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On the dynamic buckling load of spherical shells trapped by a harmonically slowly varying load

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Abstract

In this paper, the dynamic buckling load of an imperfect discretized spherical shell subjected to a slowly varying time dependent sinusoidal load is determined by means of regular perturbation The results which are given in two levels of approximation are valid for small amplitudes of the imperfection. All results are strictly asymptotic. It is found out among other things that the nonlinearity in antisymmetnc mode dominates the buckling process and that neglecting any imperfection automatically nullifies the effect of any nonlinearity in the shape of the imperfection neglected. It s additionally established that lbs effects of coupling of the buckling modes is possible only if none the imperfections in the shapes of the coupling modes is neglected. These results are of course valid for considerations at the immediate post-dynamic buckling consideration.

pp 175 - 186

1.0 Introduction

Investigations into the dynamic buckling of elastic structures have consistently received enormous patronage as evident from the vast literature on the subject available in the past thirty years. These include the pioneering works of Budiansky and Hutchinson [1-3], Svalbonas and Kalnins [4], Huyan and Simitses [5] Simitses [6,7] and Peg [8,9] among others The present study is therefore an extension of one of such works by Danielson [10] where he investigated the dynamic buckling of spherical shells under step load,

Danielson [10] discretized the spherical shell by assuming that the normal displacement W of a point on the shell surface can be written as

$$W = \xi_o(T)W_o + \xi_1(T)W + \xi_2(T)W_2 \qquad (1.0)$$

Where W_o is the pre-buckling symmetric mode while W_1 and W_2 are the axisymmetric and an arbitrary non-axisymmetric buckling modes respectively. We note that each of W_1, I = 0, 1, 2 is functionally dependent on spatial variables. Similarly $\xi_i(T)$, i = 0, 1, 2 are the respective time dependent amplitudes. To study the effects of imperfection $\overline{W}$, Danielson assumed the following

$$\overline{W} = \bar{\xi}_1 W_1 + \bar{\xi}_2 W_2 \qquad (1.2)$$

Where W_1 and W_2 still have the same meanings as before and $\bar{\xi}_1$ and $\bar{\xi}_2$ are their amplitudes assumed small relative to unity. We shall however assume $\left|\bar{\xi}_1\right| << 1$. Using (1.0) and (1.2) in a Galerkin solution of the relevant compatibility and equilibrium equations characterizing a spherical shell, he obtained the following equations (for a step loading condition) in the amplitudes $\xi_o(T), \xi_1(T)$ and $\xi_2(T)$.

$$\frac{1}{\omega_o^2}\frac{d^2\xi_o}{dT^2} + \xi_o(T) \qquad (1.3)$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_q}{dT^2}+\xi_1(1-\xi_o)-a_1\xi_1^2+a_2\xi_2^2=\xi_1\xi_o \quad (1.4)$$

$$\frac{1}{\omega_2^2}\frac{d^2\xi_q}{dT^2}+\xi_2(1-\xi_o)+\xi_1\xi_2=\xi_2\xi_o \quad (1.5)$$

Where λ is a non-dimensional step load amplitude (non-dimensionalized with respect to the classical buckling load λ_c), T is the time variable where a_1, $a_2 > 0$ are constants and ω_i, i = 0, 1, 2 are the circular frequencies of the associated modes. Using some simplifying assumptions [4,10], Danielson verified the dynamic buckling load using a two-liming singular perturbation procedure. In our present investigation the structure is no longer trapped by a step load but by a sinusoidally lime dependent slowly varying load represented by $\lambda cos\delta_o T$, $\delta_o<<1$. Thus in this case the equation corresponding to (1.3) becomes

$$\frac{1}{w_o^2}\frac{d^2\xi_o}{dT^2}+\xi_o=\lambda\cos\delta_o T \quad (1.6)$$

Various methods have over the years been employed to solve dynamic buckling problems, majority of these problems which are essentially nonlinear in nature. Relatively recent methods include numerical method [4,11] and differential quadrature method [12]. We shall however employ a two-timing regular perturbation method to analyze the problem. We remark that such problems with sinusoidal coefficients are often analyzed by using Mathieu-type of instability. However as noted by Budiansky [3,page 100], Mathieu-type of instability is usually associated with many cycles of oscillations as opposed to just one shot of oscillation that characterizes dynamic buckling The procedure to be adopted is as follows: We shall first determine uniformly valid asymptotic formulae of each of the buckling modes and determine their maximum values. We shall next determine the net maximum buckling mode ξ_2, which is the sum of the maxima of the two buckling modes. Thus if ξ_{1a} and ξ_{2a} are the maxima of ξ_1 and ξ_2 respectively then

$$\xi_m=\xi_{1a}+\xi_{2a} \quad (1.7)$$

Following the definition by Budiansky and Hutchinson [1-3] that λ_D is the maximum load parameter for which the net maximum buckling mode remains bounded for all time T>0, we shall finally determine λ_D from the maximization

$$\frac{d\lambda}{d\xi_m}=0 \quad (1.8)$$

As in [l0] we shall set $\xi_1 = 0$ and now let

$$\bar{t}=\omega_o T \quad (1.9)$$

As in some examples in [1-3] we shall ignore the inertia of the pre-buckling mode and so set $\frac{d^2\xi_o}{dT^2}=0$. Thus equations (1.6), (1.4) and (1.5) become respectively

$$\xi_o(\bar{t})=\lambda\cos\delta\bar{t} \quad (1.10)$$

$$\frac{d^2\xi_1}{d\bar{t}^2}+Q^2\xi_1(1-\lambda\cos\delta\bar{t})-Q^2a_1\xi_1^2+Q^2a_1\xi_2^2=0 \quad (1.11)$$

$$\frac{d^2\xi_2}{d\bar{t}^2}+R^2\xi_2(1-\lambda\cos\delta\bar{t})+R^2\xi_1\xi_2=\in\cos\delta\bar{t} \quad (1.12)$$

$$\xi_1(0)=\frac{d\xi_i(0)}{dt}=0;\ i=1,2 \quad (1.13a)$$

$$\delta = \frac{\delta_o}{\omega_o}, Q = \frac{\omega_1}{\omega_o}, R = \frac{\omega_2}{\omega_o}, \in = \left(\frac{\omega_2}{\omega_o}\right)^2 \bar{\xi}_2 \tag{1.13b}$$

We shall assume $|\in| << 1, 0 < \delta < 1$ and note that the formulation contains two small but unrelated parameters $\in$ and δ while the coefficients are sinusoidally slowly varying in time. We shall now assume

$$\tau = \delta\bar{t}; \frac{d\bar{t}}{d\bar{t}} = (1 - \lambda\cos\delta\bar{t})^{\frac{1}{2}} = (1 - \lambda\cos\tau)^{\frac{1}{2}} \tag{1.14a}$$

and now let

$$t = \bar{t} + \frac{1}{\delta}\left(\mu_2(\tau)\in^2 + \mu_3(\tau)\in^3 + \ldots\right), \mu_1(0) = 0,\ i = 2,3,4,\ldots \tag{1.14b}$$

Thus we have

$$\frac{d\xi_\alpha}{d\bar{t}} = (1 - \lambda\cos\tau)^{\frac{1}{2}}\xi_{\alpha,t} + \left(\mu_2' \in^2 + \mu_3' \in^3 + \ldots\right)\xi_{\alpha,t} + \delta\xi_{\alpha,\tau} \tag{1.15}$$

Where α = 1,2 and a subscript following a comma indicates partial differentiation with respect to that subscript and $\frac{d()}{d\tau} = (\ )'$. Similarly we have

$$\frac{d^2\xi_\alpha}{d\bar{t}^2} = (1 - \lambda\cos\tau)\xi_{\alpha,tt} + \left(\mu_2' \in^2 + \mu_3' \in^3 + \ldots\right)^2 \xi_{\alpha,tt} + \delta\xi_{\alpha,\tau}$$

$$+ 2(1 - \lambda\cos\tau)^{\frac{1}{2}}\left(\mu_2' \in^2 + \mu_2' \in^3 + \ldots\right)\xi_{\alpha,tt} + 2\delta\left(\mu_2' \in^2 + \mu_3' \in^2 + \ldots\right)\xi_{\alpha,t\tau} \tag{1.16}$$

$$+ 2(1 - \lambda\cos\tau)^{\frac{1}{2}}\xi_{\alpha,tr} + \frac{(\delta\lambda\sin\tau)\xi\alpha' t)}{2(1 - \lambda\cos\tau)^{\frac{1}{2}}} + \delta(\mu_2'' \in^2 + \mu_3'' \in^3 + \ldots)\xi\alpha' t$$

We shall now let $\xi_1\left(\bar{t}\right) = \eta(t, \tau, \delta, \in), \xi_2\left(t\right) = \xi(t, \tau, \delta, \in)$ and now assume the following:

$$\eta(t, \tau, \delta, \in) = \sum_{t=2}^{\infty}\sum_{j=o}^{\infty}\eta^{ij}(t, \tau)\in^i \delta^j; \zeta(t, \tau, \delta, \in) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\zeta^{ij}(t, \tau)\in^i \delta^j \tag{1.17}$$

Where the terms *ij* on η^{ij} and ζ^{ij} are superscripts and not powers. On substituting (1.17) and (1.16) into (1.11) (for α = 1) and simplifying we get

$$L\eta^{20} \equiv \eta_{tt}^{20} + Q^2\eta^{20} = -\frac{a_2 Q^2\left(\zeta^{10}\right)^2}{1 - \lambda\cos\tau} \tag{1.18a}$$

$$L\eta^{21} = -2(1 - \lambda\cos\tau)^{-\frac{1}{2}}\eta_{t\tau}^{20} - \frac{(\lambda\sin\tau)\eta_t^{20}}{2(1 - \lambda\cos\tau)^{\frac{3}{2}}} - \frac{2a_2Q^2\left(\zeta^{10}\zeta^{11}\right)}{1 - \lambda\cos\tau} \quad \text{(1.18b0)}$$

Similarly we now substitute (1.17) and (1.16) into (1.11) and for α = 2 and simplify to get

$$M\zeta^{10} \equiv \zeta_{tt}^{10} + R^2\zeta^{10} = -\frac{\lambda\cos\tau}{1 - \lambda\cos\tau} \tag{1.19a}$$

$$M\zeta^{11} = -2(1-\lambda\cos\tau)^{-\frac{1}{2}}\zeta^{20}_{t\tau} - \frac{(\lambda\sin\tau)\zeta^{10}_{'t}}{2(1-\lambda\cos\lambda)^{\frac{3}{2}}} \quad (1.19b)$$

$$M\zeta^{20} = 0 \quad (1.20a)$$

$$M\zeta^{21} = -2(1-\lambda\cos\tau)^{-\frac{1}{2}}\zeta^{20}_{'tr} - \frac{(\lambda\sin\tau)\zeta^{20}_{'t}}{2(1-\lambda\cos\tau)^{\frac{3}{2}}} \quad (1.20b)$$

$$M\zeta^{30} = -2\mu_2'(1-\lambda\cos\tau)^{\frac{1}{2}}\zeta^{10}_{'tt} - (1-\lambda\cos\tau)^{-1}R^2\eta^{20}\zeta^{10} \quad (1.21a)$$

$$M\zeta^{31} = -2\mu_2'(1-\lambda\cos\tau)^{\frac{-1}{2}}\zeta^{11}_{'t\tau} - 2\mu_2'(1-\lambda\cos\tau)^{-1}\zeta^{10}_{'t\tau} - 2(1-\lambda\cos\tau)^{-\frac{1}{2}}\xi^{30}_{t\tau}$$
$$-\frac{(\lambda\sin\tau)\zeta^{30}_{'t}}{2(1-\lambda\cos\tau)^{\frac{3}{2}}} - (1-\lambda\cos\tau)^{1}R^2\left(\eta^{21}\zeta^{10} + \eta^{20}\zeta^{11}\right) - (1-\lambda\cos\tau)^{-1}\mu_2''\zeta^{10}_{'t} \quad (1.21b)$$

The initial conditions are

$$\eta^{ij}(0,0) = 0; i = 2,3,4..., j = 0,1,2,3,..., \zeta^{rs}(0,0) = 0; r = 1,2,3,...; s = 0,1,2,3,... \quad (1.22a)$$

$$\zeta^{10}_{1,t}(0,0) = 0, \zeta^{1k}_{1,t}(0,0) + \xi^{1p}_{1,t}(0,0) = 0, p = k-1, k = 1,2,3,... \quad (1.22b)$$

$$\eta^{20}_{,t}(0,0) = 0; \eta^{2k}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\eta^{2p}_{,r}(0,0) = 0; p = k-1, k = 1,2,3... \quad (1.22c)$$

$$\zeta^{30}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\mu_2'(0)\xi^{10}_{,t}(0,0) = 0 \quad (1.22d)$$

$$\zeta^{3k}_{,t}(0,0) + (1-\lambda)^{-\frac{1}{2}}\left[\mu_2'(0)\zeta^{1k}_{,t}(0,0) + \zeta^{3p}_{,\tau}(0,0)\right] = 0 \quad (1.22e)$$
$$p = k = 1, k = 1,2,3....$$

Having set $\bar{\xi}_1 = 0$, it was found out that

$$\eta^{1j}(t,\tau) = \eta^{3j}(t,\tau) \equiv 0, j = 0,1,2,3,... \quad (1.23)$$

The solution of (1.19a) is

$$\zeta^{10} = b_1(\tau)\cos Rt + d_1(\tau)\sin Rt + B, \quad (1.24a)$$

$$B(\tau) = \frac{\lambda\cos\tau}{1-\lambda\cos\tau}; b_1(0) = -\frac{\lambda}{1-\lambda}; d_1(0) = 0, B(0) \equiv B_o = \frac{\lambda}{1-\lambda} \quad (1.24b)$$

We now substitute (1.244a) into (1.9b) and equate to zero the coefficients of *co*sRt and *sinRt* get respectively

$$d_1' + \frac{(\lambda\sin\tau)d_1}{4(1-\lambda\cos\tau)} = 0; b_1' + \frac{(\lambda\sin\tau)b_1}{4(1-\lambda\cos\tau)} = 0 \quad (1.25)$$

We solve (1.25) to get

$$d_1(\tau)=0; b_1(\tau)=b_1(0)\left(\frac{1-\lambda}{1-\lambda\cos\tau}\right)^{\frac{1}{4}} \quad (1.26)$$

The solution of the remaining equation in (1.9b) is

$$\zeta^{10}=b_2(\tau)\cos Rt+d_2(\tau)\sin Rt \quad (1.27a)$$

$$b_2(0)=0; d_2(0)=0 \quad (1.27b)$$

We have used (1.22b) (for k=1) to get (1.27b). Since we expect the final solution of (1.27a, b) to depend on the initial conditions as in (1.27b), we expect that on the final analysis the full solution of (1.27a, b) will be

$$\zeta^{11}(t,\tau)\equiv 0, \quad (1.28)$$

This follows from the fact that the full expressions for both $b_2(\tau)$ and $b_2(\tau)$ will eventually be multiplied by he initial values $b_2(0)$ and $d_2(0)$ and since these values vanish as in (1.27b) then (1.28) suffices. We also see that based on (1.26),

$$\zeta^{10}=b_1(\tau)\cos Rt+B(\tau) \quad (1.29)$$

We now substitute (1.29) into (1.18a) and solve, (using (1.22a) for i = 2, j = 0 and (1.22c) for k=1) to get

$$\eta^{20}=\gamma_2(\tau)\cos Qt+\beta_2(\tau)\sin Qt$$
$$-\frac{a_2Q^2}{(1-\lambda\cos\tau)}\left[\frac{b_1^2\cos 2Rt}{2(Q^2+4R^2)}+\frac{2Bb_1\cos Rt}{\left(Q^2-R^2\right)}+\frac{1}{Q^2}\left(\frac{b_1^2}{2}+B^2\right)\right] \quad (1.30a)$$

$$\gamma_2(0)=\frac{B_o^2a_2Q\alpha_{20}}{1-\lambda};\alpha_{20}=\left[\frac{1}{2(Q^2-4R^2)}-\frac{2}{(Q^2-R^2)}+\frac{3}{2Q^2}\right];\beta_2(0)=0 \quad (1.30b)$$

We note that Q is as defined in (1.13b) and is such that Q > 0 We next substitute into (1.18b), noting (1.28), and to ensure a uniformly valid solution we equate to zero the coefficients of *cosQt* and *sinQt* and get respectively

$$\beta_2'+\frac{(\lambda\sin\tau)\beta_2}{4(1-\lambda\cos\tau)}=0;\gamma_2'+\frac{(\lambda\sin\tau)}{4(1-\lambda\cos\tau)}=0 \quad (1.31a)$$

The solutions of (1.31a), using (1.30b), are

$$\beta_2'(\tau)\equiv 0;\gamma_2(0)\left(\frac{1-\lambda}{1-\lambda\cos\tau}\right)^{\frac{1}{4}} \quad (1.31b)$$

The remaining equation in (1.18b) is solved (using (1.22a) for i = 2, j = 1 and (1.22c) for k = 1 to get

$$\eta^{21}(t,\tau)=\gamma_3(\tau)\cos Qt+\beta_3(\tau)\sin Qt;\gamma_3(0)=0;\beta_3(0)=0 \quad (1.32a)$$

Because of the homogeneous initial conditions in (322) we expect that on full solution we shall eventually get

$$\eta^{21}(t,\tau)\equiv 0 \quad (1.32b)$$

We next substitute the relevant terms in (1.21a) and get

$$M\zeta^{30}=2\mu_2'b_1(1-\lambda\cos\tau)^{-\frac{1}{2}}R^2\cos Rt-R^2\left[I_o+I_1\cos Rt+I_2\cos 2Rt\right.$$
$$\left.+I_3\cos 3Rt+I_4\cos(R+Q)t+I_5\cos(Q-R)t+B(\alpha_2\cos Qt+\beta_2\sin Qt)\right] \quad (1.33c)$$

$$I_o = -\frac{a_2}{1-\lambda\cos\tau}\left\{Q^2\left(\frac{Bb_1^2}{Q^2-R^2}+\frac{b_1}{Q^2}\left(\frac{b_1^2}{2}+B^2\right)\right)+B\left(\frac{b_1^2}{2}+B^2\right)\right\} \quad (1.34a)$$

$$I_0(0) = -\frac{a_2 B_0^3 I_{00}}{1-\lambda}; I_{00} = \left[Q^2\left(\frac{1}{Q^2-R^2}+\frac{3}{2Q^2}\right)+\frac{3}{2}\right] \quad (1.34b)$$

$$I_1 = -\frac{a_2}{1-\lambda\cos\tau}\left\{\frac{Q^2 b_1^3}{4(Q^2-4R^2)}+\frac{2BQ^2 b1}{Q^2-R^2}\right\}; I_1(0) = -\frac{a_2 B_0^3 I_{10}}{1-\lambda};$$

$$I_{10} = \left\{\frac{Q^2}{4(Q^2-4R^2)}+\frac{2Q^2}{Q^2-R^2}\right\}, \quad (1.35)$$

$$I_2 = -\frac{a_2 Q^2 Bb_1^2}{1-\lambda\cos\tau}\left\{\frac{1}{4(Q^2-4R^2)}+\frac{1}{Q^2-R^2}\right\}; I_2(0) = -\frac{a_2 B_0^3 I_{10}}{1-\lambda};$$

$$I_{20} = \left\{\frac{1}{2(Q^2-4R^2)}+\frac{1}{Q^2-R^2}\right\}, \quad (1.36)$$

$$I_3 = -\frac{a_2 Q^2 b_1^3}{4(Q^2-AR^2)(1-\lambda\cos\tau)}; I_3(0) = -\frac{a_2 B_0^3 I_{30}}{1-\lambda}; I_{30} = \frac{Q^2}{4(Q^2-4R^2)}, \quad (1.37)$$

$$I_4 = \frac{1}{2}\alpha_2 b_1; I_4(0) = -\frac{a_2 B_0^3 I_{40}}{1-\lambda}; I_{40} = \frac{Q^2\alpha_{20}}{2}; I_5 = \frac{1}{2}\alpha_2 b_1 = I_4; I_5 = -\frac{a_2 B_0^3 I_{40}}{1-\lambda} \quad (1.38)$$

To ensure a uniformly valid solution, we equate to zero the coefficient of cosRt in (1.33) and get

$$\mu_2^{'}(\tau) = \frac{I_1(1-\lambda\cos\tau)^{\frac{1}{2}}}{2b_1} \quad (1.39)$$

The remaining equation in (1.33) is solved, (using (1.22a) for r = 3, s = 0 and (1.22d) to

$$\zeta^{30}(t,\tau) = b_3(\tau)\cos Rt + d_3(\tau)\sin Rt$$

$$-R^2\left[\frac{1_o}{R^2}-\frac{I_2\cos 2Rt}{3R^3}-\frac{1_3\cos Rt}{8R^2}-\frac{1_4\cos(Q+R)t}{Q(2R+Q)}+\frac{1_5\cos(R-Q)t}{Q(2R-Q)}+\frac{B\alpha\cos Qt}{R^2-Q^2}\right] \quad (1.40)$$

where

$$b_3(0) = \frac{a_2 R_2 B_0^3 b_{20}}{1-\lambda}; d_3(0) = 0, \quad (1.41)$$

$$b_{20} = \left[\frac{I_{oo}}{R^2}-\frac{I_{20}}{3R^2}-\frac{I_{30}}{8R^2}-\frac{I_{40}}{Q(2R+Q)}+\frac{I_{40}}{Q(2R-Q)}-\frac{Q^2\alpha_{20}}{R^2-Q^2}\right] \quad (1.42)$$

We note that

$$\eta^{21}\zeta^{10}+\eta^{20}\zeta^{11} = 0 \quad (1.43)$$

And next substitute in (1.21b) and thereafter equate to zero the coefficients of *cosRt* and *sinRt* and obtain the following

$$d_3^{'}+\frac{(\lambda\sin\tau)d_2}{4(1-\lambda\cos\tau)} = 0; b_3^{'}+\frac{(\lambda\sin\tau)b_2}{4(1-\lambda\cos\tau)} = -\frac{(1-\lambda\cos\tau)^{-\frac{1}{2}}(2\mu_2^{'}b_1^{'}+\mu_2^{''}b_1)}{2} \quad (1.44)$$

$$d_3(\tau)=0; b_3(\tau)=(1-\lambda\cos\tau)^{\frac{1}{4}}\left[C-\int_o^r(1-\lambda\cos\tau)^{-\frac{3}{4}}(2\mu_2^{'}b_1^{'}+\mu_2^{''}b_1)dr\right], \quad (1.45a)$$

$$C=b_3(0)(1-\lambda)^{\frac{1}{4}} \quad (1.45b)$$

The remaining equation in (1.21b) is simplified to yield

$$M\zeta^{31}= f_1\sin 2Rt+ f_2\sin 3Rt+ f_3\sin(Q+R)t+ f_4\sin(R-Q)t+ f_5\sin Qt \quad (1.46)$$

$$f_1=R\left\{\frac{4}{3}(1-\lambda\cos\tau)^{-\frac{1}{2}}I_2^{'}+\frac{\lambda I_2\sin\tau}{3(1-\lambda\cos\tau)^{\frac{3}{2}}}\right\};$$

$$f_2=R\left\{\frac{3}{4}(1-\lambda\cos\tau)^{-\frac{1}{2}}I_3^{'}+\frac{3\lambda I_3\sin\lambda}{16(1-\lambda\cos\tau)^{\frac{3}{2}}}\right\}, \quad (1.47a)$$

$$f_3=-R^2\left(\frac{Q+R}{Q(2R+Q)}\right)\left\{2(1-\lambda\cos\tau)^{-1/2}I_4^{'}+\frac{\lambda I_4\sin\tau}{2(1-\lambda\cos\tau)^{\frac{3}{2}}}\right\} \quad (1.47b)$$

$$f_5=-\left(\frac{R^2Q}{(R^2+Q^2)}\right)\left\{2(1-\lambda\cos\tau)^{-1/2}(B\alpha_2)'+\frac{\lambda(B\alpha_2)\sin\tau}{2(1-\lambda\cos\tau)^{\frac{3}{2}}}\right\} \quad (1.47c)$$

$$f_4=-R^2\left(\frac{R-Q}{Q(2R-Q)}\right)\left\{2(1-\lambda\cos\tau)^{-1/2}I_5^{'}+\frac{\lambda I_5\sin\tau}{2(1-\lambda\cos\tau)^{\frac{3}{2}}}\right\} \quad (1.47d)$$

$$f_1(0)= f_2(0)= f_3(0)= f_4(0)= f_5(0)=0$$

The solution of (1.46) is

$$\xi^{31}(t,\tau)=b_3(\tau)\cos Rt+d_3(\tau)\sin Rt-\frac{f_1\sin 2Rt}{3R^2}-\frac{f_2\sin 3Rt}{8R^2}-\frac{f_3\sin(Q+R)t}{Q(2R+Q)} + \frac{f_4\sin(R-Q)t}{Q(2R-Q)}+\frac{f_5\sin Qt}{R^2-Q^2} \quad (1.48)$$

$$b_3(0)=0; d_3(0)$$

$$=-\frac{(1-\lambda)^{\frac{1}{2}}}{R}\left[b_2^{'}-R^2\left\{\frac{I_o^{'}}{R^2}-\frac{I_2^{'}}{3R^2}-\frac{I_3^{'}}{8R^2}-\frac{I_4^{'}}{Q(2R+Q)}+\frac{I_5^{'}}{Q(2R-Q)} + \frac{(B\alpha_2)'}{R^2-Q^2}\right\}\right]_{\tau=0} \quad (1.49)$$

So far we write

$$\eta(t,\tau)=\eta^{20}(t,\tau)\in^2+...;\xi(t,\tau)=\in\xi^{10}+\in^3\left(\xi^{30}+\delta\xi^{31}\right)+0\left(\in\delta^2\right)+0\left(\in^2\delta^2\right) \quad (1.50)$$

2.0 Maximum Amplitude, ζ_M

We shall now determine the maximum amplitude of each of the buckling modes ζ_1 and ζ_2 and now let t_{1c} be the critical values of t for ζ_1 and ζ_2 respectively. Similarly we shall let τ_{1c} and τ_{2c}, $\bar{t}_{1c}$ and $\bar{t}_{2c}$ be the values of τ and $\bar{t}$ respectively for these two respective modes to have a maximum. We shall assume the following asymptotic series.

$$t_{1c} = t_o^{(1)} + t_{01}^{(1)}\delta + \ldots + \in \left(t_{10}^{(1)} + t_{11}^{(1)}\delta + \ldots\right) + \in^2 \left(t_{20}^{(1)} + t_{21}^{(1)}\delta + \ldots\right) + \ldots \qquad (2.1a)$$

$$t_{2c} = t_o^{(2)} + t_{01}^{(2)}\delta + \ldots + \in \left(t_{10}^{(2)} + t_{11}^{(2)}\delta + \ldots\right) + \in^2 \left(t_{20}^{(2)} + t_{21}^{(2)}\delta + \ldots\right) + \ldots \qquad (2.1b)$$

$$\bar{t}_{1c} = \bar{t}_o^{(1)} + \bar{t}_{01}^{(1)}\delta + \ldots + \in \left(\bar{t}_{10}^{(1)} + \bar{t}_{11}^{(1)}\delta + \ldots\right) + \in^2 \left(\bar{t}_{20}^{(1)} + \bar{t}_{21}^{(1)}\delta + \ldots\right) + \ldots \qquad (2.1c)$$

$$\bar{t}_{2c} = \bar{t}_o^{(2)} + \bar{t}_{01}^{(2)}\delta + \ldots + \in \left(\bar{t}_{10}^{(2)} + \bar{t}_{11}^{(2)}\delta + \ldots\right) + \in^2 \left(\bar{t}_{20}^{(2)} + \bar{t}_{21}^{(2)}\delta + \ldots\right) + \ldots \qquad (2.1d)$$

$$\tau_{1c} = \delta\bar{t}_{1c} = \delta\left\{\bar{t}_{01}^{(1)} + \bar{t}_{01}^{(1)}\delta + \ldots + \in \left(\bar{t}_{10}^{(1)} + \bar{t}_{11}^{(1)}\delta + \ldots\right) + \ldots \in^2 \left(\bar{t}_{20}^{(1)} + \bar{t}_{21}^{(1)}\delta + \ldots\right) + \ldots\right\} \qquad (2.1e)$$

$$\tau_{2c} = \delta\bar{t}_{2c} = \delta\left\{\bar{t}_{0}^{(2)} + \bar{t}_{01}^{(2)}\delta + \ldots + \in \left(\bar{t}_{10}^{(2)} + \bar{t}_{11}^{(2)}\delta + \ldots\right) + \ldots \in^2 \left(\bar{t}_{20}^{(2)} + \bar{t}_{21}^{(2)}\delta + \ldots\right) + \ldots\right\} \qquad (2.1f)$$

A necessary condition for $\eta(t,r)$ to have a maximum is following (1.5)

$$(1 - \lambda\cos\tau)^{\frac{1}{2}}\eta_{,t} + \left(u_2' \in^2 + \mu_3' \in^3 \ldots\right)\eta_{,t} + \delta\eta_{,t} = 0 \qquad (2.2a)$$

This is evaluated at the critical point. For terms of order $\in^2$, this means

$$\eta^{20}{}_{,t}(t_{1c}, \tau_{1c}) = 0 \qquad (2.2b)$$

This translates to

$$Q\alpha_{20}\sin Qt_o^{(1)} + R\left[\frac{\sin 2Rt_o^{(1)}}{Q^2 - 4R^2} - \frac{2\sin Rt_c^{(1)}}{Q^2 - R^2}\right] = 0 \qquad (2.3a)$$

In an approximate way, the least nontrivial value of $t_o^{(1)}$ is obtained by retaining just the first two terms in the expansions of the sine functions in (2.3a) getting

$$t_o^{(1)} = \frac{\sqrt{Q^2\alpha_{20} + 2R^2\left(\frac{1}{Q^2 - 4R^2} - \frac{1}{Q^2 - R^2}\right)}}{\frac{Q^4\alpha_{20}}{6} + R^4\left\{\frac{4}{3\left(Q^2 - 4R^2\right)} - \frac{1}{6\left(Q^2 - R^2\right)}\right\}} \qquad (2.3b)$$

Upon determining $t_o^{(1)}$ from (2.3a,b), the maximum amplitude ξ_{1a} of $\xi_1(\bar{t})$ (or of $\eta(t,\tau)$) is obtained η^{20} at its critical point to obtain

$$\xi_{1a} = -\frac{a_2 Q^2 B_0^2 M_1}{1 - \lambda};$$
$$M_1 = \left[\alpha_{20}\cos Qt_o^{(1)} + \frac{\cos 2Rt_o^{(1)}}{2(Q^2 - 4R^2)} - \frac{2\cos Rt_o^{(1)}}{Q^2 = R^2} + \frac{3}{2Q^2}\right] \qquad (2.4)$$

The condition for maximum amplitude ξ_{2a} of $\xi_2(\bar{t})$ or (of $\xi(t,\tau)$) is obtained by substituting ζ for η in (2.2a). For terms of order $\in$, this give

$$\xi_{t01}^{11}\left(t_0^{(2)}, 0\right) = 0; \quad t_o^{(2)} = \frac{\pi}{R} \qquad (2.5a)$$

Where we have taken the least nontrivial value of $t_o^{(2)}$. Similar evaluations give

$$t_{01}^{(2)} = t_{10}^{(2)} = t_{11}^{(2)} = 0 \quad (2.5b)$$

Thus the maximum value ζ_{2a} is

$$\xi_{2a} = 2B_o \in -\frac{\in a_1 B_0^3 M_2 R^2}{1-\lambda} + 0\left(\in \delta^2\right) + 0\left(\in^3 \delta^2\right) \quad (2.6a)$$

$$M_2 = \left[\frac{I_{00}}{R^2} + \frac{I_{20}}{3R^2} + \frac{I_{30}}{8R^2} + \frac{I_{40}\cos\left(\frac{Q\pi}{R}\right)}{Q(2R+Q)} - \frac{\cos\left(\frac{Q\pi}{R}\right)}{Q(2R-Q)} - \frac{Q^2\alpha_{20}}{R^2-Q^2}\right] \quad (2.6b)$$

The net maximum amplitude ζ_m is

$$\xi_m = \xi_{1a} + \xi_{2a} = c_1 \in + c_3 \in^3 + ... \quad (2.7a)$$

where

$$c_1 = 2B_o; c_2 = \frac{a_2 B_0^2 Q^2 M_1}{1-\lambda}; c_3 = -\frac{a_2 B_0^2 M_2 R^2}{1-\lambda} \quad (2.7b)$$

3.0 Dynamic buckling load λ_D

We determine the dynamic buckling load from the maximisation given in (1.8). For reasons noted in [3,13,14], this is accomplished by first reversing the series (2.7a,b) in the form

$$\in = s_1\xi_m + s_2\xi_m^2 + s_3\xi_m^3 + ... \quad (3.1a)$$

By substituting for ζ_m in (3.1a) from (2.7b) and equating the coefficients of powers of $\in$, we get

$$s_1 = \frac{1}{c_1}, s_2 = -\frac{c_2}{c_1^3}, s_3 = \frac{2c_2^2 - c_1c_3}{c_1^5} \quad (3.2b)$$

The final results are here presented in two levels of approximations. First by taking the two terms in (2.7b) and (3.1a) and carrying out the maximization in (1.8), we get

$$\xi_m(\lambda_D) = -\frac{s_1}{2S_2} = \frac{c_1^2}{2c_2} \quad (3.3c)$$

Which is evaluated at buckling. By now evaluating (3.1a) (first two terms) at buckling and letting $\in_D$ be the value of $\in$ at buckling we have

$$\in_D = \frac{c_1}{4c_2} \quad (3.3d)$$

On simplifying (3.3d) we get

$$(1-\lambda_D)^2 = 2a_2|\xi_2|\lambda_D\left(\frac{\omega_2}{\omega_0}\right)^2\left(\frac{\omega_1}{\omega_2}\right)^2\left[\alpha_{20}\cos Qt_0^0 + \frac{\cos 2Rt_0^1}{2\left(Q^2-4R^2\right)} - \frac{2\cos Rt_0^1}{\left(Q^2-4R^2\right)} + \frac{3}{2Q^2}\right. \quad (3.4)$$

We note that (3.4) furnishes us with the expression for evaluating the dynamic buckling load λ_D. We also note that the effects of both the quadratic term $a_1\zeta_1^2$ and the coupling term $\zeta_1\zeta_2$ are conspicuously absent in (3.4). This is due to our having dropped the imperfection parameter $\bar{\xi}_1$. As noted in [13] if we had dropped the imperfection parameter $\bar{\xi}_2$ instead of $\bar{\xi}_2$, the ensuing result would have contained the effects of the quadratic term $a_1\xi_1^2$ and excluded the effects of both the quadratic term $a_2\xi_2^2$ and the coupling term $\xi_1\xi_2$. The only condition under which the combined effects of both quadratic terms $a_1\xi_1^2$ and $a_2\xi_2^2$ as well as the effects of coupling are simultaneously felt is if both imperfection parameters are not deleted. In the case under study (that is the case where $\bar{\xi}_1 = 0$), buckling is initiated by the nonlinear term $a_2\xi_2^2$ and not by the quadratic (nonlinear) term $a_1\xi_1^2$ nor by the coupling term $\xi_1\xi_2$. The effects of coupling is thus conspicuously absent. We can relate the dynamic buckling load λ_D to the static buckling load λ_s. This is done [1-3] by dropping the inertia terms as well as the quadratic term $a_1\xi_1^2$ in the governing equations (1.3) to (1.6). Thus we have

$$\xi_o = \lambda; \xi_1(1-\lambda) + a_2\xi_2^2 = 0; \xi_2(1-\lambda) + \xi_1\xi_2 - \lambda\bar{\xi}_2 \tag{3.5a}$$

Finally the static buckling load λ_s follows from the maximisation $\frac{d\lambda}{d\xi_2} = 0$ and yields

$$(1-\lambda_s)^2 = \frac{3\lambda\sqrt{3a_2}\left|\bar{\xi}_2\right|}{2} \tag{3.5a}$$

From (3.5a) and (3.4), we obtain

$$\left(\frac{1-\lambda_D}{1-\lambda_s}\right)^2 = \frac{4\sqrt{a_2}}{3\sqrt{3}}\left(\frac{\lambda_D}{\lambda_s}\right)\left(\frac{\omega_2}{\omega_o}\right)^2\left(\frac{\omega_1}{\omega_o}\right)^2\left[\alpha_{20}\cos Qt_o^{(t)} + \frac{\cos 2Rt_o^{(1)}}{2(Q^2-4R^2} - \frac{2\cos Rt_o^{(1)}}{(Q^2-R^2} + \frac{3}{2Q^2}\right] \tag{3.6}$$

As reported in [4] the result corresponding to (3.6) obtained by Danielson for the case of step loading is

$$\frac{\lambda_D}{\lambda_s} = \frac{\frac{1}{6}\left(4 - \frac{\omega_o^2}{\omega_2^2}\right)}{\frac{\lambda_s}{\lambda_c} + \left(\frac{32}{27}\right)^2\left(\frac{10}{9}\right)\left(\frac{\omega_1}{\omega_o}\right)\left(\frac{\omega_2}{\omega_o}\right)\left(1 - \frac{\lambda_s}{\lambda_s}\right)^2} \tag{3.7}$$

where λ_c is the classical buckling load.

A rather more elaborate refinement of the result (3.6) can be made by using three terms in (3.10a,b). In this case the maximisation (1.8) yields, through (3.1a),

$$s_1 + 2s_2\xi_m + 3s\xi_m^2 = 0 \tag{3.8a}$$

From where we obtain

$$\xi_m(\lambda_D) = \frac{1}{3s_3}\left\{-s_2 \pm \left(s_2^2 - 3s_1s_3\right)^{\frac{1}{2}}\right\} \tag{3.8b}$$

From (3.11a), we obtain

$$s_1 = \frac{1}{2B_o}, s_2 = \frac{a_2Q^2M_1}{8(1-\lambda_D)B_o}; s_3 = \frac{3R^2a_2M_2M_3}{32\lambda_D}; M_3 = 1 + \frac{2a_2M_1^2}{3M_2R^2(1-\lambda_D)} \tag{3.8c}$$

A straightforward evaluation, using (3.8c) gives

$$\xi_m(\lambda_D) = \frac{4(1-\lambda_D)^{\frac{1}{2}} M_4(\lambda_D)}{3R(a_2 M_1 M_3)^{\frac{1}{2}}}; M_4 = -\frac{a_2^2 Q^2 M_1}{3R\{(1-\lambda_D)M_2 M_3\}^{\frac{1}{2}}} \pm \left\{\frac{a_2 (QM_1)^2}{9\{(1-\lambda_D)R^2 M_2 M_3\}}\right\}^{\frac{1}{2}} \tag{3.8d}$$

where the appropriate sign is chosen. From (58a), we get

$$3\in = \xi_m\left(3s_1 + 3s_2\xi_m + 3s_3\xi_m^2\right) \tag{3.8e}$$

evaluated at buckling. By making $3\xi_m^2$ the subject in (3.8a) and substituting same in (3.8e) and simplifying we get

$$3\bar{\xi}_2 = \frac{2\xi_m}{c_1}\left(1 - \frac{c_2\xi_m}{2c_1^2}\right) \tag{3.8f}$$

which is evaluated at buckling. A simplification of (3.8f) using (3.8d) yields

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{9\left(\frac{\omega_2}{\omega_o}\right)^2 \left|\bar{\xi}_2\right| \lambda_D (a_2 M_2 M_3)^{\frac{1}{2}}}{4M_4}\left[\frac{(1-\lambda_D)^{\frac{3}{2}} M_4}{6\left(\frac{\omega_2}{\omega_o}\right)\lambda_D (a_2 M_2 M_3)^{\frac{1}{2}}}\right]^{-1} \tag{3.9a}$$

We now use (3.5b) in (3.9a) to get

$$\frac{(1-\lambda_D)^{\frac{3}{2}}}{(1-\lambda_D)^2} = \frac{3\left(\frac{\omega_2}{\omega_o}\right)^2}{2\sqrt{3a_2 M_4}}\left(\frac{\lambda_D}{\lambda_s}\right)\left[1 - \frac{(1-\lambda_D)^{\frac{3}{2}} M_4}{6\left(\frac{\omega_2}{\omega_o}\right)\lambda_D (a_2 M_2 M_3)^{\frac{1}{2}}}\right]^{-1} \tag{3.9b}$$

where (3.9a,b) are evaluated at $\lambda = \lambda_D$

4.0 Summary of Results

We expect the results (3.9a,b) to be better representatives of (3.4) and (3.6) respectively. All the results are valid provided $Q \neq R$, $Q \neq 2R$, $Q \neq 0$ and $R \neq 0$. Except for the λ_D on the right hand side of (3.4) all other terms there are independent of the load parameter so that any evaluation of AD is a routine exercise. The results (6.9a,b) are implicit in λ_D especially through. The results to be principally initiated by the quadratic term $a_2\xi_2^2$. The coupling effect are not felt. Analysis reported elsewhere [13] indicates that if we had neglected the imperfection parameter $\bar{\xi}_2$, buckling would have been dominated by the quadratic term with the effects of both the coupling term $a_1\xi_2$ and the quadratic term $a_2\xi_2^2$ being absent or minimal. The same report also shows that by neglecting the imperfection parameter we automatically neglect the effects of the quadratic term $a_1\xi_1^2$. Similarly by neglecting the imperfection parameter $\bar{\xi}_2$, we automatically neglect the effects of the quadratic term $a_2\xi_2^2$. The converses of these observations are not true.

Thus neglecting both $a_1\xi_1^2$ and $\bar{\xi}_2$ (Danielson's assumption [4,10] appears indeed to be superfluous because a neglect of necessarily implies a neglect of the quadratic term (the converse not being true). It therefore stands to be concluded that for the problem at hand (and perhaps similar other cases) the only condition under which the effects of any nonlinearity of any buckling mode is felt is if the imperfection in the shape of the nonlinearity is not neglected. Similarly the coupling effects of the buckling modes are felt only if the imperfections in the shape of the coupled buckling modes are not neglected. All results are of course valid for condition at the immediate post dynamic buckling consideration.

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