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On the dynamic buckling of a weakly damped nonlinear elastic model system under a slowly varying explicitly time dependent load

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Abstract

In this paper we determine the dynamic buckling load of a strictly nonlinear but weakly damped elastic oscillatory model structure subjected to small perturbations The loading history is explicitly time dependent and varies slowly with time over a natural period of oscillation of the structure. A multiple timing regular perturbation method is used in asymptotic expansions of the variables. The elastic model structure is itself a generalization of most physical elastic structures in common use in Structural Engineering .The dynamic buckling load is obtained nontrivially and compared with related previous results of similar loading conditions. The result shows that the dynamic buckling load does not depend on any particular form of the loading function but depends on the first and second derivatives of the loading function evaluated at the initial

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1.0 Introduction

In this paper we are concerned with an asymptotic solution of a strictly nonlinear oscillatory and dissipative system of second order where the coefficients are explicitly time dependent and the loading varies slowly with time over a natural period of vibration of the system. Though the setting is in Mathematics, it is pertinent to point out that the original arena of inquiry where dynamical systems with slowly varying parameters was first analyzed was in Physics [1,2] where it was analyzed in connection with waves, rigid bodies, charged particles etc. In one of such areas of application, Kevorkian and Li [2] used this technique to study free-electron lasers (FEL) – a Mathematical problem dealing with general strictly nonlinear oscillators whose early studies were mostly Quantum Mechanical. [3]. In other areas of investigation, Li [4] investigated free –electron lasers with variable parameter wigglers using the same technique while Li and Kevorkian [5] studied the effects of wiggler taper rate and signal field gain rate in free –electron lasers.

Mathematical problem with slowly varying parameters started with the investigation by Kuzmak [6] .This was extended to higher order by Luke [7] in his work on nonlinear nearly periodic waves. The present study is therefore an application of these early investigations to dynamic buckling where the loading history is explicitly time dependent and varies slowly with time over a natural period of oscillation of the structure.

Most existing dynamic buckling investigations have tended to discuss nonlinear dynamical systems where the loading history is implicit in time. Such loadings include step loading [8-11], rectangular load [10] and impulsive loading [10] .Because the ensuing differential equations in most cases are autonomous ,a relatively easy method such as phase plane analysis is usually used to analyzed the problems .Consideration of buckling cases where time is explicit are relatively few. These include triangular load [10] and periodic load [12] .In one of such attempts, Svalbonas and Kalnins [12] developed a computer program for solving such problems and used same to determine the dynamic buckling load of a spherical shell. Similarly, Aksogan and Sofiyev [13] analyzed a case where cylindrical shells were subjected to a time dependent external pressure varying as a power function of time.

2.0 Formulation

A relatively simple elastic model structure that amply captures the essence of our objective [8-11] is a two-armed simply supported structure (column) (Figure 1) subjected to a loading F(T) at time T= 0. The structure is assumed rigid and weightless and carries a mass M at the center. The motion of M is restrained by a nonlinear "softening " spring that provides a force $KL\left(x-\beta x^3\right)$, where K >0, $\beta > 0$ and where L is the length and K is the spring constant while x is the central hinge displacement from the equilibrium position. By assuming a weak viscous damping given proportional to the velocity and assuming small angular displacement $(\cos\phi \cong 1 \ , \sin\phi \cong \phi)$, the relevant differential equation is easily found to be

$$M\frac{d^2x}{dT^2}+Q\frac{dx}{dT}+\left(1-\frac{2F(T)}{KL^2}\right)\mathrm{x} - \mathrm{KL}\beta\,\mathrm{x}^3 = \frac{2\,\bar{\mathrm{x}}\mathrm{F(T)}}{\mathrm{L}} \ , \mathrm{T}>0 \qquad (2.1)$$

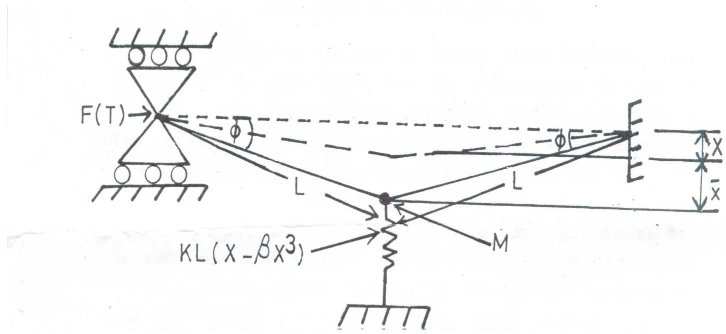

Figure 1: Simple elastic "cubic" model

where Q is the damping coefficient and $\bar{x}$ is the initial displacement which serves as initial imperfection. The following nondimensional quantities are now introduced

$$\xi=\frac{x}{L},\ \bar{\xi}=\frac{\bar{\mathrm{x}}}{\mathrm{L}}\ ,\ \bar{\mathrm{t}}=T\sqrt{\frac{\mathrm{KL}}{\mathrm{M}}}\ ,\ \lambda=\frac{2\mathrm{F}(0)}{\mathrm{KL}^2}\ ,\ \in=\frac{\mathrm{Q}}{\sqrt{\mathrm{KLM}}}\ ,\ \mathrm{f}\left(\delta\,\bar{\mathrm{t}}\right)=\frac{F(T)}{F(0)}\ ,\left(\mathrm{F}(0)\neq 0\right)$$

$$\text{where } 0<\in<1 \text{ and } 0<\delta<<1\ ,\mathrm{b}=\beta\mathrm{L}^2$$

Thus the nondimensional form of (2.1) becomes

$$\ddot{\xi}+\in\dot{\xi}+\left(1-\lambda f(\delta\,\bar{\mathrm{t}})\right)\xi-\mathrm{b}\xi^3=\lambda\bar{\xi}f(\delta\,\bar{\mathrm{t}})\ ,\ \bar{\mathrm{t}}>0 \qquad (2.2a)$$

$$\xi(0)=\dot{\xi}(0)=0 \qquad (2.2b)$$

where $\frac{d(\ )}{d\bar{t}}\equiv(\bullet)$. Here b is the imperfection-sensitivity parameter which, for a "softening" spring which we are considering is such that b>0. Without loss of generality and for ease of further analysis we shall automatically set b = 1 in (2.1). We note that λ is a non-dimensional load amplitude and satisfies the inequality 0 < λ < 1. It is the amplitude of the explicitly time dependent continuous and slowly varying load function $f\left(\delta\,\bar{\mathrm{t}}\right)$ which has right hand derivatives of all orders at $\bar{t}=0$ and satisfies the following conditions

$$f(0)=1\ ,\ \left|\,\mathrm{f}\left(\delta\,\bar{\mathrm{t}}\right)\right|\leq 1 \ \text{ for } \bar{\mathrm{t}}>0. \qquad (2.3)$$

The parameters $\in$, $\bar{\xi}$ and δ are considered small relative to unity and generally we have $\left|\bar{\xi}\right| << 1$. However we shall assume $0 < \xi < 1$. Because of the nonlinear term, ξ^3 the system (2a,b) thus represents a nonlinear cubic dissipative and oscillatory system whose solution we shall seek.

This investigation is an extension of similar (but viscously undamped) ones where in the case in [14], we had assumed that δ and $\in$ were mathematically not related. In the case in [15] we assumed that δ and $\in$ were related in the form $\delta = \bar{\xi}^2$.The results in both cases showed that the dynamic buckling load depends on $\dot{f}(0)$. In [16] we assumed the relation $\delta = \bar{\xi}$ and observed that unlike in the first two cases ,the dynamic buckling load depends on both $\dot{f}(0)$. and $\ddot{f}(0)$. In the present study we shall assume the relation

$$\delta = \bar{\xi} + \bar{\xi}^2 \tag{2.4}$$

and hope to determine the dynamic buckling load for this case. For simplicity of further analysis we shall set

$$\in = \bar{\xi}\ . \tag{2.5}$$

Similar buckling analyses were investigated by Popov [17], Zhu et al [18], Heinen [19] and Schenk [20]. In our quest for solution, we are to determine a particular value of λ, say λ_D, called the dynamic buckling load satisfying the Inequality $0 < \lambda_D < \lambda_s < \lambda_c \le 1$ for which the nonlinear cubic model structure buckles dynamically under the slowly varying explicitly time dependent load $f(\delta \bar{t})$. Here λ_s and λ_c are the static buckling load and the classical buckling load respectively of the undamped structure. The dynamic buckling load λ_D is here defined as the maximum load amplitude for which the solution of the problem remains bounded for all time $\bar{t} > 0$. Equation (2.2a) is in fact a generalization of most physical elastic systems such as columns on nonlinear elastic foundations, beams, cylindrical shells, plates and even toroidal shells to mention a few. For solution we shall first use a two- timing regular perturbation scheme to derive a uniformly valid asymptotic expression of the displacement $\xi(\bar{t})$. We shall next find the maximum value of $\xi(\bar{t})$ and lastly use another maximization to determine the dynamic buckling load λ_D.

Since the static and step loading results of the associates undamped structure are well known [8-11] we set straight to solve (2a, b). Now using (2.4) and (2.5) and setting *b* = 1 as earlier indicated we have

$$\ddot{\xi} + \bar{\xi}\dot{\xi} + \left\{1 - \lambda \bar{f}\left(\bar{\xi} + \bar{\xi}^2\right)\bar{t}\right\}\xi - \xi^3 = \bar{\xi}\lambda f\left\{\left(\bar{\xi} + \bar{\xi}^2\right)\bar{t}\right\},\ \bar{t} > 0\ , \tag{2.6a}$$

$$\xi(0) = \dot{\xi}(0) = 0. \tag{2.6b}$$

We now let
$$\tau = \left(\bar{\xi} + \bar{\xi}^2\right)\bar{t}\quad ;\quad \frac{dt}{d\bar{t}} = \left\{1 - \lambda f\left(\bar{\xi} + \bar{\xi}^2\right)\bar{t}\right\}^{\frac{1}{2}} = \left(1 - \lambda f(\tau)\right)^{\frac{1}{2}}. \tag{2.7}$$

Thus we have
$$\dot{\xi} = (1 - \lambda f)^{\frac{1}{2}}\xi_t + \left(\bar{\xi} + \bar{\xi}^2\right)\xi_\tau , \tag{2.8a}$$

$$\ddot{\xi} = (1 - \lambda f)\xi_{tt} + 2(1 - \lambda f)^{\frac{1}{2}}\left(\bar{\xi} + \bar{\xi}^2\right)\xi_{t\tau} + \left(\bar{\xi} + \bar{\xi}^2\right)^2\xi_{\tau\tau} - \frac{\left(\bar{\xi} + \bar{\xi}^2\right)\lambda f'(1 - \lambda f)^{-\frac{1}{2}}}{2} \tag{2.8b}$$

where ξ_t and ξ_τ indicate partial derivatives with respect to the given arguments and $\frac{d(\ )}{d\tau} \equiv (\ )'$. We now assume the following

$$\xi(\bar{t}) = \sum_{i=1}^{\infty} \xi^{(i)}(t,\tau)\,\bar{\xi}^{\mathrm{i}} \qquad (2.9)$$

and substitute (2.8a,b) and (2.9) into (2.6a,b) and equate to zero the coefficients of powers of $\bar{\xi}^{\,i}$, $\mathrm{i} = 1,2,3,\cdots$ and get

$$N\xi^{(1)} = \xi^{(1)}_{tt} + \xi^{(1)} = \frac{\lambda f}{(1-\lambda f)} \equiv B(\tau), \qquad (2.10)$$

$$N\xi^{(2)} = -2\,(1-\lambda f)^{-\frac{1}{2}}\,\xi^{(1)}_{t\tau} + \frac{\lambda f'\,\xi^{(1)}_{\mathrm{t}}}{2\,(1-\lambda f)^{\frac{3}{2}}} - (1-\lambda f)^{-\frac{1}{2}}\,\xi^{(1)}_{t}, \qquad (2.11)$$

$$N\xi^{(3)} = -2\,(1-\lambda f)^{-\frac{1}{2}}\,\xi^{(2)}_{t\tau} + \frac{\lambda f'\,\xi^{(2)}_{\mathrm{t}}}{2\,(1-\lambda f)^{\frac{3}{2}}}$$

$$-(1-\lambda f)^{-\frac{1}{2}}\,\xi^{(2)}_{t} - 2\,(1-\lambda f)^{-\frac{1}{2}}\,\xi^{(1)}_{t\tau} - (1-\lambda f)^{-1}\xi^{(1)}_{\tau\tau} + \frac{\lambda f'\,\xi^{(1)}_{\mathrm{t}}}{2\,(1-\lambda f)^{\frac{3}{2}}} \qquad (2.12)$$

$$-(1-\lambda f)^{-1}\xi^{(1)}_{\tau} + (1-\lambda f)^{-1}\left(\xi^{(1)}\right)^{3}.$$

etc., and the associated initial conditions evaluated at (t,τ) = (0,0) are

$$\xi^{(i)} = 0\,, \mathrm{i} = 1,2,3,\cdots;\ \xi^{(1)}_{t} = 0\,,. \qquad (2.13a)$$

$$\xi^{(2)}_{t} + (1-\lambda)^{-\frac{1}{2}}\,\xi^{(1)}_{\tau} = 0\,, \qquad (2.13b)$$

$$\xi^{(3)}_{t} + (1-\lambda)^{-\frac{1}{2}}\left\{\xi^{(2)}_{\tau} + \xi^{(1)}_{\tau}\right\} = 0 \qquad (2.13c)$$

The solution of (2.10) subject to (2.13a) for I = 1 is

$$a_1(\tau)\cos t + b_1(\tau)\sin t + B. \qquad (2.14a)$$

$$a_1(0) = -B(0) = -\frac{\lambda}{1-\lambda};\ \mathrm{b}_1(0) = 0 \qquad (2.14b)$$

Henceforth we shall let

$$B_0 = B(0)\,. \qquad (2.14c)$$

On substituting into (2.11) and equating the coefficients of cost and *sint* we obtain respectively

$$b_1' - b_1\left[\frac{\lambda\,\mathrm{f}'}{4\,(1-\lambda \mathrm{f})} + \frac{1}{2}\right] = 0,\ . \qquad (2.15a)$$

and

$$\mathrm{a}_1' - a_1\left[\frac{\lambda\,\mathrm{f}'}{4\,(1-\lambda \mathrm{f})} + \frac{1}{2}\right] = 0 \qquad (2.15b)$$

The solution of (2.15a) using (2.14a) is

$$b_1(\tau) = 0\,. \qquad (2.16a)$$

The solution of (2.15b) subject to (2.14b) is

$$a_1(\tau) = -B_0 e^{-\frac{\tau}{2}}\left(\frac{1-\lambda}{1-\lambda f}\right)^{\frac{1}{4}} . \tag{2.16b}$$

The remaining equation in (2.11) is now solved to get

$$\xi^{(2)}(t,\tau) = a_2(\tau)\cos t + b_2(\tau)\sin t\,,. \tag{2.17a}$$

$$a_2(0) = 0\,,\ \mathrm{b}_2(0) = -\frac{\mathrm{B}_0}{4(1-\lambda)^{\frac{3}{2}}}\left[4f'(0) - (2-\lambda)\right] \tag{2.17b}$$

We now substitute into (2.12) and equate to zero the coefficients of cost and *sint* and get respectively

$$b_2' - b_2\left(\frac{\lambda\,\mathrm{f}'}{4\,(1-\lambda\mathrm{f})} + \frac{1}{2}\right) = -\frac{(1-\lambda\mathrm{f})^{-\frac{1}{2}}(a_1'' + a_1')}{2} + \frac{(1-\lambda\mathrm{f})^{\frac{1}{2}}}{2}\left(3B^3 a_1 + \frac{3a_1^3}{4}\right) \tag{2.18a}$$

and

$$a_2' - a_2\left(\frac{\lambda\,\mathrm{f}'}{4\,(1-\lambda\mathrm{f})} + \frac{1}{2}\right) = -\frac{1}{2}\left(a_1' - \frac{\lambda\,\mathrm{f}'\,\mathrm{a}_1}{2\,(1-\lambda\mathrm{f})}\right). \tag{2.18b}$$

The integrating factor of (2.18a, b) is $(1-\lambda f)^{\frac{1}{4}} e^{-\frac{\tau}{2}}$ and the solution of (2.18a) subject to (2.17b) is

$$b_2(\tau) = (1-\lambda f)^{-\frac{1}{4}} e^{\frac{\tau}{2}}\left[\int_0^{\tau}\frac{1}{2}(1-\lambda f)^{-\frac{1}{4}} e^{-\frac{s}{2}}(a_1'' + a_1')ds \right.$$

$$\left. + \frac{3}{2}\int_0^{\tau}(1-\lambda f)^{\frac{3}{4}} e^{-\frac{s}{2}}\left(B^2 a_1 + \frac{a_1^3}{4}\right)ds + d_1\right] \tag{2.19a}$$

where

$$d_1 = b_2(0)(1-\lambda)^{\frac{1}{4}}. \tag{2.19b}$$

Similarly the solution of (2.18b) is

$$a_2(\tau) = (1-\lambda f)^{-\frac{1}{4}} e^{\frac{\tau}{2}}\int_0^{\tau}\left[\left(a_1' - \frac{\lambda\,\mathrm{f}'\,\mathrm{a}_1}{2\,(1-\lambda\mathrm{f})}\right)(1-\lambda\mathrm{f})^{\frac{1}{4}} e^{-\frac{s}{2}}\right]ds. \tag{2.19c}$$

The remaining equation in (2.12) is

$$N\xi^{(3)} = D(\tau) + \frac{3(1-\lambda f)^{-1} B\,a_1^2\cos 2t}{2} + \frac{(1-\lambda f)^{-1} a_1^3\cos 3t}{4}. \tag{2.20a}$$

where

$$D(\tau) = (1-\lambda f)^{-1}\left\{B^3 + \frac{3Ba_1^2}{2} - (B'' + B')\right\} \tag{2.20b}$$

Thus we have

$$\xi^{(3)}(t,\tau) = a_3(\tau)\cos t + b_3(\tau)\sin t + D - (1-\lambda f)^{-1}\left[\frac{B\,a_1^2}{2}\cos 2t + \frac{a_1^3\cos 3t}{2}\right], \quad (2.20c)$$

$$a_3(0) = -\frac{65B_0^3}{2(1-\lambda)} + \frac{B_0}{(1-\lambda)^2}\left[f''(0) + 2B_0(f'(0))^2\right], \quad (2.20d)$$

$$b_3(0) = -\frac{B_0}{4(1-\lambda)}\left[4f'(0)\left((1-\lambda)^{\frac{1}{2}} - 1\right) + 2(1-\lambda)\right] \quad (2.20e)$$

We now write $\xi(\bar{t}) = \xi(t,\tau) = \bar{\xi}\xi^{(1)} + \bar{\xi}^2\xi^{(2)} + \bar{\xi}^3\xi^{(3)} + \cdots$ (2.21)

We shall now determine the maximum displacement $\xi(\bar{t}_a) = \xi(t_a,\tau_a)$ where $\bar{t}_a, \tau_a$ and t_a are the critical values of the associated time variables at maximum displacement. We shall assume the following asymptotic expansions

$$\bar{t}_a = \bar{t}_0 + \bar{\xi}\bar{t}_1 + \bar{\xi}^2\bar{t}_2 + \cdots \quad (2.22a)$$

$$t_a = t_0 + \bar{\xi}t_1 + \bar{\xi}^2 t_2 + \cdots \quad (2.22b)$$

$$\tau_a = \left(\bar{\xi} + \bar{\xi}^2\right)\left(\bar{t}_0 + \bar{\xi}\bar{t}_1 + \bar{\xi}^2\bar{t}_2 + \cdots\right). \quad (2.22c)$$

From (2.8a) the condition for maximum displacement is

$$\xi_t + (1-\lambda f)^{1/2}\left(\bar{\xi} + \bar{\xi}^2\right)\xi_\tau = 0 \quad (2.23a)$$

evaluated at the critical values $\bar{t}_a, \tau_a$, and t_a. By substituting for $\bar{t}_a, \tau_a$ and t_a from (2.22a-c) and equating to zero the coefficients of powers of $\bar{\xi}^i$ $i = 1,2,3,\cdots$ we get, for the coefficient of $\bar{\xi}$

$$\xi_t(t_0, 0) = 0. \quad (2.23b)$$

This gives $t_0 = r\pi$; $r = 0,1,2,3,\cdots$ (2.23c)

We need the least nontrivial value of t_0 and so we set $r = 1$ and get

$$t_0 = \pi \quad (2.23d)$$

From the coefficient of $\bar{\xi}^2$, we get using (2.23d)

$$t_1\xi_{tt}^{(1)} + \xi_t^{(2)} + (1-\lambda)^{-\frac{1}{2}}\xi_\tau^{(1)} = 0. \quad (2.23e)$$

This gives

$$t_1 = -\frac{\left(\xi_t^{(2)} + (1-\lambda)^{-\frac{1}{2}}\xi_\tau^{(1)}\right)}{\xi_{tt}^{(1)}} = \frac{2f'(0)}{(1-\lambda)^{\frac{3}{2}}}. \quad (2.23f)$$

To determine the maximum displacement ξ_a, we evaluate (2.21) at the critical values and get

$$\xi_a = 2B_0\bar{\xi} + \frac{B_0\bar{t}_0(2-\lambda)\bar{\xi}^2}{4(1-\lambda)}\left\{1 + \frac{16(1-\lambda)f'(0)}{B_0\,\bar{t}_0(2-\lambda)}\right\}$$

$$+\frac{4B_0^3\bar{\xi}^3}{(1-\lambda)}\left[1-\frac{(1-\lambda)\{f''(0)+2B_0(f'(0))\}^2}{4\lambda^2}\right]+\cdots \tag{2.24}$$

To determine $\bar{t}_0$ and $\bar{t}_1$ we know from (2.7) that

$$t_a = \int_0^{\bar{t}_a}\left\{1-f\left(\bar{\xi}+\bar{\xi}^2\right)s\right\}^{\frac{1}{2}}ds$$

$$= (1-\lambda)^{\frac{1}{2}}\left[\bar{t}_a-\frac{\lambda}{2(1-\lambda)}\right]\left\{\frac{\left(\bar{\xi}+\bar{\xi}\right)^2\bar{t}_a^2 f'(0)}{2}+\frac{\left(\bar{\xi}+\bar{\xi}\right)^2\bar{t}_a^3 f''(0)}{2}\right\}+\cdots \tag{2.25a}$$

On substituting (2.22a) into (2.25) and equating the coefficients of powers of $\bar{\xi}^i, i=1,2,$ we get respectively

$$\bar{t}_0 = (1-\lambda)^{-\frac{1}{2}}t_0\ ;\quad \bar{t}_1=(1-\lambda)^{-\frac{1}{2}}t_1+\frac{\lambda\,f'(0)\bar{t}_0}{2(1-\lambda)}$$

$$= (1-\lambda)^{-\frac{1}{2}}t_1+\frac{\lambda\,f'(0)t_0}{2(1-\lambda)^{\frac{3}{2}}}\ . \tag{2.25b}$$

For further simplification of (2.24) we shall now let

$$F_1(\lambda)=1+\frac{16(1-\lambda)f'(0)}{B_0\bar{t}_0(2-\lambda)}\ ;\quad F_2(\lambda)=1-\frac{(1-\lambda)\left\{f''(0)+2B_0(f'(0))^2\right\}}{4\lambda^2} \tag{2.26a}$$

so that we have

$$\xi_a = 2B_0\bar{\xi}+\frac{B_0\bar{t}_0(2-\lambda)\bar{\xi}^2F_1}{4(1-\lambda)}+\frac{4B_0^3\bar{\xi}^3F_2}{(1-\lambda)}+0\left(\bar{\xi}^4\right). \tag{2.26b}$$

Again for ease of further analysis we let

$$C_1 = 2B_0\ ,\quad C_2=\frac{B_0\bar{t}_0(2-\lambda)F_1}{4(1-\lambda)}\ ,\quad C_3=\frac{4B_0^3F_2}{(1-\lambda)}. \tag{2.26c}$$

Thus we have

$$\xi_a = C_1\bar{\xi}+C_2\bar{\xi}^2+C_3\bar{\xi}^3+0\left(\bar{\xi}^4\right). \tag{2.26d}$$

Following [8-10], we now determine the dynamic buckling load λ_D from the maximization

$$\frac{d\lambda}{d\xi_a}=0\,. \tag{2.27}$$

For reasons noted in [21-23] we first have to reverse the series (2.26d) in the form

$$\bar{\xi} = e_1\xi_a+e_2\xi_a^2+e_3\xi_a^3+\cdots \tag{2.28}$$

By substituting for ξ_a in (2.28) from (2.26d) and equating the coefficients of powers of $\bar{\xi}$ we get

$$e_1=\frac{1}{C_1},\ e_2=-\frac{C_2}{C_1^3}=-\frac{\bar{t}_0(2-\lambda)F_1}{32B_0}\ ,\ C_3=\frac{2C_2^2-C_1C_3}{C_1^5}=-\frac{F_2F_3}{4\lambda} \tag{2.29a}$$

where

$$F_3 = \left[1 - \frac{\bar{t}_0^2 (2-\lambda)^2}{128(1-\lambda)\mathrm{B}_0^4 F_2}\right] \tag{2.29b}$$

and where each e_i depends on λ for I=1,2,3,…The maximization (2.27) is now accomplished through (2.28) to give

$$e_1 + 2e_2\xi_a + 3e_3\xi_a^2 = 0 \tag{2.30a}$$

which is evaluated at $\lambda = \lambda_D$. This gives

$$\xi_a = \frac{1}{3e_3}\left\{-e_2 + \left(e_2^2 - 3e_1e_3\right)^{\frac{1}{2}}\right\}. \tag{2.30b}$$

To simplify (2.30b) we note that

$$\left(e_2^2 - 3e_1e_3\right)^{\frac{1}{2}} = \left(\frac{3C_1C_3 - 5C_2^2}{C_1^6}\right)^{\frac{1}{2}} = \frac{\sqrt{3}(1-\lambda_D)^{\frac{1}{2}} F^{\frac{1}{2}} F_4}{2\sqrt{2}}. \tag{2.31a}$$

where $$F_4 = \left[1 - \frac{5\bar{t}_0(2-\lambda_D)^2 F_1^2(1-\lambda_D)}{384F_2\lambda_D^2}\right]^{\frac{1}{2}} \tag{2.31b}$$

where every function of λ is henceforth evaluated at λ_D.Therefore the simplification of (2.30b) now yields

$$\xi_a(\lambda_D) = \frac{2\sqrt{3}(1-\lambda_D)^{\frac{1}{2}} F_4F_5}{3\sqrt{2}\, F_2^{1/2} F_3}, \tag{2.32a}$$

where $$F_5 = 1 - \frac{\sqrt{2}\ \bar{t}_0(2-\lambda_D)(1-\lambda_D)^{\frac{1}{2}}}{16\sqrt{3}\, F_2^{1/2} F_4} \tag{2.32b}$$

and where we have taken the negative sign in (2.30b) because it is the negative sign that duplicates the step loading result when $f(\tau) \equiv 0$. On multiplying (2.28) through by 3 we get

$$3\bar{\xi} = 3\xi_a\left(e_1 + e_2\xi_a\right) + 3e_3\xi_a^2 \tag{2.33a}$$

We now make $3e_3\xi_a^2$ the subject in (2.30a) and substitute same into (2.33a) and get

$$3\bar{\xi} = \xi_a(2e_1 + e_2\xi_a) = \frac{2\xi_a}{C_1}\left(1 - \frac{C_2\xi_a}{2C_1^2}\right) \tag{2.33b}$$

and this is evaluated at $\lambda = \lambda_D$.On simplifying (2.33b) we get

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6}\lambda_D\bar{\xi}F_2^{1/2}F_3}{2F_4F_5F_6}. \qquad (2.34a)$$

where
$$F_6 = 1 - \frac{(2-\lambda_D)\bar{t}_0F_1F_4F_5(1-\lambda_D)^{\frac{1}{2}}}{8\sqrt{6}\ \lambda_D F_2^{1/2}F_3} \qquad (2.35b)$$

3.0 Analysis of the result

The results (2.34a, b) which are implicit in the load parameter λ_D are strictly asymptotic and are valid for small values of the parameter $\bar{\xi}$. They do not depend on any particular type of the load function $f(\delta\,\bar{t})$ provided equation (2.3) is satisfied but depend on both $f'(0)$ and $f''(0)$. This is unlike the results in [14, 15] which depend on $f'(0)$ only. For the purpose of comparison we recast equivalent and respective results obtained in [14-16] as follows:

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6}\,|\,\bar{\xi}\,|\,\lambda_D}{2}\left(\frac{1+\delta\,A_{31}(\lambda_D)}{1+\delta\,A_{11}(\lambda_D)}\right)^{\frac{1}{2}}, \qquad (3.1a)$$

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6}\,|\,\xi\,|\,\lambda_D}{2}\left[1+\frac{\pi(1-\lambda_D)^{\frac{3}{2}} f'(0)\left(1+\frac{\lambda_D}{2}\right)}{4\lambda_D^2}\right]^{\frac{1}{2}}, \qquad (3.1b)$$

$$(1-\lambda_D)^{\frac{3}{2}} = \frac{3\sqrt{6}\,|\,\bar{\xi}\,|\,\lambda_D}{2}\left(\frac{\hat{K}_2^{1/2}\hat{K}_3}{\hat{K}_1\hat{K}_5\hat{K}_6}\right); \hat{K}_6 = 1-\frac{\sqrt{3}\hat{K}_1\hat{K}_4\hat{K}_5}{24\sqrt{2}\,\lambda_D^2\hat{K}_2\hat{K}_5} \qquad (3.1c)$$

Where all other terms not explained are as defined in the cited works. The corresponding step loading result is obtained by setting $f'(0) = f''(0) = 0$ in (2.34a,b). It is observed that the step loading result is conservative.

References

[1] Kevorkian,J. Perturbation techniques for oscillatory systems with slowly varying coefficients", SIAM Rev 29,391-461 (1987).

[2] Kevorkian. and Li,Y.P., "Explicit approximations for strictly nonlinear oscillators with slowly varying parameters with applications to free-elections lasers", Studies in Applied Matheamatics, 78 (2), 111-165 (1988).

{3] Madey,J.M.J., " stimulated emission of bremsstrahlung in a periodic magnetic field", J. Appl. Phys. 42, 1906-1913 (1971).

[4] Li,Y.P., " Free electron lasers with variable parameters wigglers, a strictly nonlinear oscillator with slowly varying parameters", Ph.D. Dissertation, University of Washington, Seattle, 1987; Technical Rep.87-2,Dept of Applied Mathematics, University of Washington,(1987).

[5] Li,Y.P. and kevorkian,J., " The effects of Wiggler taper rate and signal field gain rate in electron lasers", IEEE J. Quantum Electron, 24 (1988).

[6] Kuzmak,G.E.," Asymptotic solutions of nonlinear second order differential equations with variable coefficients", Pure Math Manuscript 23,515-526 (1959).

[7] Luke,J.C., " A perturbation method for nonlinear dissipative wave problems", Pro. Roy. Soc. London Ser. A, 403-412 (1966).
[8] Budiansky,B and Hutchinson,J.W.," Dynamic buckling of elastic structures: criteria and estimates" in "Dynamic stability of structures", Pergamon, New York, 1966.
[9] Budiansky,B and Hutchinson,J.W.," Dynamic buckling of imperfection-sensitive structures", proceedings of the XIth congr of Appl. Mech, Springer-Verlag, Berlin, 1966.
[10] Hutchinson,J.W. and Budiansky,B.," Dynamic buckling estimates", AIAA J. 4 (3), 526-530 (1966).
[11] Danielson,D.," Dynamic buckling loads from perturbation procedures", AIAA J. 7(8), 1506-1510 (1969).
[12] Svalbonas,V. and Kalnins,A.," Dynamic buckling of shells: evaluation of various methods", Nuclear Eng. Des. 44,331-356 (1977).
[13] Aksogan,O. and Sofiyer,A.V.," Dynamic buckling of cylindrical shells with variable thickness subjected to a time dependent external pressure varying as a power of time", J. of Sound and vibration, 25 (4), 693-702 (2002).
[14] Ette, A. M.," On a two small parameter nonlinear (cubic) differential equation with slowly varying coefficients with application to dynamic buckling",(To appear).
[15] Atte, A. M.," On a certain one-parameter nonlinear equation with dynamically slowly varying coefficients with application to dynamic buckling (1)", (To appear).
[16] Ette,A.M.," On a certain one-parameter nonlinear equation with dynamically slowly varying coefficients with application to dynamic buckling (2)", (To appear).
[17] Popov,A.A.," Parametric resonance in cylindrical shells: A case study in nonlinear vibration of structural shells", Engineering structure 25 (6), 789-799 (2003).
[18] Zhu,E. , Mondal,P. and Calladine,C.R.," Buckling of cylindrical shells: An attempt to solve a paradox" , Int. J. of Mech. Science,44 (6), 1583-1601 (2002).
[19] Heinen, A. and Bullesbach, J. ,"On the influence of geometric imperfections on the stability and vibration of thin-walled shell structures", Int. J. Nonlinear Mech., 37 (4 and 5), 921-935 (2002).
[20] Schenk,C.A. and Schueller,G.I.," Buckling analysis of cylindrical shells with random imperfections", Int. J. Nonlinear. Mech.38, 1119-1132 (2003).
[21] Amazigo,J.C.," Buckling of stochastically imperfect columns on nonlinear elastic foundations", Quart. Appl. Math. 31 (1), 403-409 (1971).
[22] Ette,A.M.,":Dynamic buckling of a spherical shell under an axial impulse",Int.J. Non-Linear Mech 32 (1),201-209 (1997).
[23] Ette,A.M.,"Buckling of a cylindrical shell pressurized by an impulse",J. of the Nigerian Mathematical Society22 ,83-110 (2003).