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Influence of foundation and axial force on the vibration of thin beam under variable harmonic moving load

T. O. Awodola
Department of Mathematical Sciences
Federal University of Technology, Akure, Nigeria.

Abstract

The influence of foundation and axial force on the vibration of a simply supported thin (Bernoulli Euler) beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity is investigated in this paper. The governing equation is a fourth order partial differential equation. For the solution of this problem, in the first instance, the finite Fourier sine transformation is used to reduce the equation to a second order partial differential equation. The reduced equation is then solved using the Laplace transformation. Numerical analysis shows that the transverse deflection of the thin beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity decreases as the foundation constant increases. It also shows that as the axial force increases, the transverse deflection of the thin beam decreases.

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1.0 Introduction

Moving loads causes solid bodies to vibrate intensively, particularly at high velocities. Thus, the study of the behaviour of bodies subjected to moving loads has been the concern of several investigators. Among the earliest work in this area of study was the work of Timoshenko [1] who considered the problem of simply supported finite beams resting on an elastic foundation and traversed by moving loads. In his analysis, he assumed that the loads were moving with constant velocities along the beam. Furthermore, Kenny [2] took up the problem of investigating the dynamic response of infinite elastic beams on elastic foundation when the beam is under the influence of a dynamic load moving with constant speed. He included the effects of viscous damping in the governing differential equation of motion. More recently, Oni [3] considered the problem of a harmonic time variable concentrated force moving at a uniform velocity over a finite deep beam. The methods of integral transformations are used. In particular, the finite Fourier transform is used for the length coordinate and the laplace transform (hr the time coordinate. Series solution, which converges, was obtained for the deflection of' simply supported beams. The analysis of the solution was carried out for various speeds of the load. Oni [4] used the Galerkin method to obtain the response to several moving masses of a non-uniform beam resting on an elastic foundation. The effects of the elastic foundation on the transverse displacement of the non-uniform beam were analyzed for both the moving mass and the associated moving force problems.

Furthermore, Milormir et al [5] developed a theory describing the response of a Bernoulli-euler beam under an arbitrary number of concentrated moving masses. The theory is based on the Fourier technique and shows that, for a simply supported beam, the resonance frequency is lower with no corresponding decrease in maximum amplitude when the inertia is considered.

However, the above studies, though impressive, have failed to give the effects of lateral reinforcements on the transverse displacement of the beam.

This paper, therefore, presents the effects of foundation constant and axial force on the transverse deflection of a thin (Bernoulli-Euler) beam, resting on a uniform foundation, under the action of a harmonic load moving with variable velocity.

2.0 The Mathematical model

Consider a beam under a moving load *p(x,t)*, subjected to an axial force *N*, which remains parallel to the *x*-axis. A portion of the beam is as shown below:

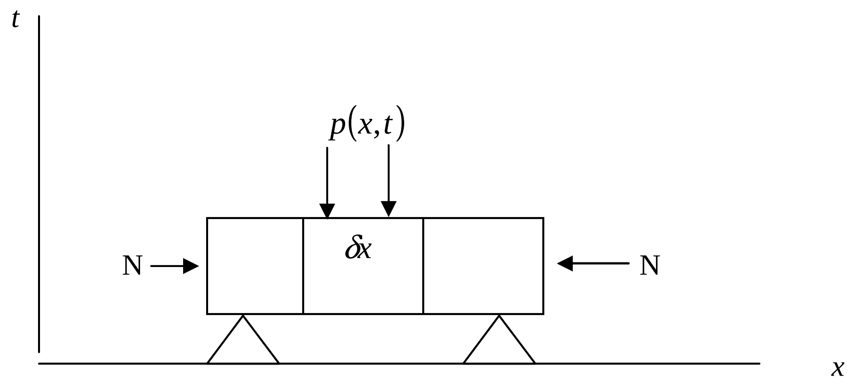

Figure 1: The beam's displacement is governed by the equation (2.6)

$$EI\frac{\partial^4 V(x,t)}{\partial x^4} - N\frac{\partial^2 V(x,t)}{\partial t^2} + \mu\frac{\partial^2 V(x,t)}{\partial t^2} + 2\mu\omega_o\frac{\partial V(x,t)}{\partial t} + KV(x,t) = P(x,t) \qquad (2.1)$$

where x is the spatial co-ordinate, t is the time, $\frac{\partial}{\partial x}$ is the partial derivative with respect to x,

V(x,t) is the transverse displacement, E is the Young's modulus, N is the axial force, I is the moment of inertia, μ is the mass per unit length of the beam, K is the foundation constant, ω_o is the circular frequency and $P(x,t)$ is the moving load.

In this paper, the beam model is taken to be simply supported and hence the boundary conditions take the form:

$$V(0,t) = V(L,t) = 0 \qquad (2.2a)$$

$$\frac{\partial^2 V(0,t)}{\partial x^2} = \frac{\partial^2 V(L,t)}{\partial x^2} = 0 \qquad (2.2b)$$

where L is the length of the beam.

For simplicity, the initial conditions shall be taken as

$$V(x,0) = 0 = \frac{\partial V(x,0)}{\partial t} \qquad (2.3)$$

In what follows, we shall consider the load traversing the Bernoulli-Euler beam model and investigate the influence of foundation constant and axial force on the vibration of the beam under a load moving with variable speed.

More specifically, we adopt the example in [3], and take our moving load to be of the form:

$$P(x,t) = P(t)\delta[x - (x_o + \beta\,\mathrm{Sin}\,\alpha t)] \qquad (2.4)$$

where $P(t)$ indicate the magnitude of the load, $sin\alpha\, t$ is the variable speed term for the moving load and x_0, β and α are constants.

The function $\delta(x)$ is defined as

$$\partial(x) = \begin{Bmatrix} 0; x \neq 0 \\ \infty; x = 0 \end{Bmatrix} \tag{2.5}$$

and is called the dirac-delta function with the property:

$$\int_a^b \delta(x-k) f(x) ds = \begin{cases} f(k); \ for \ a \prec k \prec b \\ 0: \quad for \ a \prec k \prec b \\ 0; \ for \ a \prec k \prec b \end{cases} \tag{2.6}$$

For a harmonic load, $P(t)$ is chosen to be of the form $P(t) = P cos \omega t$ and (2.4) becomes

$$P(x,t) = P \cos \omega t \delta[x - (x_o + \beta \sin \alpha t)] \tag{2.7}$$

Substituting equation (2.7) into equation (1.1) we have:

$$EI \frac{\partial^4 V(x,t)}{\partial x^4} - N \frac{\partial^2 V(x,t)}{\partial x^2} + \mu \frac{\partial^2 V(x,t)}{\partial t^2} + 2\mu\omega_o \frac{\partial V(x,t)}{\partial t} + KV(x,t) = P \cos \omega t \delta[x - (x_o + \beta \sin \alpha t)] \tag{2.8}$$

2.1 **Problem Solution**

In order to solve equation (2.8), we notice that

$$\sin \alpha t = \alpha t - \frac{(\alpha t)^3}{3!} + \frac{(\alpha t)^5}{5!} - \frac{(\alpha t)^7}{7!} + \frac{(\alpha t)^9}{9!} - ... \tag{2.9}$$

$$= \alpha t - \left[\frac{(\alpha t)^3}{3!} - \frac{(\alpha t)^5}{5!} \right] - \left[\frac{(\alpha t)^7}{7!} - \frac{(\alpha t)^9}{9!} \right] - ...$$

It is clear therefore that $\sin\alpha t \leq \alpha\, t$ since α and t are considerably small, $\sin\alpha t$ shall therefore be approximated to αt in this work. Thus equation (2.8) becomes

$$EI \frac{\partial^4 V(x,t)}{\partial x^4} - N \frac{\partial^2 V(x,t)}{\partial x^2} + \mu \frac{\partial^2 V(x,t)}{\partial t^2} + 2\mu\omega_o \frac{\partial V(x,t)}{\partial t} + KV(x,t) = P \cos \omega t \delta[x - (x_o + \beta \alpha t)] \tag{2.10}$$

A suitable method for solving diverse problems of structural dynamics is the method of integral transformation. Particularly, in order to solve equation (2.10) subject to the boundary conditions (2.2a) and (2.2b), we subject it to a finite Fourier sine transformation given by

$$V(m,t) = \int_o^L V(x,t) \sin \frac{m\pi x}{L} dx \tag{2.11a}$$

with the inverse

$$V(x,t) = \frac{2}{L} \sum_{m=1}^{\infty} V(m,t) \sin \frac{m\pi x}{L} \tag{2.11b}$$

And apply the property of the dirac-delta function (x) given in equation (6). We then obtain

$$\left[EI \frac{m^4 \pi^4}{L^4} + K \right] V(m,t) + N \frac{m^2 \pi^2}{L^2} V(m,t) + \mu r \ V_u(m,t) + 2\mu\omega_o V_t(m,t) = P \cos \omega t \sin \frac{m\pi}{L} (x_o + \beta \alpha t) \tag{2.12}$$

Subjecting equation (2.12) to a Laplace transformation using the initial conditions in (2.3), we have

$$\left[EI\frac{m^4\pi^4}{L^4}+K\right]V(m,r)+N\frac{m^2\pi^2}{L^2}V(m,t)+\mu r^2V(m,t)+2\mu\omega_o rV(m,r)$$

$$=\frac{P\sin\frac{m\pi x_o}{L}}{2\left[r+\frac{1}{r}\left(\omega+\frac{m\pi\beta\alpha}{L}\right)^2\right]}+\frac{P\sin\frac{m\pi x}{L}}{2\left[r+\frac{1}{r}\left(\omega-\frac{m\pi\beta\alpha}{L}\right)^2\right]}+\frac{P\left(\cos\frac{m\pi x_o}{L}\right)\left(\frac{m\pi x_o}{L}-\omega\right)}{2\left[r^2\left(\frac{m\pi\beta\alpha}{L}-\omega\right)^2\right]}+\frac{P\left(\cos\frac{m\pi x_o}{L}\right)\left(\frac{m\pi x_o}{L}+\omega\right)}{2\left[r^2\left(\frac{m\pi\beta\alpha}{L}+\omega\right)^2\right]} \quad (2.13)$$

Equation (2.13) can be rearranged to take the form:

$$V(m,r)=\frac{\frac{1}{2}P\sin\frac{m\pi x_o}{L}}{R_o\left[r+\frac{1}{r}\left(\omega+\frac{m\pi\beta\alpha}{L}\right)^2\right]}+\frac{\frac{1}{2}P\sin\frac{m\pi x_o}{L}}{R_o\left[r+\frac{1}{r}\left(\omega-\frac{m\pi\beta\alpha}{L}\right)^2\right]}$$

(2.14)

$$+\frac{\frac{1}{2}P\left(\cos\frac{m\pi x_o}{L}\right)\left(\frac{m\pi\beta x_o}{L}-\omega\right)\sin}{R_o\left[r^2\left(\frac{m\pi\beta\alpha}{L}-\omega\right)^2\right]}+\frac{\frac{1}{2}P\left(\cos\frac{m\pi x_o}{L}\right)\left(\frac{m\pi\beta x_o}{L}-\omega\right)}{R_o\left[r^2\left(\frac{m\pi\beta\alpha}{L}+\omega\right)^2\right]}$$

where

$$R_o=\left[EI\frac{m^4\pi^4}{L^4}+K\right]+N\frac{m^2\pi^2}{L^2}+\mu r^2+2\mu\omega_o r$$

A further rearrangement gives

$$V(m,r)=\frac{P_1\frac{r}{r^2+a^2}}{(r-k_1)(r-k_2)}+\frac{P_1\frac{r}{r^2+b^2}}{(r-k_1)(r-k_2)}-\frac{P_2\frac{b}{r^2+b^2}}{(r-k_1)(r-k_2)}+\frac{P_2\frac{a}{r^2+b^2}}{(r-k_1)(r-k_2)} \quad (2.15)$$

where

$$a=\omega+\frac{m\pi\beta\alpha}{L};b=\omega-\frac{m\pi\beta\alpha}{L};P_1=\frac{P\sin\frac{m\pi x_o}{L}}{2\mu};P_2=\frac{P\cos\frac{m\pi x_o}{L}}{2\mu}$$

$$k_1=-\frac{H_3}{2}+\sqrt{\frac{H_3^2}{4}-H_1-H_2\frac{m^2\pi^2}{L^2}};$$

$$k_2=-\frac{H_3}{2}-\sqrt{\frac{H_3^2}{4}-H_1-H_2\frac{m^2\pi^2}{L^2}}$$

where

$$H_1 = \frac{E\,\mathrm{Im}^4 \pi^4}{\mu L^4} + \frac{K}{\mu}; H_2 = \frac{N}{\mu} \text{ and } H_3 = 2\omega_o$$

But

$$\frac{1}{(r-k_1)(r-k_2)} = \frac{1}{k_1-k_2}\left[\frac{1}{r-k_1} - \frac{1}{r-k_2}\right] \qquad (2.16)$$

In order to obtain a Laplace inversion of equation (15), we adopt the following representations:

$$g_1(r) = P_1\frac{r}{r^2+a^2}; g_2(r) = P_1\frac{r}{r^2+b^2};$$

$$g_3(r) = P_2\frac{b}{r^2+b^2}; g_4(r) = P_2\frac{a}{r^2+a^2}$$

and $$f(r) = \frac{1}{(r-k_1)(r-k_2)} = \left[\frac{1}{k_1-k_2}.\frac{1}{r-k_1}\right] - \left[\frac{1}{k_1-k_2}.\frac{1}{r-k_2}\right] \qquad (2.17)$$

So that the Laplace inversion of each term of *V*(*m*,*r*) is obtained by applying the convolution theory given by

$$f(r) * g_i(r) = \int_o^t f(t-u)g_i(u)du; \qquad i = 1,2,3,4 \qquad (2.18)$$

Thus we have

$$f(r) * g_1(r) = P_1\int_o^t \frac{\left[e^{k_1(t-u)} - e^{k_2(t-u)}\right]}{k_1-k_2}\cos audu; \qquad (2.19)$$

$$f(r) * g_2(r) = P_1\int_o^t \frac{\left[e^{k_1(t-u)} - e^{k_2(t-u)}\right]}{k_1-k_2}\cos budu; \qquad (2.20)$$

$$f(r) * g_3(r) = P_2\int_o^t \frac{\left[e^{k_1(t-u)} - e^{k_2(t-u)}\right]}{k_1-k_2}\sin budu; \qquad (2.21)$$

$$f(r) * g_4(r) = P_2\int_o^t \frac{\left[e^{k_1(t-u)} - e^{k_2(t-u)}\right]}{k_1-k_2}\sin audu; \qquad (2.22)$$

The Laplace inversion of equation (2.15) is thus obtained by solving the integrals (2.19), (2.20), (2.21) and (2.22) and substituting as appropriate into equation (2.15). Thus we have

$$V(m,t) = I_1 - I_2 + I_3 - I_4 - I_5 + I_6 + I_7 - I_8 \qquad (2.23)$$

where

$$I_1 = \frac{P_1 k_1}{(k_1-k_2)(k_1^2+a^2)}\left[e^{k_1 t} + \frac{a\sin at}{k_1} - \cos at\right]; \qquad (2.24$$

$$I_2 = \frac{P_1 k_2}{(k_1-k_2)(k_2^2+a^2)}\left[e^{k_2 t} + \frac{a\sin at}{k_3} - \cos at\right]; \qquad (2.25)$$

$$I_3 = \frac{P_1 k_1}{(k_1-k_2)(k_1^2+b^2)}\left[e^{k_1 t} + \frac{b\sin bt}{k_1} - \cos bt\right]; \qquad (2.26)$$

$$I_4 = \frac{P_1 k_2}{(k_1 - k_2)(k_2^2 + b^2)}\left[e^{k_2 t} + \frac{b\sin bt}{k_2} - \cos bt\right]; \quad (2.27)$$

$$I_5 = \frac{P_2 k_1}{(k_1 - k_2)(k_1^2 + b^2)}\left[\frac{be^{k_1 t}}{k_1} - \sin bt - \frac{b\cos bt}{k_1}\right]; \quad (2.28)$$

$$I_6 = \frac{P_2 k_2}{(k_1 - k_2)(k_2^2 + b^2)}\left[\frac{be^{k_2 t}}{k_2} - \sin bt - \frac{b\cos bt}{k_2}\right]; \quad (2.29)$$

$$I_7 = \frac{P_2 k_1}{(k_1 - k_2)(k_1^2 + a^2)}\left[\frac{ae^{k_1 t}}{k_1} - \sin at - \frac{a\cos at}{k_1}\right]; \quad (2.30)$$

and

$$I_8 = \frac{P_2 k_2}{(k_1 - k_2)(k_2^2 + a^2)}\left[\frac{ae^{k_2 t}}{k_2} - \sin at - \frac{a\cos at}{k_2}\right] \quad (2.31)$$

which on Fourier sine inversion becomes

$$V(x,t) = \frac{2}{L}\sum_{m=1}^{\infty}\left\{\frac{P_1 k_1}{(k_1 - k_2)(k_1^2 + a^2)}\left[ae^{k_1 t} + \frac{a\sin at}{k_1} - \cos at\right] - \frac{P_1 k_2}{(k_1 - k_2)(k_2^2 + a^2)}\right.$$
$$\times\left[ae^{k_2 t} + \frac{a\sin at}{k_2} - \cos at\right] + \frac{P_1 k_1}{(k_1 - k_2)(k_1^2 + b^2)}\left[be^{k_1 t} + \frac{b\sin bt}{k_1} - \cos bt\right]$$
$$- \frac{P_1 k_2}{(k_1 - k_2)(k_2^2 + b^2)}\left[ae^{k_1 t} + \frac{b\sin bt}{k_2} - \cos bt\right] - \frac{P_2 k_1}{(k_1 - k_2)(k_1^2 + b^2)}$$
$$\times\left[\frac{be^{k_1 t}}{k_1} - \sin bt - \frac{b\cos bt}{k_1}\right] + \frac{P_2 k_2}{(k_1 - k_2)(k_2^2 + b^2)}\left[\frac{be^{k_2 t}}{k_2} - \sin bt - \frac{b\cos bt}{k_2}\right]$$
$$+ \frac{P_2 k_1}{(k_1 - k_2)(k_1^2 + a^2)}\left[\frac{ae^{k_2 t}}{k_1} - \sin at - \frac{a\cos at}{k_1}\right]$$
$$\left. - \frac{P_2 k_2}{(k_1 - k_2)(k_2^2 + a^2)}\left[\frac{ae^{k_2 t}}{k_2} - \sin at - \frac{a\cos at}{k_2}\right]\right\}\sin\frac{m\pi x}{L} \quad (2.32)$$

Equation (3.32) is the response of a simply supported thin reinforced bean under the action of a variable magnitude harmonic force moving with variable velocity.

3.0 Numerical Calculations and Discussion of Results

For the purpose of Numerical analysis, the length of the beam is chosen to be 12.192m while the value *EI* is chosen to be 6.068 x 10^6 kgm^3/s^2. The results are as presented in the tables below.

Table 1 presents the deflection (V) at various times t when the foundation constant *K* is varied between 0 N/m^3 and 1,000,000 N/m^3. The deflection (*V*) at various times *t* when the axial force (*N*) is chosen to be 0, 20 million and 40 million Newton respectively is presented in Table 2.

Table 1

S/N	$T(sec)$	$V(m)$ at $K = 0$	$V(m)$ at $K = 100{,}000$	$V(m)$ at $K = 1{,}000{,}000$
1	0	0	0	0
2	0.1	-1.874334E-10	-1.874296E-10	-1.873982E-10
3	0.2	-4.503662E-10	-4.503299E-10	-4.499987E-10
4	0.3	-4.450309E-10	-4.449137E-10	-4.438644E-10
5	0.4	5.140222E-11	5.161094E-11	5.348322E-11
6	0.5	1.139216E-09	1.139439E-09	1.141431E-09
7	0.6	2.792605E-09	2.79263E-09	2.792902E-09
8	0.7	4.871387E-09	4.870884E-09	4.866344E-09
9	0.8	7.15235E-09	7.150851E-09	7.137346E-09
10	0.9	9.376002E-09	9.372964E-09	9.345634E-09
11	1.0	1.129931E-08	1.129417E-08	1.124797E-08
12	1.1	1.274303E-08	1.273526E-08	1.26655E-08
13	1.2	1.36235E-08	1.361267E-08	1.351547E-08
14	1.3	1.396238E-08	1.394816E-08	1.382076E-08
15	1.4	1.387312E-08	1.385534E-08	1.369618E-08
16	1.5	1.352848E-08	1.350709E-08	1.33157E-08
17	1.6	1.311735E-08	1.309236E-08	1.286907E-08
18	1.7	1.280139E-08	1.277288E-08	1.251841E-08
19	1.8	1.268168E-08	1.264973E-08	1.236486E-08
20	1.9	1.278232E-08	1.274696E-08	1.243219E-08

Table 2

S/N	$T(sec)$	$V(m)$ at $N = 0$	$V(m)$ at $N = 20{,}000{,}000$	$V(m)$ at $N = 40{,}000{,}000$
1	0	0	0	0
2	0.1	-1.874331E-10	-1.866118E-10	-1.857923E-10
3	0.2	-4.503664E-10	-4.408951E-10	-4.314922E-10
4	0.3	-4.450311E-10	-4.105296E-10	-3.765804E-10
5	0.4	5.140199E-11	1.292244E-10	2.048364E-10
6	0.5	1.139216E-09	1.272796E-09	1.40035E-09
7	0.6	2.792605E-09	2.982935E-09	3.160431E-09
8	0.7	4.871386E-09	5.105839E-09	5.317311E-09
9	0.8	7.152348E-09	7.406051E-09	7.623628E-09
10	0.9	9.375999E-08	9.616161E-08	9.8051E-09
11	1.0	1.12993E-08	1.149132E-08	1.161664E-08
12	1.1	1.274302E-08	1.285691E-08	1.289002E-08
13	1.2	1.36235E-08	1.363919E-08	1.356396E-08
14	1.3	1.396236-08	1.387278E-08	1.368452E-08
15	1.4	1.38731E-08	1.368436E-08	1.339476E-08
16	1.5	1.352846E-08	1.325743E-08	1.288765E-08
17	1.6	1.311732E-08	1.278702E-08	1.236342E-08
18	1.7	1.280136E-08	1.24355E-08	1.198371E-08
19	1.8	1.268166E-08	1.229949E-08	1.18391E-08
20	1.9	1.278228E-08	1.239487E-08	1.19359E-08

Comparing the deflection at various values of the foundation constant, Table 1 shows, for fixed N that the transverse displacement of the beam decreases as the Foundation constant K increases. It is observed from Table 2 that, for fixed K the displacement response of the beam decreases as the effect of the axial force (N) increases.

4.0 Conclusion

The influence of Foundation and axial force on the deflection of a simply supported thin (Bernoulli-Euler) beam under the action of moving load has been studied in this work. The beam is assumed to rest on a uniform foundation and the load is moving with variable velocity.

The beam problem is solved using the Fourier sine transforms on the partial coordinate x and Laplace transform on the other partial coordinate t. The deflection (V) at time (t) For various values of axial force (N) was obtained keeping K constant. Also, for various values of K the deflection at time (t) was obtained keeping N constant. It was found that the amplitudes of vibration decrease with increasing foundation constant. Also as the axial force (N) increases, the deflection (") of the simply supported Bernoulli-Euler beam decreases.

The theory generated in this work can be applied to calculations involving prestressed or reinforced beams often encountered in structural design and construction.

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