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Dynamic response to moving concentrated masses of uniform Rayleigh beams resting on variable winkler elastic foundation

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Abstract

The response of uniform Rayleigh bean, carrying moving masses, resting on variable Winkler elastic foundations is investigated in this work. The governing equation is a fourth order partial differential equation. For the solution of this problem, in the first instance, Galerkin's method was used. The Galerkin equation representing the coefficient of the response is then solved using the modified asymptotic method of Struble. It is observed that the transverse deflections of the uniform Rayleigh beam under the actions of moving masses are higher than the deflections when only the force effects of the moving load are considered. Therefore, the moving force solution could be misleading. Also the analysis show that the response amplitudes of both moving force and moving mass problems decrease both with increasing Foundation modulus K and with increasing Rotatory inertia correction factor R^o.

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1.0 Introduction

The study of the behaviour of elastic solid bodies (beams, plates or shell) subjected to moving loads has been the concern of several researchers in applied Mathematics and Engineering. More specifically, several dynamical problems involving the response of beams on a foundation and without foundation have variously been tackled [1,2]. Among the earliest work in this area of study was the work of Stokes [3], who obtained an approximate solution for the response of a beam by neglecting the mass of the beam. This is because the introduction inertia effect of the moving mass would make the governing equation cumbersome to solve as reported in [4], recognizing this difficulty Pestel [5] applied Rayleigh – Ritz techniques to reduce the moving mass problem defined by a continuous differential equation to an approximate system of discrete differential equations with analytic coefficients. The system was reduced by a finite difference scheme for solution, but no numerical results were presented. After this, several researchers have approached this problem by assuming that the inertia of the moving load was negligible. In fact Arye et al [13] pointed out, in their summary of work done previous to 1952 that the fundamental mathematical difficulties encountered in the problem lie in the fact that one of the coefficients of the linear operator describing the motion is a function of space and time. They added that it is caused by the presence of a Dirac-Delta function as a coefficient necessary for a proper description of the motion. It is remarked at this juncture that, physically, this term represents the interplay of the inertial forces due to the discrete masses distributed over the structure during the motion [1]. Arye et al [13] also considered the problem of elastic beam under the action of moving loads. They assumed the mass of the beam to be smaller than the mass of the moving load and obtained an approximate solution to the problem. This is followed by the other extreme case when the mass of the load was smaller than the mass of the beam. In particular, the dynamic response of a simply supported beam traversed by a constant force moving at a uniform speed was first studied by Krylov [11]. He used method of expansion of Eigen function to obtain his results. Lowan [12] also considered the problem of transverse oscillations of beams under the action of moving loads for the general case of any arbitrary prescribed law of motion. He obtained his solution using Green's functions.

The problem of a load moving along elastic beam on an elastic foundation is of great theoretical and practical significance. Extensive theoretical and experimental investigations have been carried out, particularly, when the foundation modulus is constant along the span of the beam.

More recently, the problem of the dynamic response of a non-uniform beam resting on elastic foundation and under moving concentrated masses was tackled by Oni [8]. Analysis of his results show that the response amplitude of both moving force and moving mass decrease with increasing foundation moduli. Similarly, Oni and Omolofe [10] investigated the dynamic behaviour of a finite Bernoulli-Euler beam on an elastic foundation to masses moving at non-uniform velocities. They concluded that for all variants of classical boundary conditions, when the axial force is fixed, the displacements of a uniform Bernoulli-Euler beam resting on elastic foundation and traversed by masses travelling at varying speeds decrease as the foundation moduli increase.

In all the aforementioned investigations, problems have been largely restricted to the case when the foundation stiffness varies along the span of the station. Thus, this paper considers the response to moving concentrated masses of uniform Rayleigh beam resting on variable elastic foundation.

2.0 The governing equation

The differential equation for the deflection curve of an elastic Rayleigh beam under a moving load when the beam is of constant flexural rigidity *EI* and supported by a variable elastic foundation is given by:

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 U(x,t)}{\partial x^2}\right]+\mu\frac{\partial^2 U(x,t)}{\partial t^2}-\frac{\partial \mu R^o}{\partial x}\left[\frac{\partial^3 U(x,t)}{\partial x\partial t^2}\right]+K(x)U(x,t)=P(x,t) \qquad (2.1)$$

ehere E is the young's modulus, $U(x,\ t)$ is the transverse displacement. $K(x)$ is the variable elastic foundation, R^o is the measure of Rotatory inertia correction factor and $x,\ t$ are respectively spatial and time coordinates.

When the effect of the mass of the moving load on the beam is considered, P(x,t) takes the form:

$$p(x,t)=M\delta(x-ct)\left[g-\left(\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right)U(x,t)\right] \qquad (2.2)$$

For $K(x)$, the example in [7] shall be adopted and we have:

$$K(x)=K(4x-3x^2+x^3) \qquad (2.3)$$

At this juncture, the boundary conditions for our dynamical system are arbitrary and the initial conditions are

$$U(x,0)=0=\frac{\partial U(x,0)}{\partial t} \qquad (2.4)$$

3.0 Analytical solution procedure

Substituting equations (12) and (2.3) into (2.1), simplifying and arranging yields

$$EI\frac{\partial^4 U(x,t)}{\partial x^4}+\mu\frac{\partial^2 U(x,t)}{\partial t^2}-\mu R^0\frac{\partial^4 U(x,t)}{\partial x^2\partial t^2}$$
$$+M\delta(x-ct)\left(\frac{\partial^2}{\partial t^2}+2c\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right)Ux,t+K\left(4x-3x^2+x^3\right)U(x,t)=Mg\delta(x-ct) \qquad (3.1)$$

Evidently, the method of separation of variables is inapplicable because it becomes difficult to get separate equations where functions are functions of a single variable. In the, a closed form solutionof the above singular differential equation (3.1) does riot exist. As a result, an approximate solution is sought. Thus, the Galerkin's technique described in [7] is employed.

This elegant technique requires that the solution of the equation (3.1) eakes the form:

$$U_n(x,t) = \sum_{m=1}^{n} W_m(t)V_m(x) \tag{3.2}$$

where $Vm(x)$ is a function chosen such that the appropriate boundary conditions are satisfied. When equation (3.2) is substituted into (3.1) and simplified one obtains

$$\begin{aligned}
&\sum_{m=1}^{n}\Big\{EIV_m^{iv}(x)W_m(t) + \mu V_m(x)\ddot{W}_m(t) - \mu R^0 V_m^{ii}(x)\ddot{W}_m(t)\\
&M\delta(x-ct)\Big[V_m(x)\ddot{W}_m(t) + 2cV_m^{i}(x)\dot{W}_m(t) + c^2V_m^{ii}(x)W_m(t)\Big]\\
&+K\left(4x-3x^2+x^3\right)V_m(x)W_m(t) - Mg\delta(x-ct)\Big\} = 0
\end{aligned} \tag{3.3}$$

In order to determine $W_m(t)$ it is required that the expression on the left hand side of equation (3.3) be orthogonal to the functions $V_k(x)$. Hence

$$\begin{aligned}
&\int_0^L \Big\langle \sum_{m=1}^{n}\Big\{EIV_m^{iv}(x)W_m(t) + \mu V_m(x)\ddot{W}_m(t) - \mu R^0 V_m^{ii}(x)\ddot{W}_m\\
&+M\delta(x-ct)\Big[V_m(x)\ddot{W}_m(t) + 2cV_m^{i}(x)\dot{W}_m(t) + c^2V_m^{ii}(x)W_m(t)\Big]\\
&+K\left(4x-3x^2+x^3\right)V_m(x)W_m(t) - Mg\delta(x-ct)\Big\}\Big\rangle V_k(x)dx = 0
\end{aligned} \tag{3.4}$$

Simplification and rearrangement of (3.4) yields

$$\begin{aligned}
&\sum_{m=1}^{n}\Bigg\{\Omega_0\ddot{W}_m(t) + \left[\frac{EI}{\mu}\Omega_{1A} + \frac{K}{\mu}\left(4\Omega_{1B} - 3\Omega_{1C} + \Omega_{1D}\right)\right]W_m(t)\\
&+\frac{M}{\mu}\Big[\Omega_2(t)\ddot{W}_m(t) + 2c\Omega_3(t)\dot{W}_m(t) + c^2\Omega_4(t)W_m(t)\Big]\Bigg\} = \frac{MgV_k(ct)}{\mu}.
\end{aligned} \tag{3.5}$$

where

$$\Omega_0(m_1k) = \int_0^L V_m(x)V_k(x)dx - R^0\int_0^L V^{ii}_{m\,m}(x)V_k(x)dx;\ \Box\Omega_{1A} = \int_0^L V_m^{ii}(x)V_k(x)dx;$$

$$\Omega_{1B} = \int_0^L xV_m(x)V_k(x)dx;\ \Omega_{1C} = \int_0^L x^2V_m(x)V_k(x)dx;\ \Omega_{1D} = \int_0^L x^3V_m(x)V_k(x)dx;$$

$$\Omega_2(t) = \int_0^L \delta(x-ct)V_m(x)V_k(x)dx;\ \Omega_3(t) = \int_0^L \delta(x-ct)V^{i}_{m\,m}(x)V_k(x)dx$$

and $\Omega_4(t) = \int_0^L \delta(x-ct)V^{ii}_{m\,m}(x)V_k(x)dx.$

Using the property of the Dirac-delta function as an even function, it can easily be shown that

$$\delta(x-ct) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi x}{L} \tag{3.6}$$

When use is made of (3.6) in one obtains

$$\sum_{m=1}^{n}\left\{\Omega_0(m,k)\ddot{W}_m(t)+\Omega_1(m,k)W_m(t)+\frac{M}{L\mu}\left[\left(\Omega_{2A}(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\Omega_{2B}(n,m,k)\right)\ddot{W}_m(t)\right.\right.$$

$$+2c\left(\Omega_{3A}(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\Omega_{3B}(n,m,k)\right)W_m(t)$$

$$\left.\left.+c^2\left(\Omega_{4A}(m,k)+2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\Omega_{4B}(n,m,k)W_m(t)\right)W_m(t)\right]\right\}=\frac{MgV_k(ct)}{\mu} \tag{3.7}$$

where

$$\Omega_1(m,k)=\frac{EI}{\mu}\Omega_{1A}+\frac{K}{\mu}[4\Omega_{1B}-3\Omega_{1C}+\Omega_{1D}]\Omega_{2A}(m,k)=\int_0^L V_m(x)V_k(x)dx;$$

$$\Omega_{3A}(m,k)=\int_0^L V_m^i(x)V_k(x)dx;\ \ \Omega_{4A}(m,k)=\int_0^L V_m^{ii}(x)V_k(x)dx;$$

$$\Omega_{2B}(n,m,k)=\int_0^L\cos\frac{n\pi x}{L}V_m(x)V_k(x)dx;\ \ \Omega_{3B}(n,m,k)=\int_0^L\cos\frac{n\pi x}{L}V_m^i(x)V_k(x)dx;$$

and $\Omega_{4A}(n,m,k)=\int_0^L\cos\frac{n\pi x}{L}V_m^{ii}(x)V_k(x)dx.$

It is remarked at this juncture that as the boundary conditions are arbitrary. The most suitable form of function $V_m(x)$ is the beam function:

$$V_m(x)=\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sin\text{h}\frac{\lambda_m x}{L}+C_m\cos h\frac{\lambda_m x}{L} \tag{3.8}$$

where the constant A_m, B_m, C_m and the mode number λ_m are determined by using desired ends support conditions. Thus substituting (3.8) into equation (3.7) yields

$$R_0(m,k)\ddot{W}_m(t)+R_1(m,k)W_m(t)+\Gamma\left[\left(R_{2A}(m,k)+R_{2B}(n,m,k)\ddot{W}_m(t)\right)\right.$$

$$\left.+2c\left(R_{3A}(m,k)+R_{3B}(n,m,k)\dot{W}_m(t)+c^2\left(R_{4A}(m,k)+R_{2B}(n,m,k)\right)W_m(t)\right)\right.$$

$$=\frac{Mg}{\mu}\left[\sin\frac{\lambda_k ct}{L}+A_k\cos\frac{\lambda_k ct}{L}+B_2\sin\text{h}\frac{\lambda_k ct}{L}\right] \tag{3.9}$$

where

$$\Gamma=\frac{M}{L\mu} \tag{3.10}$$

$$R_{2B}(n,m,k)=2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\Omega_{2B}(n,m,k);\ \ R_{3B}(n,m,k)=2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\Omega_{3B}(n,m,k);$$

$$R_{4B}(n,m,k)=2\sum_{n=1}^{\infty}\cos\frac{n\pi ct}{L}\Omega_{4B}(n,m,k);$$

and Ω_0 (m,k), $\Omega_1(m,k)$, $\Omega_{2A}(m,k)$, $\Omega_{2B}(n,m,k)$, $\Omega_{3A}(m,k)$, $\Omega_{3B}(n,m,k)$, $\Omega_{4A}(m,k)$ and $\Omega_{4B}(n,m,k)$ become respectively $R_0(rn,k)$, $R_1(m,k)$, $R_{2A}(m,k)$, $\Omega_{2B}(n,rn,k)$, $R_{3A}(m,k)$, $\Omega_{3B}(n,m,k)$, $R_{4A}(m.k)$ and $\Omega_{4B}(n,m,k)$ after the substitution of(3.8) and summation sign neglected.

Equation (3.9) is the transformed equation governing the problem of a uniform Rayleigh beam on a variable Winkler elastic foundation. This second order differential equation holds for all variants of the classical boundary conditions.

Evidently, an exact solution to equation (3.9) is not possible. Consequently, a modification of Struble's technique described in [9] is employed. Huts equation (3.9) is rearranged to take the form:

$$\ddot{W}_m(t) + \frac{2\Gamma c\left[R_{3A}(m,k) + R_{3B}(n,m,k)\right]\dot{W}_m(t)}{R_0(m,k) + \Gamma\left[R_{2A}(m,k) + R_{2B}(n,m,k)\right]} + \frac{\left\{R_1(m,k) + \Gamma c^2\left[R_{4A}(m,k) + R_{4B}(n,m,k)\right]\right\}W_m(t)}{R_0(m,k) + \Gamma\left[R_{2A}(m,k) + R_{2B}(n,m,k)\right]} = \frac{\Gamma g L\left[\sin\frac{\lambda_k ct}{L} + A_k\cos\frac{A_k ct}{L} + B_k \sin h\frac{\lambda_k ct}{L} + c_k \cos h\frac{\lambda_k ct}{L}\right]}{R_0(m,k) + \Gamma\left[R_{2A}(m,k) + R_{2B}(n,m,k)\right]} \tag{3.11}$$

First, we shall consider the homogenous part of (3.11) and obtain a modified frequency corresponding to the frequency of the free system due to presence of the moving mass- An equivalent free system operator defined by the modified frequency then replaces equation (3.11). To this end, the right-hand side of (3.11) is set to zero and a parameter $\mathcal{E} < 1$ is considered for any arbitrary mass ratio r defined as:

$$\varepsilon = \frac{\Gamma}{1+\Gamma} \tag{3.12}$$

It is then clear that

$$\Gamma = \varepsilon + o(\varepsilon^2) \tag{3.13}$$

Consequently,

$$\frac{1}{R_0(m,k) + \varepsilon\left[R_{2A}(m,k) + R_{2B}(n,m,k)\right]} = \frac{1}{R_0(m,k)}\left\{1 - \frac{\varepsilon}{R_0(m,k)}\left[R_{2A}(m,k) + R_{2B}(n,m,k)\right] + o(\varepsilon^2)\right\} \tag{3.14}$$

where

$$\left|\frac{\varepsilon}{R_0(m,k)}\left[R_{2A}(m,k) + R_{2B}(n,m,k)\right]\right| < 1 \tag{3.15}$$

which implies that all the coefficients of $W_m(t)$ and its derivatives in equation (3.11) can be written in terms of the parameter $\mathcal{E}$. When $\mathcal{E} = 0$, a case corresponding to the case when the inertia, effect of the mass of the system is neglected is obtained. In such a case the solution is of the form:

$$W_{mA} = (t) = c_1\cos(\theta_m t - \phi_m) \tag{3.16}$$

where c_1 and ϕ_m are constants and

$$\theta_m^2 = \frac{R_1(m,k)}{R_0(m,k)} \tag{3.17}$$

Since $\mathcal{E} < 1$ for nay arbitrary mass ratio Γ. Struble's technique requires that the asymptotic solution of the homogenous part of (3.11) be of the form [7]:

$$W_m(t) = \phi(m,t)\cos\left[\theta_m t - \Omega(m,t)\right] + \varepsilon W_1(t) \tag{3.18}$$

When use is made of (3.13) and (3.14), equation (3.11) takes the form:

$$\ddot{W}_m(t)+\frac{2c\varepsilon}{R_0(m,k)}[R_{3A}(m,k)+R_{3B}(n,m,k)]\dot{W}_m(t)+\left\{\frac{R_1(m,k)}{R_0(m,k)}\left[1-\frac{\varepsilon}{R_0(m,k)}[R_{2A}(m,k)+R_{2B}(n,m,k)]\right]\right.$$
$$\left.+\frac{c^2\varepsilon}{R_0(m,k)}[R_{4A}(m,k)+R_{4B}(n,m,k)]\right\}W_m(t)$$
$$=\frac{\varepsilon gL}{R_0(m,k)}\left[\sin\frac{\lambda_k ct}{L}+A_k\cos\frac{\lambda_k ct}{L}+B_k\sinh\frac{\lambda_k ct}{L}+C_k\cosh\frac{\lambda_k ct}{L}\right] \quad (3.19)$$

to $0(\mathcal{E})$ only.

When equation (3.18) is substituted into the homogenous part of (3.19) one arrives at:

$$2\theta_m\phi(m,t)\Omega(m,t)\cos[\theta_m t-\Omega(m,t)]-2\theta_m\dot{\phi}(m,t)\sin[\theta_m t-\Omega(m,t)]$$
$$-\phi(m,t)\theta_m^2\cos[\theta_m t-\Omega(m,t)]$$
$$+\frac{2c\varepsilon}{R_0(m,k)}[R_{3A}(m,k)+R_{3B}(n,m,k)] \quad (3.20)$$
$$\times\{-\phi(m,t)\theta_m\sin[\theta_m t-\Omega(m,t)]\}+\left\{\frac{R_1(m,k)}{R_0(m,k)}-\frac{\varepsilon R_1(m,k)}{R_0^2(m,k)}[R_{2A}(m,k)+R_{2B}(n,m,k)]\right.$$
$$\left.+\frac{c^2\varepsilon}{R_0(m,k)}[R_{4A}(m,k)+R_{4B}(n,m,k)]\right\}[\phi(m,t)\cos[\theta_m t-\Omega(m,t)]]=0$$

retaining terms to $0(\mathcal{E})$ only.

The variational equations are obtained by equating the coefficients of $\sin[\theta_m t-\Omega(m,t)]$ *and* $\cos[\theta_m t-\Omega(m,t)]$ terms on both sides of the above equation. Thus, neglecting those terms that do not contribute to the variational equations, equation (3.20) reduces to:

$$\left[2\theta_m\phi(m,t)\Omega(m,t)-\varepsilon\phi(m,t)\theta_m^2\frac{R_{2A}(m,k)}{R_0(m,k)}+\varepsilon c^2\phi(m,t)\frac{R_{4A}(m,k)}{R_0(m,k)}\right]\cos[\theta_m t-\Omega(m,t)]$$
$$-\left[2\theta_m\dot{\phi}(m,t)+\frac{2\varepsilon cR_{3A}(m,k)}{R_0(m,k)}\theta_m\phi(m,t)\right]\sin[\theta_m t-\Omega(m,t)]=0 \quad (3.21)$$

From (3.21), the variational equations are obtained respectively as:

$$2\theta_m\phi(m,t)\Omega(m,t)-\varepsilon\phi(m,t)\theta_m^2\frac{R_{2A}(m,k)}{R_o(m,k)}+\varepsilon c^2\phi(m,t)\frac{R_{4A}(m,k)}{R_o(m,k)}=0 \quad (3.22)$$

and
$$\theta_m\dot{\phi}(m,t)+\frac{\varepsilon cR_{3A}(m,k)}{R_o(m,k)}\theta_m\phi(m,t)=0 \quad (3.23)$$

Rearranging (3.22) and (2.23), we have:

$$\Omega(m,t)=\frac{\varepsilon\left[\theta_m^2R_{2A}(m,k)-c^2R_{4A}(m,k)\right]}{2R_o(m,k)\theta_m} \quad (3.24)$$

and
$$\dot{\phi}(m,t)=-\frac{\varepsilon c\phi(m,t)R_{3A}(m,k)}{R_o(m,k)} \quad (3.25)$$

respectively,

Equations (3.24) and (3.25) when solved respectively yield:

$$\Omega(m,t) = \frac{\varepsilon\left[\theta_m^2 R_{2A}(m,k) - c^2 R_{4A}(m,k)\right]}{2\theta_m R_o(m,k)} + \Omega_m \tag{3.26}$$

where Ω_m is a constant and $\phi(m,t) = c^o e^{-\lambda t}$

where $$\lambda = \frac{\varepsilon c R_{3A}(m,k)}{R_o(m,k)} \text{ and } c^o \text{ is a constant} \tag{3.28}$$

Therefore, when the effect of the mass of the particle is considered, the first approximation to the homogeneous system is:

$$W_m(t) = \phi(m,t)\cos\left[\beta_m t - \Omega_m\right] \tag{3.29}$$

where $$\beta_m = \theta_m - \frac{\varepsilon\left[\theta_m^2 R_{2A}(m,k) - c^2 R_{4A}(m,k)\right]}{2\theta_m R_o(m,k)}$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. Therefore, the differential operator which acts on $W_m(t)$ can be replaced by the equivalent free system operator defined by the modified frequency β_m. That is:

$$\frac{d^2 W_m(t)}{dt^2} + \beta_m^2 W_m(t) = R_m\left[\sin\frac{\lambda_k ct}{L} + A_k\cos\frac{\lambda_k ct}{L} + B_k\sinh\frac{\lambda_k ct}{L} + C_k\cosh\frac{\lambda_k ct}{L}\right] \tag{3.30}$$

where $$R_m = \frac{\varepsilon L g}{R_o(m,k)}$$

Using the Laplace transformation technique and the convolution theory, expression for $W_m(t)$ is obtained. Thus, in view of (3.2), one obtains:

$$U_n(x,t) = \sum_{m=1}^{n} \frac{R_m}{\beta_m(\beta_m^4 - \theta_k^4)}\Big\{\left(\beta_m^2 - \theta_k^2\right)\left[C_k\beta_m(\cosh\theta_k t - \cos\beta_m t) + B_k(\beta_m\sinh\theta_k t - \theta_k\sin\beta_m t)\right]$$
$$+ \left(\beta_m^2 + \theta_K^2\right)\left[A_k\beta_m(\cos\theta_k t - \cos\beta_m t) - (\theta_k\sin\beta_m t - \beta_m\sin\theta_k t)\right]\Big\}$$
$$\times\left[\sin\frac{\lambda_m x}{L} + A_m\cos\frac{\lambda_m x}{L} + B_m\sinh\frac{\lambda_m x}{L} + C_m\cosh\frac{\lambda_m x}{L}\right] \tag{3.31}$$

where $$P_m = \frac{Mg}{\mu R_o(m,k)},\ \theta_m^2 = \frac{R_1(m,k)}{R_o(m,k)} \text{ and } \omega_k = \frac{\lambda_k c}{L}$$

4.0 Some examples of classical end Conditions

For the purpose of illustrating the foregoing analysis, the following examples of classical end conditions are considered namely:

(a) Simply supported
(b) Clamped-clamped
(c) Clamped-free

4.1 Simply supported uniform Rayleigh beam

For a simply supported uniform Rayleigh beam, the boundary conditions are:

$$U(0,t) = 0 = U(L,t) \text{ and } U_{xx}(0,t) = 0 = U_{xx}(L,t) \tag{4.1}$$

It then follows that, for normal modes:

$$V_m(0) = 0 = V_m(L) \text{ and } V_{m},xx(0) = 0 = V_{m,xx}(L) \tag{4.2}$$

which implied that:

$$V_k(0) = 0 = V_k(L) \text{ and } V_{k,xx}(0) = 0 = V_{k,xx}(L) \tag{4.3}$$

Applying (4.2) and (4.3), we have:

$$A_m = A_k = 0;\ B_m = B_k = 0;\ C_m = C_k = 0 \tag{4.4}$$

which implies $\lambda_m = m\pi$ and $\lambda_k = k\pi$

Substituting equations (4.4) and (4.5) into equation (3.20), rearranging and following the same arguments with those in previous section, Struble's technique is used to obtain:

$$\gamma_m = \theta_m - \frac{\varepsilon\left[4L^2\theta_m^2 + c^2m^2\pi^2\right]}{4L^2\theta_m P_o} \tag{4.6}$$

where $$P = 1 + \frac{R^o m^2\pi^2}{L^2},\ \theta_m^2 = \frac{2R(k,m)}{LP_o} \tag{4.7}$$

where

$$R(k,m) = \left\{\frac{E\,\mathrm{I}m^4\pi^4 L}{2\mu L^4} + \frac{kL^2}{\mu\pi^4(k-m)^4(k+m)^4}\left[48L^2km(k^2+m^2) - 16\pi^2 km(k-m)^2(k+m)^2\right.\right.$$
$$\left.\left. + 6kLm\pi^2(L-2)(k-m)^2(k+m)^2(-1)^{k+m}\right]\right\} \tag{4.8}$$

as the modified frequency of the free system due to the presence of the moving mass of the model. θ_m is the frequency of the corresponding moving force problem.

Thus, the moving mass problem reduces to:

$$\frac{d^2W_m(t)}{dt^2} + \gamma_m^2 W_m(t) = \frac{2\varepsilon g}{P_o}\sin\frac{k\pi ct}{L} \tag{4.9}$$

which when solved in conjunction with the initial conditions yields expression for $W_m(t)$. Thus, using (3.2), one obtains:

$$U_n(x,t) = \sum_{m=1}^{n}\frac{2\varepsilon g}{P_o\gamma_m\left[\gamma_m^2 - \left(\frac{k\pi c}{L}\right)^2\right]}\left[\gamma_m\sin\frac{k\pi ct}{L} - \frac{k\pi c}{L}\sin\gamma_m t\right]\sin\frac{m\pi x}{L} \tag{4.10}$$

(4.10) represents the transverse-displacement response to a moving mass of a simply supported uniform Rayleigh beam on a variable Winkler elastic foundation.

The corresponding moving force solution is:

$$U_n(x,t) = \sum_{m=1}^{n}\frac{2Mg}{LP_o\mu\theta_m\left[\theta_m^2 - \left(\frac{k\pi c}{L}\right)^2\right]}\left[\theta_m\sin\frac{k\pi ct}{L} - \frac{k\pi c}{L}\sin\theta_m t\right]\sin\frac{m\pi x}{L} \tag{4.11}$$

4.2. Clamped-clamped uniform Rayleigh beam

At a clamped end, both deflection and slope vanish. Thus for a clamped-clamped uniform Rayleigh beam, the boundary conditions are:

$$U(0,t) = 0 = U(L,t);\ U_x(0,t) = 0 = U_x(L,t) \tag{4.12}$$

Thus, for normal modes:

$$V_m(0) = 0 = V_m(L);\ V_{m,x}(0) = 0 = V_{m,x}(L) \tag{4.13}$$

which follows that:

$$V_k(0) = 0 = V_k(L);\ V_{k,x}(0) = 0 = V_{k,x}(L) \tag{4.14}$$

Applying (4.13) to (3.8) one obtains:

$$A_m = \frac{\sinh\lambda_m - \sin\lambda_m}{\cos\lambda_m - \cosh\lambda_m} = \frac{\cos\lambda_m - \cosh\lambda_m}{\sin\lambda_m + \sinh\lambda_m} = -C_m \tag{4.15}$$

and
$$B_m = -1 \tag{4.16}$$

The frequency equation can simply be obtained from (4.15) as:

$$\cos\lambda_m \cosh\lambda_m = 1 \tag{4.17}$$

Expressions for A_k, B_k, C_k and the corresponding frequency equation are obtained by substituting k for m in (4.15), (4.16) and (4.17). Thus, the general solutions of the associated moving mass and moving force problems are obtained by substituting relevant results in (4.15), (4.16) and (4.17) into (3.31) and (3.32) respectively.

4.3 Clamped-free uniform Rayleigh beam

Next at $x = 0$, the Rayleigh beam is taken to be clamped and at $x = L$, the beam model is free. Thus, the boundary conditions of the Rayleigh beam can be written as:

$$U_{xx}(L,t) = 0 = U_{xxx}(L,t) \text{ and } U(0,t) = 0 = U_x(0,t) \tag{4.18}$$

Similarly, for normal modes

$$V_{m,xx}(L) = 0 = V_{m,xxx}(L) \text{ and } V_m(0) = 0 = V_{m,x}(0) \tag{4.19}$$

which implies that

$$V_{k,xx}(L) = 0 = V_{k,xxx}(L) \text{ and } V_k(0) = 0 = V_{k,x}(0) \tag{4.20}$$

Using (4.19), it is straight forward to show that at $x = 0$,

$$A_m = -C_m \text{ and } B_m = -1 \tag{4.21}$$

also at $x = L$,
$$A_m = -\frac{\sin\lambda_m - \sinh\lambda_m}{\cos\lambda_m + \cosh\lambda_m} = -\frac{\cos\lambda_m - \cosh\lambda_m}{\sinh\lambda_m - \sin\lambda_m} = -C_m \tag{4.22}$$

and the frequency equation for both end conditions is $\cos\lambda_m \cosh\lambda_m = -1$ (4.23)

such that $\lambda_1 = 1.875, \lambda_2 = 4.694, \lambda_3 = 7.855$ and so on (4.24)

Substituting (4.22), (4.23) and (4.24) into equations (3.31) and (3.32), one obtains the displacement response respectively to a moving force and a moving mass of a uniform clampled-free ends Rayleigh beam on a variable elastic foundation.

5.0 Analysis of closed form solutions

In studying undamped system such as this, it is desirable to examine the phenomenon of resonance. Equation (4.11) clearly shows that the simply uniform Rayleigh beam on a variable Winkler elastic foundation and traversed by a moving force reaches a state of resonance whenever:

$$\theta_m = \frac{k\pi c}{L} \tag{5.1}$$

While equation (4.10) shows that the same beam under the action of a moving mass experiences resonance when:

$$\gamma_m = \frac{k\pi c}{L} \tag{5.2}$$

where
$$\gamma_m = \theta_m \left\{ 1 - \varepsilon \left[\frac{1}{2\left(1 + \frac{R^o m^2 \pi^2}{L^2}\right)} + \frac{c^2 m^2 \pi^2}{2L^2 \theta_m^2 \left(1 + \frac{R^o m^2 \pi^2}{L^2}\right)} \right] \right\} \tag{5.3}$$

From equations (5.2) and (5.3), it can be shown that:

$$\frac{\theta_m\left[1+\frac{R^o m^2\pi^2}{L^2}-\frac{\varepsilon}{2}\left(1+\frac{c^2m^2\pi^2}{L^2\theta_m^2}\right)\right]}{1+\frac{R^o m^2\pi^2}{L^2}}=\frac{k\pi c}{L} \tag{5.4}$$

Since $$1+\frac{R^o m^2\pi^2}{L^2}>1+\frac{R^o m^2\pi^2}{L^2}-\frac{\varepsilon}{2}\left[1+\frac{c^2m^2\pi^2}{L^2\theta_m^2}\right] \text{ for all } m,$$

It can be deduced from equation (5.4) that, for the same natural frequency, the critical speed (and the natural frequency) for the system of a simply supported uniform Rayleigh beam traversed by a moving mass is smaller than that of same system traversed by a moving force. Thus, for the same natural frequency of the uniform Rayleigh beam, the resonance is reached earlier when we consider the moving mass system than when we consider the moving force system.

Furthermore, from equation (3.32) it is evident that for other classical boundary conditions, the uniform Rayleigh beam on a variable Winkler elastic foundation and traversed by a moving force encounters a resonant effect when:

$$\theta_m=\frac{\lambda_k c}{L} \tag{5.5}$$

while equation (3.31) shows that the same beam under the action of a moving mass experiences resonant effect whenever:

$$\beta_m=\frac{\lambda_k c}{L} \tag{5.6}$$

where

$$\beta_m=\theta_m-\frac{\varepsilon\left[\theta_m^2\Omega_{2A}(m,k)-c^2\Omega_{4A}(m,k)\right]}{2\theta_m\Omega_o(m,k)} \tag{5.7}$$

This implies that:

$$\beta_m=\frac{\theta_m\left[\Omega_o(m,k)-\frac{\varepsilon}{2}\left(\Omega_{2A}(m,k)-\frac{c^2\Omega_{4A}(m,k)}{\theta_m^2}\right)\right]}{\Omega_o(m,k)}=\frac{\lambda_k c}{L} \tag{5.8}$$

Consequently from equations (5.5) and (5.8), the same results and analysis obtained in the case of a Rayleigh beam simply supported at both ends are obtained for the other examples of end support conditions.

6.0 Discussion of numerical results

In order to present the calculations of practical interests in dynamics of structures and Engineering design for all the illustrative examples, and elastic uniform Rayleigh beam of length 12.192m has been considered. It is assumed that the mass travels at the constant velocity 8.123*m/s*. Also *EI* and ε are chosen to be 6.068 x $10^6 m^3/s^2$ and 0.25 respectively. The results are as shown on the various graphs below for the classical boundary conditions considered.

Figure 6.01 display the effect of Rotary inertia (R^o) on the transverse deflection of the simply supported uniform beam in the case of moving mass. The graph shows that the response amplitude of the uniform beam decreases as the value of the Rotatory inertia correction factor increases. Values of R^o between 0m and 20*m* are used.

The effect of foundation constant K on the transverse deflection of moving mass displayed in Figure 6.02 shows that an increase in the value of the foundation constant K decreases the deflection of the simply supported uniform Rayleigh beam. Here, values of K between $0N/m^3$ and $1mN/m^3$ are used.

Figure 6.03 displays the effect of R^o on the transverse deflection of the clamped-clamped uniform Rayleigh beam in the case of moving mass problem. It is clear that as the value of R^o increases, the deflection of the beam decreases.

Figure 6.04 shows that, an increase in the value of foundation moduli K reduces the transverse deflection of moving mass problem of the uniform Rayleigh beam with clamped ends.

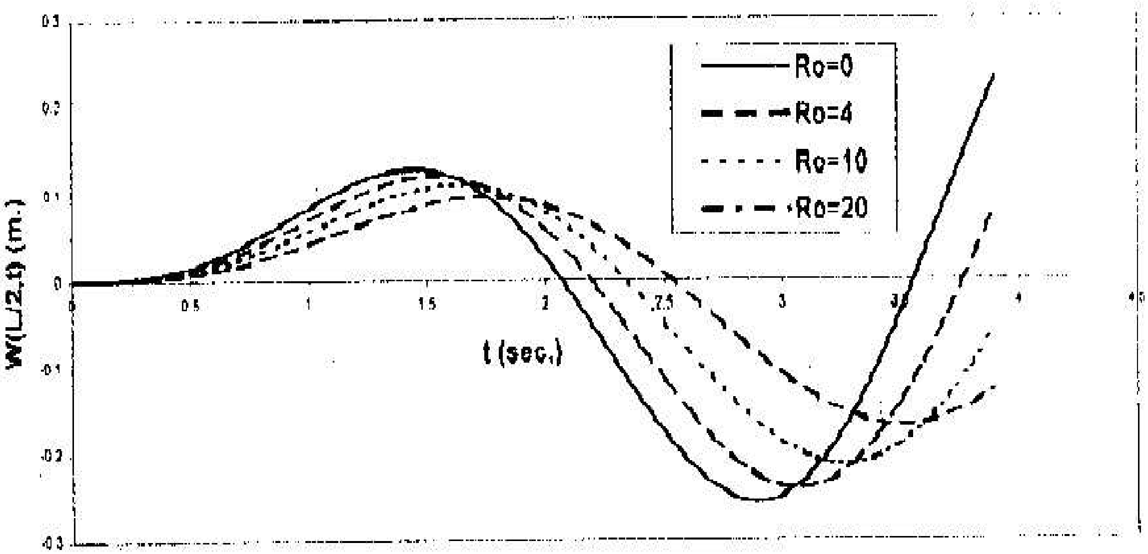

Figure 6.01: Deflection profile of moving mass for simply supported uniform Rayleigh beam for various values of Ro.

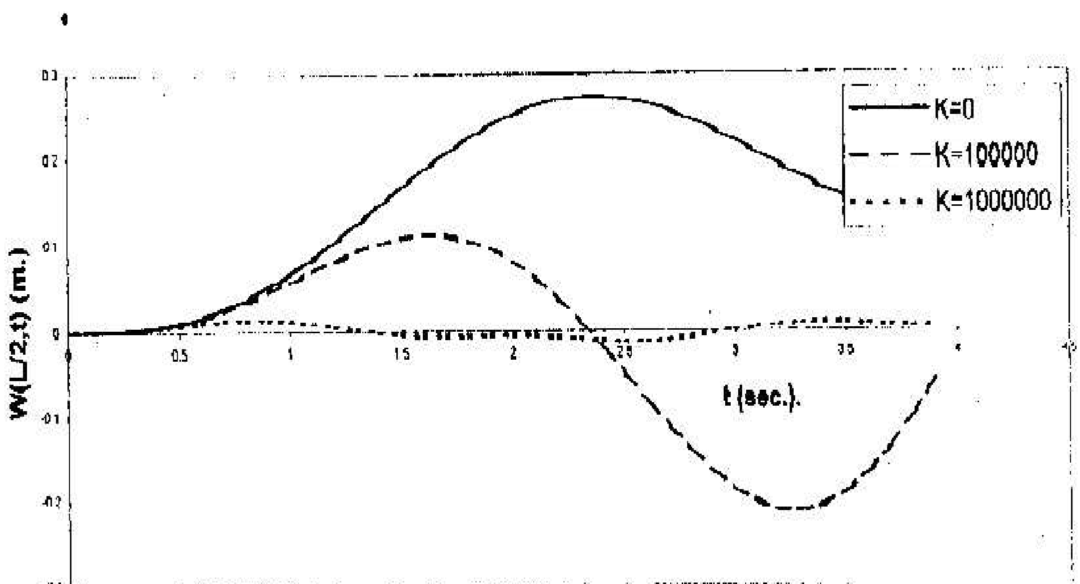

Figure 6.02: Displacement response of moving mass for simply supported uniform Rayleigh beam for various values of foundation moduli K.

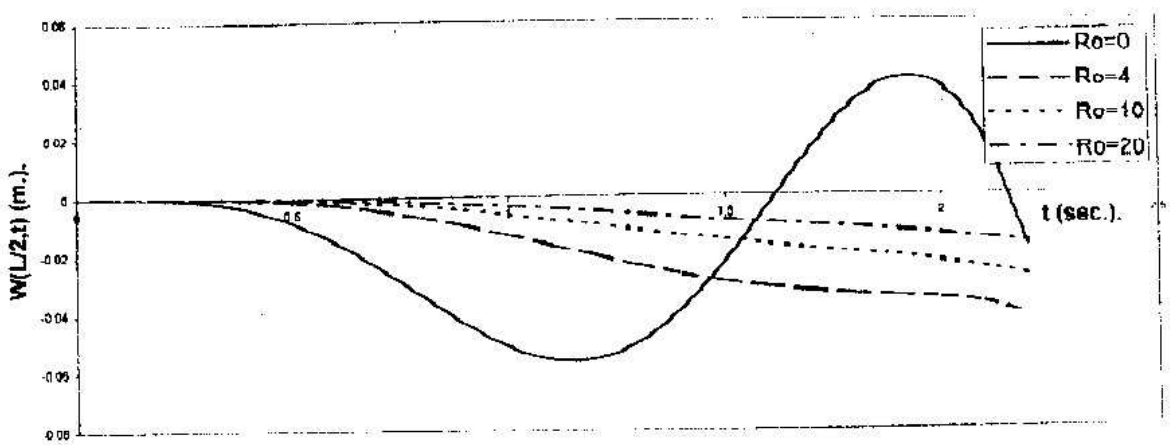

Figure 6.03: Deflection profile of moving mass for clamped-clamped uniform beam for various values of Ro.

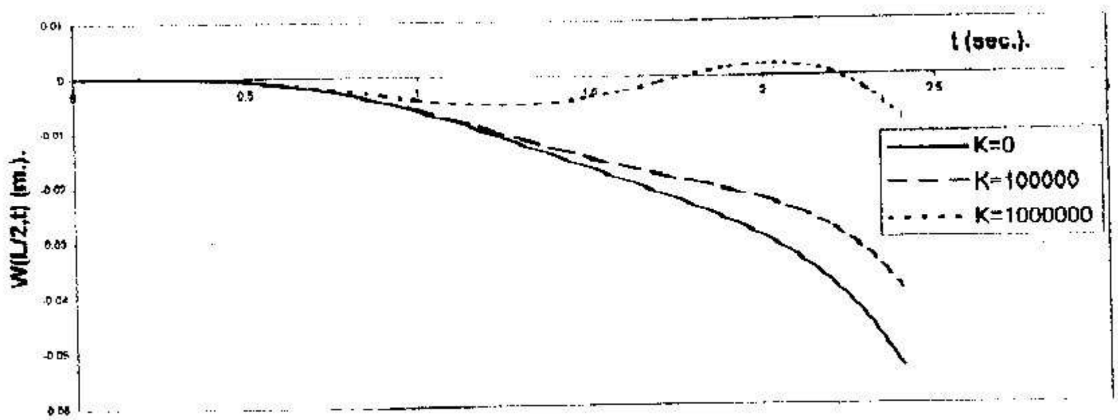

Figure 6.04: Displacement response of moving mass for clamped-clamped uniform Rayleigh beam for various values of foundation moduli K.

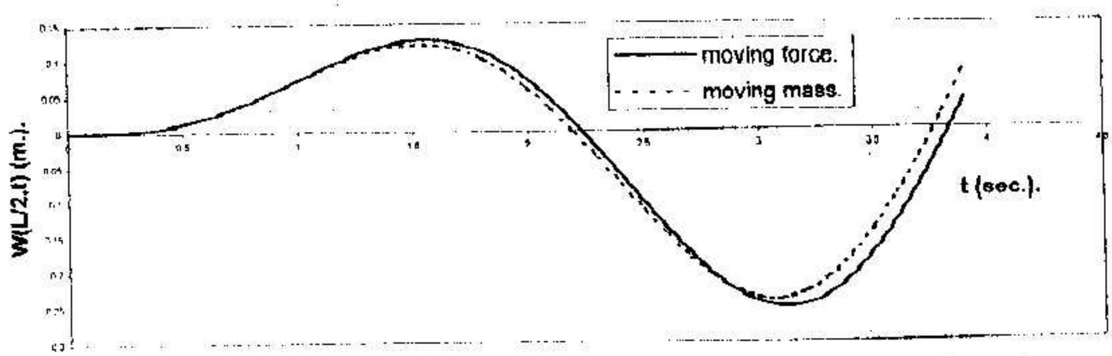

Figure 6.05: Comparison of the deflection of moving force and moving mass for simply supported uniform Rayleigh beam.

T(*sec*)	*Displacement, W(L/2,t)(m) (Moving force)*	*Displacement, W(L/2,t)(m) (Moving mass*
0	0	0
0.1	-3.244287E-06	-3.244333E-06
0.2	-4.83799E-05	-4.838319E-06
0.3	-2.268494E-04	-2.268851E-04
0.4	-6.595776E-04	-6.597672E-04
0.5	-1.470749E-03	-1.471429E-03
0.6	-2.764083E-03	-2.765986E-03
0.7	-4.60339E-03	-4.607862E-03
0.8	-6.998967E-03	-7.008212E-03
0.9	-0.009901	-9.918297E-03
1.0	-1.320056E-02	-1.323046E-02
1.1	-1.673623E-02	-1.678663E-02
1.2	-2.373675E-02	-2.039382E-02
1.3	-2.373675E-02	-2.384521E-02
1.4	-2.679335E-02	-2.694557E-02
1.5	-2.933279E-02	-2.953885E-02
1.6	-3.126631E-02	-3.153646E-02
1.7	-3.260056E-02	-3.294464E-02
1.8	-3.346309E-02	-3.389017E-02
1.9	-3.412639E-02	-3.464428E-02

Table 6.01: Comparison of the displacement of moving force mass for clamped-clamped uniform Rayleigh bean

For the purpose of comparison, the displacements of the moving force and moving mass for both simply supported and clamped-clamped uniform Rayleigh beams with $R° = 4$ and $K = 100000N/m$ are presented in Figure 6.05 and Table 6.01 respectively. It can be noted that the response amplitude of a moving mass is greater than that of a moving force problem for both simply supported and clamped-clamped uniform Rayleigh beam. This result also holds for clamped-free uniform Rayleigh beam. The same thing obtains for other choices of R^o and *K*.

7.0 Conclusion

The objective of this work has been to study the problem of the dynamic response to moving concentrated masses of uniform Rayleigh beams on variable Winkler elastic foundations. In particular, the closed form solutions of the fourth order partial differential equations with variable and singular coefficients of uniform Rayleigh beam moving mass problem is obtained for the solution of the problem, Galerkin's method and a modification of the Strubles technique are employed. For the three illustrative examples considered, the moving force solution is not an upper bound for the accurate solution of the moving mass solution in uniform Rayleigh beam moving mass problem Also, as the Rotatory inertia correction factor increases, the response amplitudes of the beam decreases whether beam is simply supported, clamped at both ends or clamped-free. It was also observed that for fixed Rotatory inertia correction factor and Foundation moduli, the response amplitude for the moving mass problem is greater than that of the moving force problem for the illustrative end conditions considered. Similarly, in Vie illustrative examples considered, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem.s

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