{20}+\delta\, \tilde{t}_{21}\cdots\right)+\cdots\\
\bar{t}_a &= \bar{t}_0+\delta\, \bar{t}_{01}+\cdots+\in\left(\bar{t}_{10}+\delta\, \bar{t}_{11}\cdots\right)+\in^2\left(\bar{t}_{20}+\delta\, \bar{t}_{21}\cdots\right)+\cdots\\
\tau_a &= \delta\, \bar{t}_a=\delta\left\{\bar{t}_0+\delta\, \bar{t}_{01}+\cdots+\in\left(\bar{t}_{10}+\delta\, \bar{t}_{11}\cdots\right)+\in^2\left(\bar{t}_{20}+\delta\, \bar{t}_{21}\cdots\right)+\cdots\right\}
\end{aligned} \qquad (3.2a,b,c,d)$$

On substituting for $U(x,t,\tau,\in,\delta)$ from (2.48) into (3.1), using (3.2a-d), we get ,from the first of (3.1)

$$\cos m x_a = 0\ ;\ x_a=\frac{\pi}{2m},\ (\text{m odd}) \qquad (3.3)$$

By substituting for x_a from (3.3) into the expansions of the second equation in (3.1) we now get the following equations corresponding to the indicated coefficients.

$$(\in 1): U^{10}_{m,t}=0$$

$$(\in\delta): t_{01}U^{10}_{m,tt}+U^{11}_{m,t}+\bar{t}_0 U^{10}_{m,t\tau}+\omega^{-\frac{1}{2}}(0)\mu_1'(0)U^{10}_{m,\tau}=0 \qquad (3.4a,b)$$

$$\left(\in^2 1\right): t_{10}U^{10}_{m,tt}+\sum U^{20}_{m,t}+\omega^{-\frac{1}{2}}(0)\mu_1'(0)U^{10}_{m,t}=0 \qquad (3.4c)$$

$$\left(\in^2\delta\right): t_{11}U^{10}_{m,tt}+t_{10}U^{10}_{m,t\tau}+t_{10}U^{11}_{m,tt}+\sum\left(t_{01}U^{20}_{m,tt}+\bar{t}_0 U^{20}_{m,t\tau}\right)+\sum U^{21}_{m,t}$$

$$+\bar{t}_0\left(\mu_1'\,\omega^{-\frac{1}{2}}U^{10}_{m,t}\right)_{,\tau}+\mu_1'\,\omega^{-\frac{1}{2}}t_{01}U^{10}_{m,tt}+\mu_2'\,\omega^{-\frac{1}{2}}t_{10}U^{10}_{m,t\tau}+\mu_2'\,\omega^{-\frac{1}{2}}\sum U^{20}_{m,\tau}=0 \qquad (3.4d)$$

$$\left(\in^3 1\right): t_{20}U^{10}_{m,tt}+t_{10}\sum U^{20}_{m,tt}+\left(U^{30}_{m,t}-U^{30}_{3m,t}\right)+t_{10}\omega^{-\frac{1}{2}}(0)\mu_1'(0)U^{10}_{m,tt}$$

$$+\mu_2'\,\omega^{-\frac{1}{2}}\sum U^{20}_{m,t}+\mu_2'\,\omega^{-\frac{1}{2}}U^{10}_{m,t}=0 \qquad (3.4e)$$

From (3.4a) we have $\sin t_0=0$; $t_0=r\pi$, r = 0,1,2,3,… . Since we need the least nontrivial value of t_0 we set r = 1 and get

$$t_0=\pi \qquad (3.5a)$$

From (3.4b,c) we get respectively

$$t_{01}=\omega^{-\frac{1}{2}}(0),\ t_{10}=0 \qquad (3.5b)$$

From (3.4d) we have

$$t_{11} = \sum \left\{ \frac{40\,B_0}{9\pi\,m\,\omega^{\frac{3}{2}}(0)} - \frac{8}{3\pi\,m\,\omega^{\frac{3}{2}}(0)} \right\} \omega^{-\frac{1}{2}}(0)\,\mu_1'(0) t_{01} \qquad (3.5c)$$
$$+\,\omega^{-\frac{1}{2}}(0)\,\mu_2'(0) \sum \frac{8\alpha(1-B_0)}{3\pi\,m\,\omega(0)} -$$

Every other term vanishes in (3.4e) and we finally get

$$t_{20} = \frac{U^{30}_{3m,t}}{U^{10}_{m,t\ \mathrm{t}}} = \frac{B_0^2 Q_0 \alpha_3 \phi \sin\phi\,\pi}{4\,\omega(0)} \qquad (3.5d)$$

We shall eventually need $\bar{t}_0{}'\bar{t}_{01}$ and $\bar{\mathrm{t}}_{20}$ which we now determine as follows: From (2.4b) evaluated at the critical point we have

$$t_a = \tilde{t}_a + \frac{\left\{ \in \mu_1(\tau_a) + \in^2 \mu_2(\tau_a) + \cdots \right\}}{\delta} = \tilde{t}_a + \in \left\{ \bar{t}_a \mu_1'(0) + \frac{\mu_1'(0)\delta\,\bar{\mathrm{t}}_{\mathrm{a}}^2}{2} \right\} \qquad (3.6a)$$
$$+ \in^2 \bar{t}_a \mu_2'(0) + \cdots$$

By using (3.2a-d) in (3.6a) and equating the relevant coefficients, we have

$$t_0 = \pi = \tilde{t}_0, \quad \mathrm{t}_{10} = 0 = \tilde{t}_{10} + \bar{t}_0 \mu_1'(0), \quad \mathrm{t}_{20} = \tilde{t}_{20} + \bar{t}_{10} \mu_1'(0) \qquad (3.6b)$$

From the second of (2.4a) we have

$$\tilde{t}_a = \int_0^{\bar{t}_a} \omega^{\frac{1}{2}}(s) ds = \int_0^{\bar{t}_a} \left(m^4 - 2m^2 \lambda \cos \mathrm{r} + 1\right)^{\frac{1}{2}} dr$$
$$= \left(m^4 - 2m^2\lambda + 1\right)^{\frac{1}{2}} \left\{ \bar{t}_a - \frac{m^2 \lambda\,\delta^2 t^3}{3\left(m^4 - 2m^2\lambda + 1\right)} \right\} + \cdots \qquad (3.6c)$$

By equating the relevant coefficients of (3.6c), using (3.2a-d) we have

$$\tilde{t}_0 = \left(m^4 - 2m^2\lambda + 1\right)^{\frac{1}{2}} \bar{t}_0\ , \tilde{\mathrm{t}}_{10} = \left(m^4 - 2m^2\lambda + 1\right)^{\frac{1}{2}} \bar{t}_{10}\ , \tilde{\mathrm{t}}_{20} = \left(m^4 - 2m^2\lambda + 1\right)^{\frac{1}{2}} \bar{t}_{2\,0} \qquad (3.6d)$$

From (3.6b) and (3.6d) we get

$$\bar{t}_0 = \left(m^4 - 2m^2\lambda + 1\right)^{-\frac{1}{2}} \pi\ , \ \bar{\mathrm{t}}_{10} = -\bar{t}_0 \mu_1'(0) \left(m^4 - 2m^2\lambda + 1\right)^{-\frac{1}{2}} \qquad (3.6e)$$

$$\bar{t}_{20} = \left(m^4 - 2m^2\lambda + 1\right)^{-\frac{1}{2}} \left(t_{20} - \mu_1'(0)\,\bar{t}_{10}\right) \qquad (3.7)$$

By evaluating (2.48) at the critical values, using (3.2a) - (3.3) and (3.5a)-(3.7), we determine the expression for the maximum displacement U_a as follows (though most of these terms actually vanish on substitution):

$$
\begin{aligned}
U_a = & \in \left\{ U_m^{10} + \delta\left(t_{01} U_{m,t}^{10} + \bar{t}_0 U_{m,\tau}^{10} + U_m^{11}\right) \right\} + \in^2 \Big[t_{10} U_{m,t}^{10} \\
& + \sum (U_m^{20}) + \delta \Big\{ t_{11} U_{m\,t}^{10} + \bar{t}_{10} U_{m,\tau}^{10} + t_{01} t_{10} U_{m,\,tt}^{10} + t_{10} U_{m,t}^{11} \\
& + \sum t_{01} U_{m,t}^{20} + \bar{t}_0 \sum U_{m,\tau}^{20} + \sum U_m^{21} \Big\} \Big] + \in^3 \Big[t_{20} U_{m,t}^{10} + \frac{1}{2} t_{10}^2 U_{m,\,tt}^{10} \\
& + t_{10} \sum U_{m,t}^{20} + \left(U_m^{30} - U_{3m}^{30}\right) + \delta \Big\{ t_{21} U_{m,t}^{10} + \bar{t}_{20} U_{m,\tau}^{10} + \left(t_{01} t_{20} + t_{10} t_{11}\right) U_{m,tt}^{10} \\
& + t_{20} U_{m,t}^{11} + \frac{1}{2} t_{10}^2 U_{m,tt}^{11} + t_{11} \sum U_{m,t}^{20} + \bar{t}_{10} \sum U_{m,\tau}^{20} + t_{01} t_{10} \sum U_{m,tt}^{20} \\
& + t_{10} \sum U_{m,t}^{21} + t_{01}\left(U_{m,t}^{30} - U_{3m,t}^{30}\right) + \bar{t}_0 t_{01}\left(U_{m,\tau}^{30} - U_{3m,\tau}^{30}\right) + \left(U_m^{31} - U_{3m}^{31}\right) \Big\} \Big] \\
& + 0\left(\in \delta^2\right) + 0\left(\in^2 \delta^2\right) + 0\left(\in^3 \delta^2\right)
\end{aligned} \qquad (3.8)
$$

Having neglected $U_m^{31} -_{3m}^{31}$ as indicated earlier, we simplify (3.8) as follows:

$$
U_a = 2\,\mathrm{B}_0 \in + \frac{64 B_0^2 \alpha \in^2 F_1}{9\pi\,\mathrm{m}\,\omega(0)} + \frac{15 \alpha_3 B_0^3 F_2 \in^3}{4\,\omega(0)} + 0\left(\in^4\right) \qquad (3.9a)
$$

where $$F_1 = 1 - \frac{3\,\delta\,(1-\mathrm{B}_0)}{8\mathrm{B}_0} + \frac{9\pi\,\mathrm{m}\,\omega(0)}{64 B_0^2} \sum \left\{ \frac{64 B_0^2}{9\pi\,\mathrm{m}\,\omega(0)} + \frac{8 B_0\,\delta\,(1-\mathrm{B}_0)}{3\pi\,\mathrm{m}\,\omega(0)} \right\} \qquad (3.9b)$$

$$
\begin{aligned}
F_2 = & 1 - \frac{Q_0\left(\cos\phi\,\pi - 1\right)}{15} + \frac{8\,\alpha\,\omega(0)}{15 \alpha_3 B_0^3} \sum \left\{ \frac{48 B_0^3}{9\pi\,\mathrm{m}^2\,\omega(0)} + \frac{32 \mu_1'(0) B_0^3}{27\pi\,\mathrm{m}\omega^{\frac{3}{2}}(0)} \right\} \\
& + \frac{4\omega(0)\,\delta}{15 \alpha_3 B_0^3} \left\{ \left\{ \bar{t}_{10} \sum \left\{ \frac{8 B_0 \alpha\,(1-\mathrm{B}_0)}{3\pi\,\mathrm{m}\,\omega(0)} \right\} - t_{01} t_{20} B_0 + t_{20} B_0 \omega^{-\frac{1}{2}}(0) \right. \right. \\
& \left. \left. + \frac{t_{01} \alpha_3 B_0^3 Q_0 \phi \sin\phi\pi}{4\,\omega(0)} + \bar{t}_0 \left\{ \frac{(\omega(0))^{-\frac{1}{2}} h_2(0)}{2} + \sum \frac{20\,\alpha\,B_0^2 (1-B_0)}{9\pi\,\mathrm{m}\,\omega(0)} \right\} \right\} \right\}
\end{aligned} \qquad (2.9c)
$$

We shall now let $$U_a = C_1 \in + \mathrm{C}_2 \in^2 + C_3 \in^3 + \cdots \qquad (3.10a)$$

where $$C_1 = 2B_0\,, \quad \mathrm{C}_2 = \frac{64\,\mathrm{B}_0^2 \alpha\,F_1}{9\pi\,\mathrm{m}\,\omega(0)}\,, \quad \mathrm{C}_3 = \frac{15\,\alpha_3 B_0^3 F_2}{4\,\omega(0)} \qquad (3.10b)$$

4.0 Dynamic Buckling Load λ_D

We shall now determine the dynamic buckling load λ_D which, according to Budiansky and Hutchinson [14] and Ette [15] is obtained from the maximization

$$\frac{d\lambda}{dU_a} = 0 \qquad (4.1)$$

Before applying (4.1), we first have to [10, 15] reverse the series (3.10a,b) and now write

$$\in = e_1 U_a + e_2 U_a^2 + e_3 U_a^3 \quad (4.2a)$$

By substituting for U_a in (4.2a) from (3.9a,b) and equating the coefficients of powers of $\in$ we have

$$e_1 = \frac{1}{C_1}, \; e_2 = -\frac{C_2}{C_1^3}, \quad e_3 = \frac{2C_2^2 - C_1 C_3}{C_1^5} \quad (4.2b)$$

The maximization (4.1) easily follows from (4.2a,b) where we obtain

$$e_1 + 2e_2 U_a + 3e_3 U_a^2 = 0 \quad (4.3a)$$

where (4.3a) is evaluated at $\lambda = \lambda_D$. From (4.3a) we get

$$U_a(\lambda_D) = \frac{1}{3e_3}\left[-e_2 \pm \left(e_2^2 - 3e_1 e_3\right)^{\frac{1}{2}} \right]\Bigg|_{\lambda=\lambda_D} \quad (4.3b)$$

where we shall take the negative root sign, the positive sign having no physical significance . We now evaluate the terms in (4.3b) as follows:

$$\left(e_2^2 - 3e_1 e_3\right)^{\frac{1}{2}} = \sqrt{\frac{3C_1 C_3}{C_1^3}} \left(1 - \frac{5C_2^2}{3C_1 C_3}\right)^{\frac{1}{2}} \quad (4.4a)$$

Therefore we have $-e_2 - \left(e_2^2 - 3e_1 e_3\right)^{\frac{1}{2}} = -\sqrt{\frac{3C_1 C_3}{C_1^3}} \left\{ \left(1 - \frac{5C_2^2}{3C_1 C_3}\right)^{\frac{1}{2}} + \frac{C_2}{\sqrt{3C_{1_3} C_3}} \right\}$ (4.4b)

Thus we obtain

$$U_a(\lambda_D) = \frac{C_1^{\frac{3}{2}} F_3}{\sqrt{3C_3}}, \; F_3 = \frac{\left(1 - \frac{5C_2^2}{3C_1 C_3}\right)^{\frac{1}{2}} + \frac{C_2}{\sqrt{3C_1 C_3}}}{1 - \frac{2C_2^2}{C_1 C_3}} \quad (4.4c)$$

Now if we multiply (4.2a) by 3 we get

$$3\in = 3U_a e_1 + 3e_2 U_a^2 + 3e_3 U_a^3 = U_a\left(3e_1 + 3e_2 U_a + 3e_3 U_a^2\right) \quad (4.5a)$$

On making $3e_3 U_a^2$ the subject in (4.3a) and substituting same into (4.5a) we get

$$3\in = U_a\left(2e_1 + e_2 U_a\right) = \frac{2U_a}{C_1^2}\left(1 - \frac{C_2 U_a}{2C_1^2}\right) \quad (4.5b)$$

which is evaluated at $\lambda = \lambda_D$. On substituting for $U_a(\lambda_D)$ from (4.4c) into (4.5b), we get

$$3\in = \frac{2C_1^{\frac{1}{2}} F_3}{\sqrt{3C_1}}\left[1 - \frac{C_2 F_3}{2C_1^{\frac{3}{2}}\sqrt{3C_3}}\right] \quad (4.5c)$$

which is again evaluated at $\lambda = \lambda_D$ .On substituting for C_1 and C_3 from (3.10b) into (4.5c), using (2.4a) and (2.15a) and simplifying we get

$$\left(m^4 - 2m^2\lambda_D + 1\right)^{\frac{3}{2}} = \frac{9\sqrt{10}\ (\alpha_3 F_2)^{\frac{1}{2}} \lambda_D m^2 |\in \bar{a}_m|}{4 F_3} \left[1 - \frac{32\sqrt{2}\ (\alpha F_1 F_3)}{27\alpha_3 \pi \sqrt{5}\ \omega^{\frac{1}{2}}(0) m F_2}\right] \quad (4.6)$$

5.0 Discussion

The result (4.6) is asymptotic and is valid for small values of the $\in$ and δ . In this investigation we have not limited our inquiry (as is usually the case) to the limitation that the imperfection be always in the shape of buckling mode. However our result even caters for this special limitation. For example by neglecting both $U_{3m}^{30} \sin 3mx$ and $U_{3m}^{31} \sin 3mx$ from our result we see that the resultant result is for the case where the buckling mode is strictly in the shape of the imperfection. If the column were on a cubic nonlinear elastic foundation (and not on a quadratic-cubic nonlinear elastic foundation) then we set $\alpha = 0, C_2 = 0$ So that we get

$$U_a(\lambda_D) = \frac{C_1^{\frac{3}{2}}}{\sqrt{3C_3}}\ , F_3 = 1\ ,\ 3\in = \frac{2C_1^{\frac{1}{2}}}{\sqrt{3C_3}} \quad (5.1a)$$

The result in this case becomes

$$\left(m^4 - 2m^2\lambda_D + 1\right)^{\frac{3}{2}} = \frac{9\sqrt{15}\quad (\alpha_3 F_2)^{\frac{1}{2}} m^2 \lambda_D |\in \bar{a}_m|}{4\sqrt{2}} \quad (5.1b)$$

and for the case m=1 we get

$$\left(1 - \lambda_D\right)^{\frac{3}{2}} = \frac{9\sqrt{15}\ (\alpha_3 F_2)^{\frac{1}{2}} \lambda_D |\in \bar{a}_1|}{16} \quad (5.1c)$$

6.0 Conclusion

We have utilized a generalization of Lindsted–Poincare method in an asymptotic solution of a strictly nonlinear oscillatory system which depends on two small but unrelated parameters and which has dynamically and sinusoidally slowly varying coefficients. We have particularized the result to that of a column on a cubic nonlinear elastic foundation. It is hoped that the method can be found useful in analyzing more complex elastic structures under similar loading conditions.

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