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On the dynamic buckling of a lightly damped elastic cubic model structure modulated by sinusoidally slowly varying load

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Abstract

In this paper, we employ a generalization of Lindsted-Poincare technique to determine the dynamic buckling load of a lightly and viscously damped elastic cubic model structure modulated by a sinusoidally slowly varying dynamic load. The imperfect elastic cubic (nonlinear) structure is itself a generalization of most elastic physical structures that have been investigated over the years. The formulation contains two small but mathematically unrelated parameters upon which asymptotic expansions are initiated. The dynamic buckling load is obtained asymptotically and is related to the result corresponding to that of the static loading. This process by-passes the labour of repeating the entire process for different imperfection parameters.

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1.0 Introduction

The exposition contained here is aimed at finding a strictly analytical solution to the problem of determining the dynamic buckling load of an imperfect elastic cubic model structure modulated by an explicitly time dependent sinusoidal load. It is strictly a nonlinear oscillatory and dissipative dynamical system where the effects of a small viscous damping is investigated using a generalization of Lindsted–Poincare procedure in a regular perturbation analysis. In other words the ensuing coefficients in the equations characterizing the dynamic equilibrium of the structure become sinusoidally and dynamically slowly varying. Our objective is to determine the dynamic buckling load of the imperfect structure under the instance of the prescribed slowly varying load.

We remark that strictly nonlinear dynamical problems with slowly varying parameters were first investigated by Kuzmak [1]. Later, Luke [2] , in his work n nonlinear nearly periodic waves, extended it to higher orders. In some other application of slowly varying parameters in nonlinear dynamical systems Collinge and Ockendon [3] discussed the case of transition through resonance of a Duffing oscillator. We however remark that relatively recent analyses of nonlinear oscillatory and dynamical systems with slowly varying coefficients have been primarily discussed on the platform of Physics (not Mathematics) in connection with waves, rigid bodies, charged particles etc. In one of such investigations, Kevorkian [4] used this technique to study free-electron lasers – a purely Mathematical problem dealing with general strictly nonlinear oscillations whose earlier studies were mostly Quantum Mechanical [5] .Later Li [6] used the same technique to investigate free-electron lasers with variable parameters while Li and Kevorkian [7] similarly studied the effects of wiggler taper rate and signal field gain rate in free electron lasers.

2.0 Formulation

The elastic cubic model structure which we are about to investigate was originally investigated by Budiansky and Hutchinson [8-10]. They considered a two-armed simply-supported column (Figure 1), subjected to a time dependent load F(T) applied at time T = 0 . The structure is assumed weightless and carries a mass M at the center .The motion of M is restrained by a nonlinear

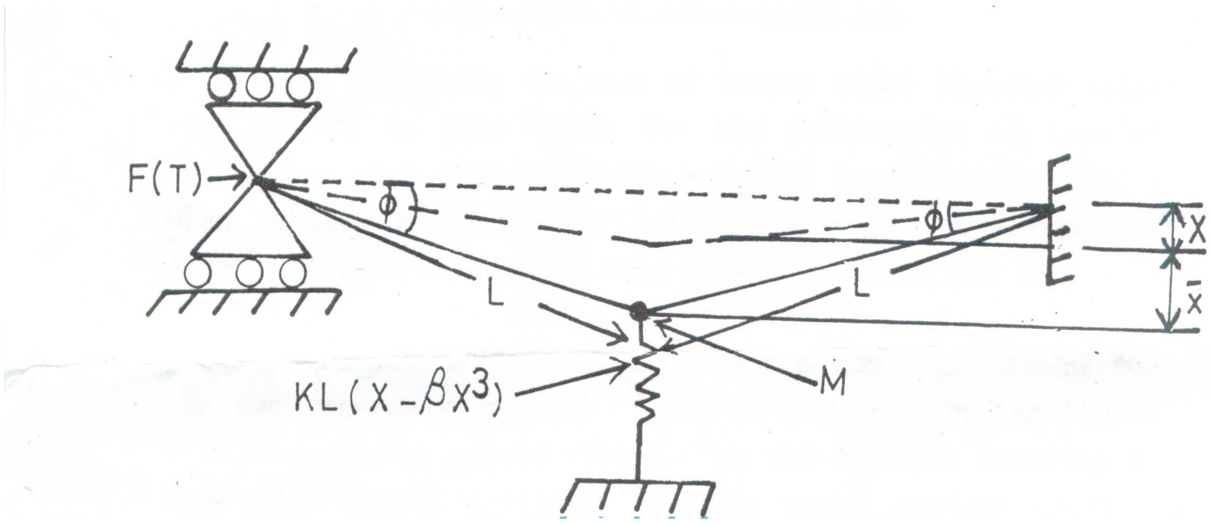

Figure 1: Simple elastic "cubic" model

"softening" spring which provides a restoring force given by

$$\mathrm{KL}\left(\mathrm{x} - \beta\, \mathrm{x}^3\right)$$

where $\mathrm{K} > 0, \beta > 0$ and L is the length of the column while x is the central hinge displacement from the equilibrium position. In a major refinement of the works in [8-10] we shall introduce a light viscous damping taken proportional to the first degree of the velocity. We shall assume small angular displacement characterized by $\cos\phi \approx 1, \sin\phi \approx \phi$. Using these assumptions the governing equation of dynamic equilibrium of the structure becomes

$$M\frac{d^2x}{dT^2} + Q\frac{dx}{dT} + \left(1 - \frac{2F(T)}{KL^2}\right)x - KL\beta x^3 = \frac{2\bar{x}F(T)}{L}, \quad \mathrm{T} > 0 \qquad (2.1)$$

where Q is the damping constant and $\bar{x}$ is the initial displacement which serves as the initial imperfection . We shall now introduce the following nondimensional quantities:

$$\xi = \frac{x}{L},\ \bar{\xi} = \frac{\bar{x}}{L},\ \hat{\mathrm{t}} = T\sqrt{\frac{\mathrm{KL}}{\mathrm{M}}},\ \lambda = \frac{2F(0)}{KL^2},\ \in = \frac{\mathrm{Q}}{\sqrt{\mathrm{KLM}}},$$

$$f(\in \hat{t}) = \frac{F(T)}{F(0)} = \cos \in \hat{t} \qquad (2.2a)$$

$$(F(0) \neq 0), \mathrm{b} = \beta \mathrm{L}^2 \qquad (2.2b)$$

where $0 < \in << 1, 0 < \bar{\xi} << 1$ and $0 < \lambda < 1$. However the analysis is generally valid for $\left|\bar{\xi}\right| << 1$. Here λ is a load amplitude whose particular value at buckling we are to determine. We easily note that $\in$ and $\bar{\xi}$ are two small but Mathematically unrelated parameters and b is the imperfection-sensitivity parameter which is such that for $b < 0$, the structure is said to be imperfection–insensitive whereas for $b >$ 0,the structure is said to be imperfection - sensitive. Using (2.2a,b), the nondimensional form of (2.1) together with the initial conditions becomes

$$\ddot{\xi} + \in \dot{\xi} + (1 - \lambda \cos \in \hat{t})\xi - \mathrm{b}\xi^3 = \lambda\bar{\xi}\cos \in \hat{t}, \hat{\mathrm{t}} > 0 \qquad (2.3a)$$

$$\xi(0) = \dot{\xi}(0) = 0$$

where
$$\frac{\mathrm{d}(\ )}{\mathrm{d}\hat{\mathrm{t}}} \equiv \dot{(\ )} \qquad (2.3b)$$

The undamped and step loading case $(\in = 0)$ was investigated in [8-10] .It is however our considered opinion that most buckling phenomena are in some way affected by some element of damping which is usually not taken into account in most buckling considerations . The problem in (2.3a,b) becomes a two-small parameter non-autonomous one with harmonically and dynamically slowly varying coefficients. We remark that problems with sinusoidal coefficients are, at the best of times, solved using Mathieu-type of instability. However as noted by Budiansky [8, page 100], Mathieu-type of instability is usually associated with many cycles of oscillations as opposed to just one cycle of oscillation that is usually associated with dynamic buckling. This has thus necessitated the need for an alternative approach which has resulted in the method of generalization of Lindsted–Poincare technique.

The simple elastic cubic model structure characterized by (2.3a,b) has a lot of practical applications in Science and Engineering . It is infact a generalization of equations satisfied by most physical systems under various dynamic loading histories. Such systems include (a) a finite (or infinite) imperfect column lying on elastic cubic (or quadratic-cubic or cubic-quatic) nonlinear foundations, (b) a finite (or infinite) imperfect cylindrical shell trapped by any dynamic load and (c) an imperfect toroidal shell segment under any dynamic loading history, among others. The structure satisfying equation (2.3a) is said to be cubic because of the cubic nonlinearity in the displacement. Relevant literatures include Zhu et al [11], Heinen and Bullesbach [12] , Schenk and Schueller [13] and Ette [14-16] .Henceforth we shall ,without loss of generality ,set $b = 1$.

3.0 The associated static problem

In this case we set $\in = 0$ and also ignore the inertia term in (3a).The resultant equation is

$$(1-\lambda)\xi - \xi^3 = \lambda\bar{\xi}\ . \qquad (3.1a)$$

If the structure were perfect then $\bar{\xi} = 0$ and the associated classical buckling load λ_C takes the value λ_C =1. For the imperfect structure we have $\bar{\xi} \neq 0$ and following [8-10], the condition for obtaining the static buckling load λ_S is $\frac{d\lambda}{d\xi} = 0$. This gives

$$(1-\lambda_S)^{\frac{3}{2}} = \frac{3\sqrt{3}\,|\bar{\xi}|\,\lambda_S}{2} \qquad (3.1b)$$

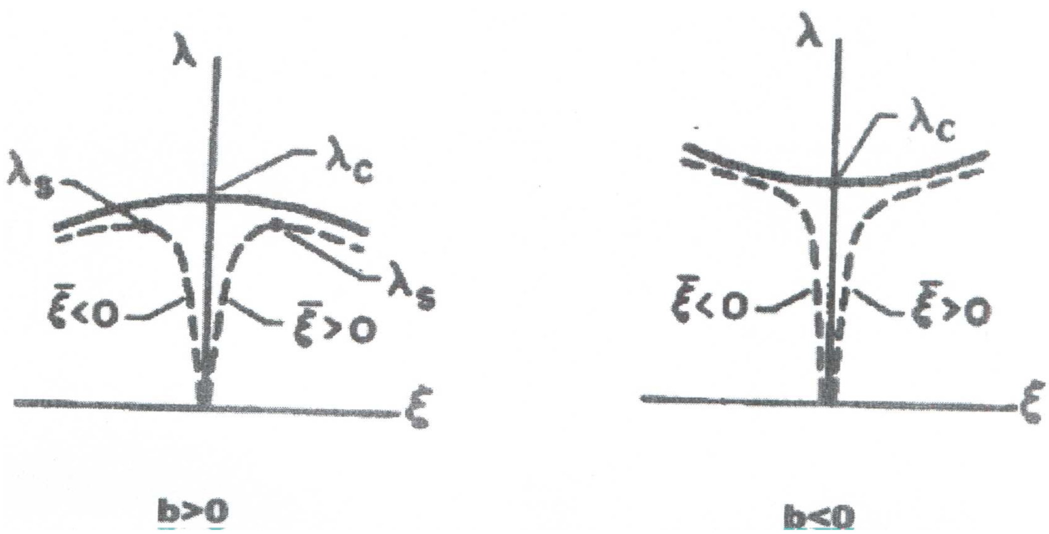

Figure 2: Load-deflection curves of a cubic model structure

In (Figure 2a,b) we see the schematic plots of the various load deflections for various ranges of the imperfection parameter *b*.

4.0 The dynamic cases

The undamped structure under step loading consideration satisfies the equation

$$\ddot{\xi} + (1-\lambda)\xi - \xi^3 = \lambda\bar{\xi} \tag{4.1a}$$

$$\xi(0) = \dot{\xi}(0) \tag{4.1b}$$

The dynamic buckling load for this case was investigated by Budiansky and Hutchinson [8-10], using phase-plane analysis. The dynamic buckling load λ_D in this autonomous case satisfies the equation

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6}\,|\bar{\xi}|\,\lambda_D}{2} \tag{4.2}$$

While the phase-plane method was readily available and sufficient in analyzing the autonomous case (4.1a,b) , the same cannot be said of the non-autonomous case in (2.3a,b) which we vividly recast thus

$$\ddot{\xi} + \epsilon\,\dot{\xi} + (1-\lambda\cos\epsilon\,\hat{t})\xi - \xi^3 = \lambda\bar{\xi}\cos\epsilon\,\hat{t} \tag{4.3a}$$

$$\xi(0) = \dot{\xi}(0) = 0 \tag{4.3b}$$

In the analysis that follows, we shall first determine a uniformly valid asymptotic value of the displacement variable $\xi(\hat{t})$. We shall next determine the maximum ξ_a of $\xi(\hat{t})$. Lastly we shall determine the dynamic buckling load λ_D from the maximization [8-10] $\frac{d\lambda}{d\xi_a} = 0$. We define the dynamic buckling load λ_D as the largest load parameter for which the solution of the problem (4.3a,b) remains bounded for all time $\hat{t} > 0$. We shall now let

$$\frac{d\tilde{t}}{d\hat{t}} = (1-\lambda\cos\epsilon\,\hat{t})^{\frac{1}{2}}, \quad \tau = \epsilon\,\hat{t}, \quad t = \tilde{t} + \frac{(\omega_2\bar{\xi}^2 + \omega_3\bar{\xi}^3 + \cdots)}{\epsilon} \tag{4.4a}$$

$$\omega_i = \omega_i(\tau), i = 2,3,4,\cdots;\ \omega_i(0) = 0,\ t = 0 \text{ for } \tilde{t} = \hat{t} = 0 \tag{4.4b}$$

Thus we have $\dfrac{d\xi}{d\hat{t}} = (1-\lambda\cos\tau)^{\frac{1}{2}}\xi_{,t} + \left(\omega_2'\bar{\xi}^2 + \omega_3'\bar{\xi}^3 + \cdots\right)\xi_{,t} + \in \xi_{,\tau}$ (4.5a)

$$\frac{d^2\xi}{d\hat{t}^2} = (1-\lambda\cos\tau)\xi_{,tt} + \left(\omega_2'\bar{\xi}^2 + \omega_3'\bar{\xi}^3 + \cdots\right)^2\xi_{,tt} + \in^2 \xi_{,\tau\tau}$$

$$+2(1-\lambda\cos\tau)^{\frac{1}{2}}\left(\omega_2'\bar{\xi}^2 + \omega_3'\bar{\xi}^3 + \cdots\right)\xi_{,t\tau} + 2\in\left(\omega_2'\bar{\xi}^2 + \omega_3'\bar{\xi}^3 + \cdots\right)\xi_{,t\tau} \qquad (4.5b)$$

$$+2\in(1-\lambda\cos\tau)^{\frac{1}{2}}\xi_{,t\tau} + \frac{\in(\lambda\sin\tau)(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi_{,t}}{2} + \left(\omega_2''\bar{\xi}^2 + \omega_3''\bar{\xi}^3 + \cdots\right)\xi_{,t}$$

We shall now let $\xi(\hat{t}) = \xi(t,\tau,\in,\bar{\xi}) = \displaystyle\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\xi^{ij}(t,\tau)\bar{\xi}^i \in^j$ (4.6)

where the ij in $\xi^{ij}(t,\tau)$ are superscripts and not powers. On substituting (4.6) into (4.3a,b), using (4.5a,b) and equating the coefficients of $\bar{\xi}^i \in^j, i = 1,2,3,\cdots; j = 0,1,2,3,\cdots$ we get the following sequence of equations $N\xi^{10} \equiv \xi^{10}_{,tt} + \xi^{10} = \dfrac{\lambda\cos\tau}{1-\lambda\cos\tau} \equiv B(\tau)$ (4.7a)

$$N\xi^{11} = -(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{10}_{,t} - 2(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{10}_{,t\tau} - \frac{(\lambda\sin\tau)(1-\lambda\cos\tau)^{-\frac{3}{2}}\xi^{10}_{,t}}{2} \qquad (4.7b)$$

$$N\xi^{12} = -(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{11}_{,t} - 2(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{11}_{,t\tau} - \frac{(\lambda\sin\tau)(1-\lambda\cos\tau)^{-\frac{3}{2}}\xi^{11}_{,t}}{2} - (1-\lambda\cos\tau)^{-1}\xi^{10}_{,\tau\tau} \qquad (4.7c)$$

$$N\xi^{20} = 0 \qquad (4.8a)$$

$$N\,\xi^{21} = -(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{20}_{,t} - 2(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{20}_{,t\tau} - \frac{(\lambda\sin\tau)(1-\lambda\cos\tau)^{-\frac{3}{2}}\xi^{20}_{,t}}{2} \qquad (4.8b)$$

$$N\,\xi^{22} = -(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{21}_{,t} - 2(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{21}_{,t\tau} - \frac{(\lambda\sin\tau)(1-\lambda\cos\tau)^{-\frac{3}{2}}\xi^{21}_{,t}}{2} - (1-\lambda\cos\tau)^{-1}\xi^{20}_{,\tau\tau} \quad (4.8c)$$

$$N\xi^{30} = -(1-\lambda\cos\tau)^{-1}\left(\xi^{10}\right)^3 - 2(1-\lambda\cos\tau)^{-\frac{1}{2}}\omega_2'\xi^{10}_{,tt} \quad (4.9a)$$

$$N\xi^{31} = -3(1-\lambda\cos\tau)^{-1}\left(\xi^{10}\right)^2\xi^{11} - 2(1-\lambda\cos\tau)^{-\frac{1}{2}}\omega_2'\xi^{11}_{,tt}$$

$$-2(1-\lambda\cos\tau)^{-1}\omega_2'\xi^{10}_{,t\tau} - 2(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{30}_{,t\tau} - \frac{(\lambda\sin\tau)(1-\lambda\cos\tau)^{-\frac{3}{2}}\xi^{30}_{,t}}{2} \quad (4.9b)$$

$$-(1-\lambda\cos\tau)^{-1}\omega_2'\xi^{10}_{,t} - \omega_2''(1-\lambda\cos\tau)^{-1}\xi^{10}_{,t} - (1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{30}_{,t}$$

The associated initial conditions are $\xi^{ij}(0,0) = 0 \;\forall i,j$ (4.10a)

$$\xi^{10}_{,t}(0,0) = 0;\; \xi^{1k}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\xi^{1p}_{,\tau}(0,0) = 0;\quad p = k-1,\; k = 1,2,3,\cdots \quad (4.10b)$$

$$\xi^{20}_{,t}(0,0) = 0;\; \xi^{2k}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\xi^{2p}_{,\tau}(0,0) = 0;\quad p = k-1,\; k = 1,2,3,\cdots \quad (4.10c)$$

$$\xi^{30}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\omega_2'(0)\xi^{10}_{,t}(0,0) = 0 \quad (4.10d)$$

$$\xi^{3k}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\omega_2'(0)\xi^{1k}_{,t}(0,0) = (1-\lambda)^{-\frac{1}{2}}\xi^{3p}_{,\tau}(0,0) = 0; \quad (4.10e)$$

$$p = k-1,\; k = 1,2,3,\cdots$$

We now solve (4.7a), using (4.10a) and the first of (4.10b) to get

$$\xi^{10}(t,\tau) = a_1(\tau)\cos t + b_1(\tau)\sin t + B \quad (4.11a)$$

$$a_1(0) = -B_0,\; \mathrm{B}_0 = \frac{\lambda}{1-\lambda};\; \mathrm{b}_1(0) = 0 \quad (4.11b)$$

We now substitute (4.11a,b) into (4.11b) and to ensure a uniformly valid solution, equate to zero the coefficients of cost and *sint* and get respectively

$$b_1' + \frac{1}{2}\left[1 + \frac{\lambda \sin\tau}{2(1-\lambda\cos\tau)}\right]\mathrm{b}_1 = 0\,;\ \mathrm{a}_1' + \frac{1}{2}\left[1 + \frac{\lambda \sin\tau}{2(1-\lambda\cos\tau)}\right]\mathrm{a}_1 = 0 \quad (4.12a)$$

The integrating factor for each of (4.12a) is $(1-\lambda\cos\tau)^{\frac{1}{2}} e^{\frac{\tau}{2}}$ and so the solutions of (4.12a) together with (4.11b) are

$$b_1(\tau) = b_1(0)e^{-\frac{\tau}{2}}(1-\lambda\cos\tau)^{-\frac{1}{2}} \equiv 0,\ \mathrm{a}_1(\tau) = a_1(0)e^{-\frac{\tau}{2}}(1-\lambda\cos\tau)^{-\frac{1}{2}} \quad (4.12b)$$

We can now write $$\xi^{10} = a_1(\tau)\cos t + B \quad (4.12c)$$

The remaining equations in (4.7b) are $$N\xi^{11} = 0 \quad (4.13a)$$

$$\xi^{11}(0,0) = 0, \xi^{11}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\xi^{10}_{,\tau}(0,0) = 0 \quad (4.13b)$$

The solution of (4.13a,b) is $$\xi^{11} = a_2(\tau)\cos t + b_2(\tau)\sin t \quad (4.14a)$$

$$a_2(0) = 0\,,\ \ \mathrm{b}_2(0) = -\frac{B_0}{2}(1-\lambda)^{-\frac{1}{2}} \quad (4.14b)$$

where we have used the fact that $$a_1'(0) = \frac{B_0}{2},\ \mathrm{B}'(0) = 0 \quad (4.14c)$$

We now substitute into (4.7c) and equate to zero the coefficients of cost and *sin t* and get respectively

$$b_2' + \frac{1}{2}\left[1 + \frac{\lambda \sin\tau}{2(1-\lambda\cos\tau)}\right]\mathrm{b}_2 = -\frac{a_2''}{2}\,;\ \mathrm{a}_2' + \frac{1}{2}\left[1 + \frac{\lambda \sin\tau}{2(1-\lambda\cos\tau)}\right]\mathrm{a}_2 = 0 \quad (4.15a)$$

The solutions of (4.15a), using (14.4b), are

$$b_2(\tau) = (1-\lambda\cos\tau)^{-\frac{1}{2}} e^{-\frac{\tau}{2}}\left[b_2(0)(1-\lambda)^{\frac{1}{2}} - \int_0^\tau \frac{a_1'' e^{\frac{s}{2}}(1-\lambda\cos s)^{\frac{1}{2}}\,ds}{2}\right] \quad (4.15b)$$

We thus conclude that $$\xi^{11}(t,\tau) = b_2(\tau)\sin t \quad (4.15c)$$

The remaining equation in (4.14c) is solved to get

$$\xi^{12} = a_3(\tau)\cos t + b_3(\tau)\sin t - B''(1-\lambda\cos\tau)^{-1} \quad (4.16a)$$

$$a_3(0) = -(1-\lambda)^{-1}B''(0),\ \mathrm{b}_3(0) = 0. \quad (4.16b)$$

The solutions of (4.8a-c) yield $$\xi^{2j}(t,\tau) = 0\,,\ \mathrm{j} = 0,1,2,3,\cdots \quad (4.17)$$

On substituting into (4.9a) and simplifying we get

$$N\xi^{30} = (1-\lambda\cos\tau)^{-1}\left[\frac{a_1^3\cos 3t}{4} + \frac{3a_1^2 B\cos 2t}{2} + \left(3a_1B^2 + \frac{3a_1^3}{4}\right)\cos t + B^3\right]$$
$$+2\omega_2'(1-\lambda\cos\tau)^{-\frac{1}{2}}a_1'\cos t \quad (4.18a)$$

$$\xi^{30}(0,0)=0 \quad ; \quad \xi^{30}_{,t}(0,0)+(1-\lambda)^{-\frac{1}{2}}\omega_2'(0)\xi^{10}_{,t}(0,0)=0 \tag{4.18b}$$

To ensure a uniformly valid solution, we equate to zero in (4.18a) the coefficient of cost and get

$$\omega_2'(\tau)=-3(1-\lambda\cos\tau)^{-\frac{1}{2}}\left(B^2+\frac{a_1^2}{4}\right), \omega_2'(0)=-\frac{15B_0^2(1-\lambda)^{-\frac{1}{2}}}{8},$$

$$\omega_2''(0)=-\frac{3B_0^2(1-\lambda)^{-\frac{1}{2}}}{8} \tag{4.18c}$$

The remaining equation in (4.18a) is solved to get

$$\xi^{30}=a_4(\tau)\cos t+b_4(\tau)\sin t$$
$$+(1-\lambda\cos\tau)^{-1}\left[-\frac{a_1^3\cos 3t}{32}-\frac{a_1^2B\cos 2t}{2}+\left(\frac{3}{2}a_1^2B+B^3\right)\right] \tag{4.19a}$$

$$a_4(0)=-\frac{65B_0^3(1-\lambda)^{-1}}{32}, \quad \mathrm{b}_4(0)=0 \tag{4.19b}$$

We next substitute the relevant terms into (4.9b) and re-arrange to get

$$N\xi^{31}=A_{11}\sin 3t+A_{12}\sin 2t+A_{13}\sin t+A_{14}\cos t \tag{4.20a}$$

$$\xi^{31}(0,0)=0 \;, \xi^{31}_{,\mathrm{t}}(0,0)+(1-\lambda)^{-\frac{1}{2}}\omega_2'(0)\xi^{11}_{,t}(0,0)+(1-\lambda)^{-\frac{1}{2}}\xi^{31}_{,\tau}(0,0)=0 \tag{4.20b}$$

where
$$A_{11}=3(1-\lambda\cos\tau)^{-1}\frac{b_2a_1^2}{4}-\frac{3}{16}(1-\lambda\cos\tau)^{-\frac{1}{2}}\left\{a_1^3(1-\lambda\cos\tau)^{-1}\right\}$$
$$-\frac{3a_1^3(\lambda\sin\tau)(1-\lambda\cos\tau)^{-\frac{5}{2}}}{64}-\frac{3a_1^3(1-\lambda\cos\tau)^{-\frac{3}{2}}}{32} \tag{4.20c}$$

$$A_{12}=3(1-\lambda\cos\tau)^{-1}Bb_2a_1-2(1-\lambda\cos\tau)^{\frac{1}{2}}\left\{a_1^2(1-\lambda\cos\tau)^{-1}\right\}$$
$$-\frac{Ba_1^2(\lambda\sin\tau)(1-\lambda\cos\tau)^{-\frac{5}{2}}}{2}-Ba_1^2(1-\lambda\cos\tau)^{-\frac{3}{2}} \tag{4.20d}$$

$$A_{13}=3(1-\lambda\cos\tau)^{-1}b_2\left(B^2+\frac{a_1^2}{4}\right)-2(1-\lambda\cos\tau)^{-\frac{1}{2}}\left\{\omega_2'(b_2'+a_1')+a_4'\right.$$
$$\left.+\frac{(\lambda\sin\tau)(1-\lambda\cos\tau)^{-1}a_4}{4}+\frac{a_4}{2}\right\}+(1-\lambda\cos\tau)^{-1}(\omega_2'a_1+\omega_2''a_1) \tag{4.20e}$$

$$A_{14} = -\frac{(\lambda\sin\tau)(1-\lambda\cos\tau)^{-\frac{3}{2}} b_4}{2} - 2(1-\lambda\cos\tau)^{-\frac{1}{2}} b_4' - (1-\lambda\cos\tau)^{-\frac{1}{2}} b_4 \quad (4.20f)$$

To ensure a uniformly valid solution, we set $A_{13}=A_{14}=0$ and get

$$a_4' + \frac{a_4}{2}\left(1 + \frac{\lambda\sin\tau}{2(1-\lambda\cos\tau)}\right) = h(\tau) \quad (4.21a)$$

$$h(\tau) = -\omega_2'\left(b_2^2 + a_1'\right) - \frac{(1-\lambda\cos\tau)^{-\frac{1}{2}}}{2}\left[(\omega_2' + \omega_2'')a_1 + 3b_2\left(B^2 + \frac{a_1^2}{4}\right)\right] \quad (4.21b)$$

$$a_4'(0) = \frac{53B_0^3}{64(1-\lambda)} + 0(1-\lambda)^{-\frac{1}{2}} \quad (4.21c)$$

$$b_4' + \frac{b_4}{2}\left(1 + \frac{\lambda\sin\tau}{2(1-\lambda\cos\tau)}\right) = 0 \quad (4.21d)$$

The solutions of (4.21a-d) subject to (4.19b) are

$$a_4(\tau) = (1-\lambda\cos\tau)^{-\frac{1}{2}} e^{-\frac{\tau}{2}}\left[a_4(0)(1-\lambda)^{\frac{1}{2}} - \int_0^\tau (1-\lambda\cos\tau)^{\frac{1}{2}} e^{\frac{s}{2}} h(s)\,ds\right], \quad b_4(\tau) \equiv 0 \quad (4.22a)$$

We shall use the following later

$$\xi_{,\tau}^{30}(0,0) = a_4'(0) - \frac{61(1-\lambda)^{-1}B_0^3}{64} + 0(1-\lambda)^{-\frac{1}{2}} = -\frac{B_0^3}{8(1-\lambda)} \quad (4.23)$$

The remaining equation in (4.20a) is solved to get

$$\xi^{31}(t,\tau) = a_5(\tau)\cos t + b_5(\tau) - \frac{A_{11}\sin 3t}{8} - \frac{A_{12}\sin 2t}{3} \quad (4.24a)$$

$$a_5(0) = 0, \quad b_5(0) \neq 0 \quad (424b)$$

We shall not need expressed determination of $b_5(0)$. Thus the summary so far is

$$\xi(t,\tau) = \bar{\xi}\left(\xi^{10} + \in \xi^{11}\right) + \bar{\xi}^3\left(\xi^{30} + \in \xi^{31}\right) + 0\left(\bar{\xi} \in^2\right) + 0\left(\bar{\xi}^3 \in^2\right) \quad (4.25$$

We shall now determine the maximum displacement ξ_a and the condition for this is

$$\xi_{,t} + (1-\lambda\cos\tau)^{-\frac{1}{2}}\left[\omega_2'\bar{\xi}^2\xi_{,t} + \in \xi_{,\tau}\right] = 0 \quad (4.26)$$

We shall let $\hat{t}_a, \tilde{t}_a, t_a$ *and* τ_a be the critical values of $\hat{t}, \tilde{t}, t$ and τ respectively at maximum point and now assume the following asymptotic series.

$$\hat{t}_a = \hat{t}_0 + \in \hat{t}_{01} + \bar{\xi}^2\left(\hat{t}_{20} + \in \hat{t}_{21}\right) + \cdots \quad (4.27a)$$

$$\tilde{t}_a = \tilde{t}_0 + \in \tilde{t}_{01} + \bar{\xi}^2\left(\tilde{t}_{20} + \in \tilde{t}_{21}\right) + \cdots \quad (4.27b)$$

$$t_a = t_0 + \in t_{01} + \bar{\xi}^2 (t_{20} + \in t_{21}) + \cdots \qquad (4.27c)$$

$$\tau_a = \in t_a = \in \{t_0 + \in t_{01} + \bar{\xi}^2 (t_{20} + \in t_{21}) + \cdots\} \qquad (4.27d)$$

We now substitute (4.27a-d) into (4.26), using (4.25) and equate to zero the coefficients of $\bar{\xi}^i \in^j, i = 1,2,3,\cdots$, $j = 0,1,2,3,\cdots$. From the coefficients of $\bar{\xi}, \bar{\xi}\in$ and $\bar{\xi}^2$ we get respectively

$$\xi^{10}_{,t}(t_0,0) = 0, \qquad t_{01}\xi^{10}_{,tt} + \xi^{11}_{,t} + t_0 \xi^{10}_{,t\tau} = 0$$

$$t_{20}\xi^{10}_{,tt} + \xi^{30}_{,t} + (1-\lambda)^{-\frac{1}{2}} \omega_2' \xi^{10}_{,t} = 0 \qquad (4.28a,b,c)$$

From (4.27a) we get, using (4.12ff), $\sin t_0 = 0$; $t_0 = n\pi$, n = 0,1,2,3,⋯

Since we need the least nontrivial value of t_0 we set n = 1 and obtain $t_0 = \pi$ (4.29a)

From (4.28b) we get
$$t_{01} = -\frac{\xi^{11}_{,t}(t_0,0)}{\xi^{10}_{,tt}(t_0,0)} = -(1-\lambda)^{-\frac{1}{2}} \qquad (4.29b)$$

From (4.28c) we get
$$t_{20} = 0 \qquad (4.29c)$$

To arrive at all these values we note that if $\psi(t_a, \tau_a)$ is an arbitrary function of the indicated arguments, then it has the following Taylor series expansion

$$\psi(t_a,\tau_a) = \psi(t_0,0) + \in (t_{01}\psi_{,t} + t_0\psi_{,\tau}) + \bar{\xi}^2 \Big[t_{20}\psi_{,t} + \in \Big\{ t_{21}\psi_{,t} + t_{20}\psi_{,\tau} + \frac{1}{2}(t_{20}t_{01}\psi_{,tt} + 2t_{20}t_0\psi_{,t\tau}) \Big\} \Big] + \cdots \qquad (4.29d)$$

where (4.29d) is evaluated at $(t_0,0)$. We shall now determine the maximum displacement ξ_a of (4.25),using (4.27a-d) to get

$$\xi_a = \xi(t_a,\tau_a) = \bar{\xi}\{\xi^{10} + \in (\xi^{11}_{,t} + t_0\xi^{10}_{,\tau})\} + \bar{\xi}^3 \big[\xi^{30} + \in \{ \xi^{31} + t_0\xi^{30}_{,\tau} + t_{20}\xi^{11}_{,t} + t_{20}\xi^{10}_{,\tau} + t_{01}\xi^{30}_{,t} + t_{01}t_{20}\xi^{10}_{,tt} \} \big] + 0(\bar{\xi}\in^2) + 0(\bar{\xi}^3\in^2) \qquad (4.30a)$$

where (4.30a) is evaluated at (t_0,0).It is obvious that most of the terms in (4.30a) vanish on evaluation and so the eventual simplification of (4.30a) yields $\xi_a = \bar{\xi}C_1 + \bar{\xi}^3 C_3$ (4.30b)

where
$$C_1 = 2B_0\left(1 + \frac{\pi\in}{4}\right), \quad C_3 = \frac{4B_0^3}{1-\lambda}\left[1 - \frac{57\pi\in}{32}\right] \qquad (4.30c)$$

According to [8-10], the criterion for dynamic buckling is
$$\frac{d\lambda}{d\xi_a} = 0 \qquad (4.31)$$

As in [14-16], we first have to reverse the series (4.30b,c) in the form

$$\bar{\xi} = e_1\xi_a + e_3\xi_a + \cdots \qquad (4.32a)$$

By substituting for ξ_a in (37a) from (35b,c) and equating the coefficients of $\in$ and $\in^3$, we get

$$e_1 = \frac{1}{C_1}, \quad e_3 = -\frac{C_3}{C_1^3} \qquad (4.32b)$$

The maximization (4.31) is now easily accomplished through (4.32a,b) to yield

$$\bar{\xi} = \frac{2}{3}\sqrt{\frac{C_1}{3C_3}} \tag{4.33a}$$

On substituting for C_1 and C_3 in (4.33a) from (4.30c), we get

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6}\,|\bar{\xi}|\,\lambda_D}{2}\sqrt{\frac{1-\frac{57\pi\in}{32}}{1+\frac{\pi\in}{4}}} \tag{4.33b}$$

5.0 Analysis of result

The result (4.33b) is asymptotic in nature and is valid for small values of both $\bar{\xi}$ and $\in$. The terms multiplying $\in$ in (4.33b) clearly show the contributions to dynamic buckling of damping. If $\in = 0$ (that is no damping), then from (4.33b) we have

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6}\,|\bar{\xi}|\lambda_D}{2} \tag{5.1}$$

which is the same result already obtained in (4.2) for step loading situation obtained using phase - plane analysis. This method also clearly shows the supremacy of the generalized Lindsted-Poincare asymptotic method (which we have adopted) over Phase-plane method because it is applicable to both autonomous and non – autonomous systems. In an approximate way, we can determine the dynamic buckling load λ_D directly by taking the first three terms in the Binomial expansion of the right hand side of (4.35b) to get

$$\frac{3}{8}\lambda_D^2 - \lambda_D\left(\bar{\xi}S + \frac{3}{2}\right) + 1 = 0 \tag{5.2a}$$

$$S = \frac{3\sqrt{6}}{2}\left[\frac{1-\frac{57\pi\in}{32}}{1+\frac{\pi\in}{4}}\right] \tag{5.2b}$$

This gives

$$\lambda_D \approx \frac{4}{3}\left(S\bar{\xi} + \frac{3}{2}\right)\left[1-\sqrt{1-\frac{3}{2\left(S\bar{\xi}+\frac{3}{2}\right)^2}}\right] \tag{5.3}$$

where we have taken the negative root sign ; the positive root sign having no physical significance. To the level of approximation retained in (5.3) we easily observe that the dynamic buckling load λ_D decreases with increased damping and increases with decreased damping for both $\bar{\xi} < 0$ and $\bar{\xi} > 0$ .However the decrease in the value of λ_D is more in the range $\bar{\xi} < 0$ than in the range $\bar{\xi} > 0$ for the same damping parameter $\in$. We easily eliminate the imperfection parameter $\bar{\xi}$ in (4.35b) using (3.2b) to get

$$\left(\frac{1-\lambda_D}{1-\lambda_s}\right)^{\frac{3}{2}} = \sqrt{2}\left(\frac{\lambda_D}{\lambda_s}\right)\sqrt{\frac{1-\frac{57\pi\in}{32}}{1+\frac{\pi\in}{4}}} \tag{5.4}$$

Thus, by using (5.4), we avoid the labour of repeating the entire process for different imperfection parameters $\overline{\xi}$.

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