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# On the Transverse motions under heavy loads of thin beams with variable prestress

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**Abstract**

***In this paper, the effect of variable axial force on the dynamic response of elastic beam resting on elastic foundation and subjected to concentrated moving loads is investigated. The fourth order partial differential equation with variable and singular coefficients governing the motion of the elastic thin beam is solved using the Generalized Galerkin's Method and the Modified Asymptotic Method of Struble. It is established by both Analytical solution and Numerical analysis that, the higher the values of axial force N and the foundation rigidity K, the lower the response amplitudes of the elastic thin beam with variable prestress when it is under the action of the concentrated moving loads. Furthermore, it is found that the critical velocity for the system under the influence of the moving force is greater than that under the influence of the moving mass first approximation and moving mass entire beam model. Hence, resonance is reached earlier in the latter.***

**Key words:** Thin Beam, Moving force, Moving Mass, prestress, Critical Velocity, Resonance.
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## 1.0 Introduction

This paper is concerned with the problem of assessing the effects of variable axial force on the dynamic response to moving concentrated load of thin beam resting on elastic foundation.

Various structures, ranging from bridges and roads to space vehicles and submarines are constantly acted upon by moving masses and hence, the problem of analyzing the dynamic behaviour of elastic structures under the action of moving masses. Pertinent to investigators in the field of structural dynamics is the quest for an effective and reliable method in accurately determining the response of an elastic structure under the actions of heavy masses traversing it at various speeds.

Several authors have extensively studied the flexural vibration of prismatic or non prismatic rods, beams, plates and shells [1, 2, 3, 4 5]. In most of these studies, it has been tacitly assumed that prestress in beams or rods have been uniform all through the length of the elastic members. The account of the effects of axial force when it is non-uniform on the frequencies has been neglected. In most cases also, the beams considered have been idealized by one whose mass is approximately negligible. The case in which the moving load has mass commensurable with the mass of the elastic structure has suffered neglect. Among the earliest progress in this area of research is the work of Stanisic et al [6] who studied the two dimensional vibration of plate under the actions of moving masses. They considered only the inertia term which measures the effect of local acceleration in the direction of the deflection. This work was taken up much later by Gbadeyan and Oni [7] who studied the dynamic analysis of an elastic structure on an elastic Pasternak foundation when it is under an arbitrary number of concentrated masses. All the components of the inertia terms were considered and the rectangular plate was assumed to be simply supported. In a more recent development, Oni

[8] and Gbadeyan and Idowu [9] studied the one-dimensional moving loads problems that involve axial force effects. In their study, they assumed that prestress is constant along the length of the beam. The more realistic beam having variable prestress and resting on elastic foundation when it is under the action of moving concentrated masses has not been considered. Evidently, in practice, prestress varies from a point to another along a structural member. Thus, this study considers the effect of variable axial force on the dynamic response to moving concentrated loads of a Thin Beam resting on elastic foundation.

## 2.0 The Model

The transverse displacement response of a thin beam under the passage of moving concentrated load is governed by a fourth order partial differential equation given by

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}+\overline{m}\frac{\partial^2 W(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left[N(x)\frac{\partial W(x,t)}{\partial x}\right] + KW(x,t)=P_f(x,t)\left[1-\frac{\Theta}{g}W(x,t)\right] \tag{2.1}$$

where $x$ is the position coordinate in the axial direction, t is the time, $\frac{\partial}{\partial x}$ is the partial derivative with respect to x, $W(x,t)$ is the transverse displacement, $P_f(x,t)$ is the moving force, EI is the flexural stiffness $N(x)$ is the variable axial force, I is the moment of inertia and $\overline{m}$ is the constant mass per unit length of the beam and K is the stiffness of the elastic beam.

The structure under consideration is assumed to be under tensile stress, resting on elastic foundation and executing vibrations according to simple Bernoulli-Euler beam theory of flexure. Furthermore, the beam has simple support at both ends.

Thus the boundary conditions are

$$W(0,t)=0=W(L,t)\,,\ \frac{\partial^2 W(0,t)}{\partial x^2}=0=\frac{\partial^2 W(L,t)}{\partial x^2} \tag{2.2}$$

and the initial conditions of the motion are

$$W(x,0)=0=\frac{\partial W(x,0)}{\partial t} \tag{2.3}$$

The operator $\Theta$ is defined as

$$\Theta=\frac{\partial^2}{\partial t^2}+2v_i\frac{\partial^2}{\partial t\partial x}+v_i^2\frac{\partial^2}{\partial x^2} \tag{2.4}$$

and the velocity of the $i^{th}$ concentrated load is denoted by $v_i$. and $P_f(x,t)$ is defined as

$$P_f(x,t)=\sum M_i g\,\delta(x-v_i t) \tag{2.5}$$

The time t is such that $0\le tv_i\le L$, where L is the length of the beam.

Using equations (2.4) and (2.5) in equation (2.1), the governing equation of motion takes the form

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}+\overline{m}\frac{\partial^2 W(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left[N(x)\frac{\partial W(x,t)}{\partial x}\right]+KW(x,t)$$

$$+M_i\delta(x-v_it)\left[\frac{\partial^2 W(x,t)}{\partial t^2}+2v_i\frac{\partial^2 W(x,t)}{\partial t\partial x}+v_i^2\frac{\partial^2 W(x,t)}{\partial x^2}\right]=\sum_{i=1}^{N} M_i g\delta(x-v_it) \quad (2.6)$$

As an example, let the variable axial force be defined as

$$N(x)=N_0\left(1+Sin\frac{\pi x}{L}\right)^3 \quad (2.7)$$

where $N_0$ is the average value of the axial force of the elastic beam.
When equation (2.7) is substituted into equation (2.6) after some rearrangements, the equation of motion can be rewritten as

$$EI\frac{\partial^4 W(x,t)}{\partial x^4}+\overline{m}\frac{\partial^2}{\partial t^2}-\frac{\partial}{\partial x}\left[\frac{N_0}{4}\left(10+15Sin\frac{\pi x}{L}-6Cos\frac{2\pi x}{L}-Sin\frac{3\pi x}{L}\right)\frac{\partial W(x,t)}{\partial x}\right]$$

$$+KW(x,t)+M_i\delta(x-v_it)\left[\frac{\partial^2 W(x,t)}{\partial t^2}+2v_i\frac{\partial^2 W(x,t)}{\partial x\partial t}+v_i^2\frac{\partial^2 W(x,t)}{\partial x^2}\right] \quad (2.8)$$

$$=\sum_{i=1}^{N} M_i g\delta(x-v_it)$$

Equation (2.8) is a non-homogeneous partial differential equation with variable coefficient. Evidently, an exact analytical solution of the above equation does not exist.

## 3.0 Solution Procedure

The fourth order partial differential equation (2.8), apart from having variable coefficient, it is singular. This is as a result of the presence of the Dirac delta function in the governing equation which is not defined at certain values of the independent variable. In view of the above, it becomes impossible to obtain a closed form solution of the equation (2.8) by any conventional analytical method. As a result of the foregoing difficulty, an approximate analytical method of solution is sought. One of the methods suited for solving diverse problems in dynamic of structures is the Galerkin's method. This method requires that the solution of equation (2.8) be of the form

$$U_n(x,t)=\sum_{i=1}^{n} Y_m(t)V_m(x) \quad (3.1)$$

where $V_m(x)$ is chosen such that all the boundary conditions stated in (2.2) are satisfied. Since our elastic system has simple supports at the edges x = 0 and x = L, we choose

$$V_m(x)=Sin\frac{m\pi x}{L} \quad (3.2)$$

Substituting equations (3.1) and (3.2) into equation (2.8) leads to

$$\sum_{m=1}^{\infty}\left\{\overline{m}Sin\frac{m\pi x}{L}\ddot{Y}_m(t)+EI\left(\frac{m\pi}{L}\right)^2 Sin\frac{m\pi x}{L}Y_m(t)+N_0\frac{m\pi}{4L}\left(10\frac{m\pi}{L}Sin\frac{m\pi x}{L}-15\frac{\pi}{L}Cos\frac{\pi x}{L}Cos\frac{m\pi x}{L}\right.\right.$$

$$+15\frac{m\pi}{L}Sin\frac{\pi x}{L}Sin\frac{m\pi x}{L}-12\frac{\pi}{L}Sin\frac{2\pi x}{L}Cos\frac{m\pi x}{L}-6Cos\frac{2\pi x}{L}Sin\frac{m\pi x}{L}$$

$$\left.+3\frac{\pi}{L}Cos\frac{3\pi x}{L}Cos\frac{m\pi x}{L}-\frac{m\pi}{L}Sin\frac{3\pi x}{L}Sin\frac{m\pi x}{L}\right)Y_m(t)$$

$$+M_i\left[\delta(x-v_i t)\ddot{Y}_m(t)Sin\frac{m\pi x}{L}+2v_i\frac{m\pi}{L}\delta(x-v_i t)\dot{Y}_m(t)Cos\frac{m\pi x}{L}\right.$$

$$\left.\left.-v_i^2\left(\frac{m\pi}{L}\right)^2\delta(x-v_i t)Y_m(t)Sin\frac{m\pi x}{L}\right]\right\}-\sum_{i=1}^{N}M_i g\delta(x-v_i t)=0 \tag{3.3}$$

In order to determine $Y_m(t)$ it is required that the expression on the left hand side of equation (3.3) be orthogonal to the function

$$V_k(x)=Sin\frac{k\pi x}{L} \tag{3.4}$$

Consequently, using (3.4) in (3.3) one obtains

$$\sum_{m=1}^{n}\left\{H_0\ddot{Y}_m(t)+(H_1+H_2)Y_m(t)\right.$$

$$\left.+\sum_{i=1}^{N}\frac{M_i}{\overline{m}}\left[H_3(t)\ddot{Y}_m(t)+2v_iH_4(t)\dot{Y}_m(t)-v_i^2H_5(t)Y_m(t)\right]\right\} \tag{3.5}$$

$$=\sum_{i=1}^{N}\frac{M_i g}{\overline{m}}H_6(t)$$

where

$$H_0=\int_0^L Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{3.6}$$

$$H_1=EI\left(\frac{m\pi}{L}\right)^4\int_0^L Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{3.7}$$

$$H_2=G_1-G_2+G_3-G_4-G_5+G_6-G_7 \tag{3.8}$$

$$G_1=\frac{10N_0}{4}\left(\frac{m\pi}{L}\right)^2\int_0^L Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{3.9}$$

$$G_2=\frac{15N_0}{4m}\left(\frac{m\pi}{L}\right)^2\int_0^L Cos\frac{\pi x}{L}Cos\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{3.10}$$

$$G_3=\frac{15N_0}{4}\left(\frac{m\pi}{L}\right)^2\int_0^L Sin\frac{\pi x}{L}Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{3.11}$$

$$G_4 = \frac{12N_0}{4m}\left(\frac{m\pi}{L}\right)^2 \int_0^L Sin\frac{2\pi x}{L} Cos\frac{m\pi x}{L} Sin\frac{k\pi x}{L} dx \tag{3.12}$$

$$G_5 = \frac{6N_0}{4}\left(\frac{m\pi}{L}\right)^2 \int_0^L Cos\frac{2\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L} dx \tag{3.13}$$

$$G_6 = \frac{3N_0}{4}\left(\frac{m\pi}{L}\right)^2 \int_0^L Cos\frac{3\pi x}{L} Cos\frac{m\pi x}{L} Sin\frac{k\pi x}{L} dx \tag{3.14}$$

$$G_7 = \frac{N_0}{4}\left(\frac{m\pi}{L}\right)^2 \int_0^L Sin\frac{3\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L} dx \tag{3.15}$$

$$H_3(t) = \int_0^L \delta(x - v_i t) Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L} dx \tag{3.16}$$

$$H_4(t) = \frac{m\pi}{L}\int_0^L \delta(x - v_i t) Cos\frac{m\pi x}{L} Sin\frac{k\pi x}{L} dx \tag{3.17}$$

$$H_5(t) = \left(\frac{m\pi}{L}\right)^2 \int_0^L \delta(x - v_i t) Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L} dx \tag{3.18}$$

$$H_6(t) = \int_0^L \delta(x - v_i t) Sin\frac{k\pi x}{L} dx \tag{3.19}$$

when integrals (3.6) to (3.19) are evaluated, one obtains a series of coupled second order ordinary differential equations called Galerkin's equations governing the coefficients of all lower and higher modes of the beam.

Equation (3.5) will be solved by considering only a mass moving with velocity v thus, after some simplifications and rearrangements equation (3.5) reduces to

$$\begin{aligned}\sum_{m=1}^{n} \Big\{\ddot{Y}_m(t) + \omega_{nf}^2 Y_m(t) + \eta * \big[\ddot{Y}_m(t) + S_0(k,m)\ddot{Y}_m(t) + S_1(k,m)\dot{Y}_m(t) \\ + S(k,m,n)\dot{Y}_m(t) - S_2(k,m)Y_m(t) + S_0 S_2 Y_m(t)\big]\Big\} = \frac{Mg}{\overline{m}L} Sin\frac{k\pi vt}{L}\end{aligned} \tag{3.20}$$

where

$$\omega_{nf}^2 = \frac{2(H_1 + H_2)}{\overline{m}L},\ S_0 = 2Sin\frac{m\pi vt}{L} Sin\frac{k\pi vt}{L},\ S_1(k,m) = \frac{8mvk}{L(k^2 - m^2)} \tag{3.21a}$$

$$S*(k,m,n) = \sum_{n=1}^{\infty} \frac{16mvk(k^2 - m^2 - n^2)}{L[k^2 - (n+m)^2][k^2 - (n-m)^2]} Cos\frac{n\pi vt}{L},\ S_2(k,m) = \frac{v^2 m^2 \pi^2}{2L}$$

and $\eta^* = \frac{M}{\overline{m}L}$ (3.21b)

equation (3.20) is the transformed equation governing the mode response of a thin beam with variable axial force, under tensile stress and subjected to moving concentrated masses. Evidently, equation (3.20) is not amenable to any known analytical method of solution; hence an exact analytical solution to this equation is impossible. In what follows three special cases of equation (3.20) are considered.

### 3.1 The Moving Force Prestressed Thin Beam Problem.

In solving the moving force problem of equation (3.20), when the inertia term effect of the moving mass m is neglected one obtains the classical case of a moving force problem and equation (3.20) reduces to

$$\frac{d^2Y_m(t)}{dt^2} + \omega_{nf}^2 Y_m(t) = \frac{2Mg}{\overline{m}L} Sin\frac{k\pi vt}{L} \tag{3.22}$$

To obtain the solution of the equation (3.22), it is subjected to a Laplace transform defined as

$$(\tilde{\cdot}) = \int_0^\infty (\cdot) e^{-st} dt \tag{3.23}$$

where $s$ is the Laplace parameter. Applying the initial conditions (2.3), one obtains the simple algebraic equation given by

$$Y_m(s) = \frac{2Mg}{\overline{m}L\omega_{nf}} \left[ \frac{\alpha}{s^2 + \alpha^2} \cdot \frac{\omega_{nf}}{s^2 + \omega_{nf}^2} \right] \tag{3.24}$$

where

$$\alpha = \frac{k\pi v}{L} \tag{3.25}$$

Thus, the problem reduces to that of obtaining the Laplace inversion of (3.24). To do this we adopt the following representations:

$$\tilde{f}(s) = \frac{\omega_{nf}}{s^2 + \omega_{nf}^2}, \qquad \tilde{g}(s) = \frac{\alpha}{s^2 + \alpha^2} \tag{3.26}$$

so that the Laplace inversion of $Y_m(s)$ is the convolution of $f(s)$ and $g(s)$ defined as

$$f(s) * g(s) = \int_0^t f(t-u) g(u) du \tag{3.27}$$

In view of (3.27), $Y_m(t)$ is expressed as

$$Y_m(t) = \frac{2Mg}{\overline{m}L\omega_{nf}} \int_0^t Sin\omega_{nf}(t-u) Sin\alpha u du \tag{3.28}$$

Equation (3.28) after some rearrangements takes the form

$$Y_m(t) = \frac{Mg}{\overline{m}L\omega_{nf}} \left[ I_a Cos\omega_{nf} t + I_b Sin\omega_{nf} t - I_c Cos\omega_{nf} t + I_d Sin\omega_{nf} t \right] \tag{3.30}$$

where

$$I_a = \int_0^t Cos(\omega_{nf} + \alpha) u du, \qquad I_b = \int_0^t Sin(\omega_{nf} + \alpha) u du, \tag{3.31}$$

and

$$I_c = \int_0^t Cos(\alpha - \omega_{nf})udu, \qquad I_d = \int_0^t Sin(\alpha - \omega_{nf})udu \tag{3.32}$$

Evaluating integrals in (3.31) and (3.32), one obtains

$$I_a = \frac{Sin(\omega_{nf} + \alpha)t}{\omega_{nf} + \alpha}, \quad I_b = -\frac{(Cos(\omega_{nf} + \alpha)t - 1)}{\omega_{nf} + \alpha}, \quad I_c = \frac{Sin(\alpha - \omega_{nf})t}{\alpha - \omega_{nf}} \tag{3.33}$$

and

$$I_d = -\frac{(Cos(\alpha - \omega_{nf})t - 1)}{\alpha - \omega_{nf}} \tag{3.34}$$

Substituting (3.33) and (3.34) into equation (3.30), after some simplifications and rearrangements we obtain an expression for $Y_m(t)$ as

$$Y_m(t) = \frac{2Mg}{\overline{m}L\omega_{nf}\left(\alpha^2 - \frac{2(H_1+H_2)}{\overline{m}L}\right)}\left(\alpha Sin\left(\sqrt{\frac{2(H_1+H_2)}{\overline{m}L}}\right)t - \left(\sqrt{\frac{2(H_1+H_2)}{\overline{m}L}}\right)Sin\alpha t\right) \tag{3.35}$$

Thus, in view of equations (3.1) and (3.2), one obtains

$$U_n(x,t) = \sum_{m=1}^{n}\left(\frac{P_0^*}{\left(\alpha^2 - \frac{2(H_1+H_2)}{\overline{m}L}\right)}\left(\alpha Sin\left(\sqrt{\frac{2(H_1+H_2)}{\overline{m}L}}\right)t - \left(\sqrt{\frac{2(H_1+H_2)}{\overline{m}L}}\right)Sin\alpha t\right)\right)Sin\frac{m\pi x}{L} \tag{3.36}$$

where

$$P_0^* = \frac{2Mg}{\overline{m}L\sqrt{\frac{2(H_1+H_2)}{\overline{m}L}}} \tag{3.37}$$

Equation (3.36) represents the transverse-displacement response of an elastic thin beam under the passage of moving concentrated loads due to moving force.

## 3.2 The Moving Mass- First Approximation of prestressed thin beam problem.

If we consider only the linear inertia term, equation (3.20) after some rearrangements becomes

$$\frac{d^2Y_m(t)}{dx^2} + \gamma_{FA}^2 Y_m(t) = P_{FA}^* Sin\frac{k\pi vt}{L} \tag{3.38}$$

where

$$\gamma_{FA}^2 = \frac{\omega_{nf}^2 - \frac{v^2 m^2 \pi^2}{L^2}}{1+\eta} \tag{3.39}$$

and

$$P_{FA}^* = \frac{2Mg}{mL(1+\eta)} \tag{3.40}$$

This model represents an approximation to the moving load problem when coupled terms are neglected. The general solution of equation (3.38) is directly analogous to equation (3.22). Applying the initial condition (2.3) one obtains

$$Y_m(t) = \frac{P_{FA}^*}{\alpha^2 - \frac{\omega_{nf}^2 - \frac{v^2m^2\pi^2}{L^2}}{1+\eta}}\left(\alpha Sin\left(\sqrt{\frac{\omega_{nf}^2 - \frac{v^2m^2\pi^2}{L^2}}{1+\eta}}\right)t - \left(\sqrt{\frac{\omega_{nf}^2 - \frac{v^2m^2\pi^2}{L^2}}{1+\eta}}\right)Sin\alpha t\right.$$

$$\left. - \left(\sqrt{\frac{\omega_{nf}^2 - \frac{v^2m^2\pi^2}{L^2}}{1+\eta}}\right)Sin\alpha t\right)\Bigg) Sin\frac{m\pi x}{L} \tag{3.41}$$

Thus, by equation (3.1), we have

$$U_n(x,t) = \sum_{m=1}^{n}\left(\frac{P_{FA}^*}{\alpha^2 - \frac{\omega_{nf}^2 - \frac{v^2m^2\pi^2}{L^2}}{1+\eta}}\left(\alpha Sin\left(\sqrt{\frac{\omega_{nf}^2 - \frac{v^2m^2\pi^2}{L^2}}{1+\eta}}\right)t\right.\right. \tag{3.42}$$

## 3.3 The Moving Mass-Entire Equation of Prestressed Thin Beam Problem.

The next attempt is to solve the entire coupled moving load problem. An exact closed form solution to this equation is not possible. Thus, to solve equation (3.20), we resort to the analytical solution technique which is a modification of the asymptotic method of Struble. This technique requires that the equation (3.20) be rewritten in the form

$$\ddot{Y}_m(t) + \frac{\Delta_1(n,t,\eta)}{\Delta_0(m,t,\eta)}\dot{Y}_m(t) + \frac{\Delta_2(n,t,\eta)}{\Delta_0(m,t,\eta)}Y_m(t) = \frac{P_{mm}^*}{\Delta_0(m,t,\eta)}Sin\frac{k\pi vt}{L} \tag{3.43}$$

where

$$\Delta_0(m,t,\eta) = 1+\eta\left(1+2Sin\frac{m\pi vt}{L}Sin\frac{k\pi vt}{L}\right) \tag{3.44}$$

$$\Delta_1(n,t,\eta) = \eta\left(\frac{8mvk}{L\left(k^2-m^2\right)} + \sum_{n=1}^{\infty} \frac{16mvk\left(k^2-m^2-n^2\right)}{L\left[k^2-(n+m)^2\right]\left[k^2-(n-m)^2\right]}Cos\frac{n\pi vt}{L}\right) \tag{3.45}$$

$$\Delta_2(n,t,\eta) = \omega_{nf}^2 + \eta\left(\frac{v^2m^2\pi^2}{2L} + \frac{v^2m^2\pi^2}{L}Sin\frac{m\pi vt}{L}Sin\frac{k\pi vt}{L}\right) \tag{3.46}$$

Specifically, by means of this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the moving load. An equivalent free system operator defined by the modified frequency then replaces equation (3.43). Thus, we set the right hand side of (3.43) to zero and consider a parameter $\tau_0 < 1$ for any arbitrary mass ratio $\eta *$ defined as

$$\tau_0 = \frac{\eta *}{1+\eta *} \tag{3.47}$$

which implies that,

$$\eta * = \tau_0 + O\left(\tau_0^2\right) \tag{3.48}$$

and

$$\frac{1}{1+\eta *\left(1+2Sin\frac{m\pi vt}{L}Sin\frac{k\pi vt}{L}\right)} = \left[1-\tau_0\left(1+2Sin\frac{m\pi vt}{L}Sin\frac{k\pi vt}{L}\right)+O\left(\tau_0^2\right)\cdots\right] \tag{3.49}$$

Using (3.53), equation (3.47) becomes

$$\begin{aligned}&\ddot{Y}_m(t) + \tau_0\left[S_1(k,m) + S*(k,m,n)\right]\dot{Y}_m(t) + \left[\omega_{nf}^2 - \tau_0\omega_{nf}^2(1+S_0(k,m)\right. \\ &\left. - \tau_0\left(S_2(k,m) + S_0(k,m)S_2(k,m)\right)Y_m(t)\right] = \frac{2\tau_0 Lg}{\overline{m}L}Sin\frac{k\pi vt}{L}\end{aligned} \tag{3.50}$$

when $\tau_0$ is set to zero in equation (3.50), a situation corresponding to the case in which the inertia effect of the mass of the system is regarded as negligible is obtained. In such case, the solution is of the form

$$Y_m(t) = C_0^* Cos\left[\omega_{nf}t - \phi_j\right] \tag{3.51}$$

where $C_0^*$, $\omega_{nf}$ and $\phi_j$ are constants. Furthermore, as $\tau_0 < 1$, the Struble's technique requires that the solution of equation (3.50) be of the form

$$Y_m(t) = Q(m,t)Cos\left[\omega_{nf}t - \Omega(m,t)\right] + \tau_0 Y_{m1}(t) + O\left(\tau_0^2\right) \tag{3.52}$$

where $Q(m,t)$ and $\Omega(m,t)$ are slowly varying functions of time.

In order to obtain the modified frequency, equation (3.52) and its derivatives are substituted into the homogeneous part of the equation (3.50). Thereafter, we extract only the variational part of the equation describing the behaviour of $Q(m,t)$ and $\Omega(m,t)$ during the motion of the mass. Thus, making this substitution and neglecting terms that do not contribute to the variational equations we obtain

$$\begin{aligned}&-2\omega_{nf}\dot{Q}(m,t)Sin\left[\omega_{nf}t - \Omega(m,t)\right] + 2Q(m,t)\omega_{nf}\dot{\Omega}(m,t)Cos\left[\omega_{nf}t - \Omega(m,t)\right] \\ &- S_1(k,m)Q(m,t)\omega_{nf}Sin\left[\omega_{nf}t - \Omega(m,t)\right] - \omega_{nf}^2\tau_0 Q(m,t)Cos\left[\omega_{nf}t - \Omega(m,t)\right] \\ &- \tau_0 S_2(k,m)Q(m,t)Cos\left[\omega_{nf}t - \Omega(m,t)\right] = 0\end{aligned} \tag{3.53}$$

to $O(\tau_0)$ only.

The variational equations of our problem are obtained by setting coefficients of $Sin\lfloor\omega_{nf} t - \Omega(m,t)\rfloor$ and $Cso\lfloor\omega_{nf} t - \Omega(m,t)\rfloor$ in equation (3.53) to zero respectively. Thus, we have

$$-2\omega_{nf}\dot{Q}(m,t) - S_1(k,m)\omega_{nf}\tau_0 Q(m,t) = 0 \tag{3.54}$$

and

$$2Q(m,t)\omega_{nf}\dot{\Omega}(m,t) - \omega_{nf}^2\tau_0 Q(m,t) - \tau_0 S_2(k,m)Q(m,t) = 0 \tag{3.55}$$

Thus, equations (3.54) and (3.55) respectively lead to

$$\frac{\dot{Q}(m,t)}{Q(m,t)} = -\frac{S_1(k,m)\tau_0}{2} \tag{3.56}$$

and

$$\dot{\Omega}(m,t) = \frac{\tau_0\omega_{nf}^2 + S_2(k,m)\tau_0}{2\omega_{nf}} \tag{3.57}$$

Solving equations (3.60) and (3.61) respectively, one obtains

$$Q(m,t) = C^* e^{-\frac{S_1(k,m)\tau_0}{2}} \tag{3.58}$$

and

$$\Omega(m,t) = \frac{\tau_0\lfloor\omega_{nf}^2 + S_2(k,m)\rfloor}{2\omega_{nf}}t + \psi_m \tag{3.59}$$

where $C^*$ and $\psi_m$ are constants. Thus,

$$Y_m(t) = C^* e^{-\frac{S_1(k,m)\tau_0}{2}} Cos\lfloor\omega_{mf} t - \psi_m\rfloor \tag{3.60}$$

where

$$\omega_{mf} = \omega_{nf}\left[1 - \frac{\tau_0}{2}\left(1 + \frac{S_2(k,m)}{\omega_{nf}^2}\right)\right] \tag{3.61}$$

is called the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. Thus, the homogeneous part of equation (3.50) can be written as

$$\frac{d^2 Y_m(t)}{dt^2} + \omega_{mf}^2 Y_m(t) = 0 \tag{3.62}$$

Hence, equation (3.50) finally takes the form

$$\frac{d^2 Y_m(t)}{dt^2} + \omega_{mf}^2 Y_m(t) = \frac{2\tau_0 g}{\overline{m}} Sin\frac{k\pi vt}{L} \tag{3.63}$$

Evidently, equation (3.63) is analogous to equation (3.22); thus, solving equation (3.63) in conjunction with the initial conditions one obtains

(3.64)

$$Y_m(t)=\frac{2\tau_0 g}{\overline{m}\left(\alpha^2-\varpi_{mf}^2\right)}\left(aSin\alpha\left(\varpi_{nf}\left[1-\frac{\tau_0}{2}\left(1+\frac{S_2(k,m)}{\varpi_{nf}^2}\right)\right]\right)t\right.$$

$$\left.-\left(\omega_{nf}\left[1-\frac{\tau_0}{2}\left(1+\frac{S_2|k,m|}{\omega_{nf}^2}\right)\right]\right)Sin\alpha t\right) \qquad (3.64)$$

In view of equation (3.1) and (3.2) we therefore obtain

$$U_n(x,t)=\sum_{m=1}^{n}\left(\frac{2\tau_0 g}{\overline{m}\left(\alpha^2-\omega_{mf}^2\right)}\alpha\sin\left(\varpi_{nf}\left[1-\frac{\tau_0}{2}\left(1+\frac{S_2(k,m)}{\varpi_{nf}^2}\right)\right]\right)t\right.$$

$$\left.-\left(\varpi_{nf}\left[1-\frac{\tau_0}{2}\left(1-\frac{S_2(k,m)}{\varpi_{nf}^2}\right)\right]\right)Sin\alpha\right)\right)Sin\frac{m\pi x}{L} \qquad (3.65)$$

which represents the transverse displacement response of an elastic Thin beam under the passage of moving concentrated loads due to moving mass.

## 4.0 Discussion of the Closed Form Solution

In a work such as this, it is imperative that the resonance phenomenon is investigated, because the transverse displacement of elastic solid structures may grow without bound.

Equation (3.36) clearly shows that the prestressed elastic beam resting on elastic foundation and transverse by a moving force will grow without bound whenever

$$\omega_{nf}^2=\alpha^2 \qquad (4.10)$$

while equation (3.42) depicts that the same beam under the action of a moving mass only the linear inertia terms are considered experiences resonance when

$$\gamma_{FA}^2=\alpha^2 \qquad (4.11)$$

Similarly, equation (3.65) shows that the prestressed beam under the action of the entire moving mass when no term is neglected experiences resonance whenever

$$\omega_{mf}^2=\alpha^2 \qquad (4.12)$$

For the problem of moving mass first approximation of prestressed Euler-Bernoulli beam problem the natural frequency is given by

$$\gamma_{FA}^2=\frac{\omega_{nf}^2-\dfrac{v^2m^2\pi^2}{L^2}}{1+\eta} \qquad (4.13)$$

which implies

$$\omega_{nf}=\frac{\alpha^2}{\dfrac{1}{1+\eta}\left[\omega_{nf}-\dfrac{v^2m^2\pi^2}{\omega_{nf}L^2}\right]} \qquad (4.14)$$

but, in the case of the Moving mass when no term is neglected in the transformed governing equation the modified natural frequency is obtained as

$$\omega_{mf} = \omega_{nf}\left[1-\frac{\tau_0}{2}\left(1+\frac{S_2(k,m)}{\omega_{nf}^2}\right)\right] \tag{4.15}$$

which implies

$$\omega_{nf} = \frac{\alpha^2}{1-\frac{\tau_0}{2}\left(1+\frac{S_2(k,m)}{\omega_{nf}^2}\right)} \tag{4.16}$$

Thus, from equations (4.11) and (4.14) and equations (4.12) and (4.16) it is easily deduced, that for the same natural frequency, the critical speed for the system consisting of an elastic beam resting on an elastic foundation and traversed by a moving force is greater than that of the moving mass first approximation and moving mass when no term is neglected. Hence, resonance is reached earlier in the case of a moving mass problem.

## 5.0 Comments on the Numerical Results and Analysis.

In this section, we illustrate the theory presented in this paper numerically. Elastic thin beam of length 12.2m is considered, other parameters used are stated as follows, $\gamma = 2\times10^{-4}m$, $\beta = \frac{3\pi}{4}$, $\frac{EI}{\mu} = 2200m^4/s^2$ and the ratio of the mass of the load to the mass of the beam is 0.25. The values of axial force N and subgrade K, are between 0 and 3,500,000 N and 0 and $5500{,}000N/m^3$ respectively. Figures 1 depicts the deflection profiles of the elastic beams with a non-uniform axial force under the actions of concentrated moving loads for various values of axial force N and for fixed K. The figure shows that as N increases, response amplitudes of the prestressed beam decrease. Similarly, as the foundation modulli K increases, for fixed value of N, the displacement response of the moving force of the simply supported uniform beams subjected to moving heavy masses decrease as shown in Figure 2.

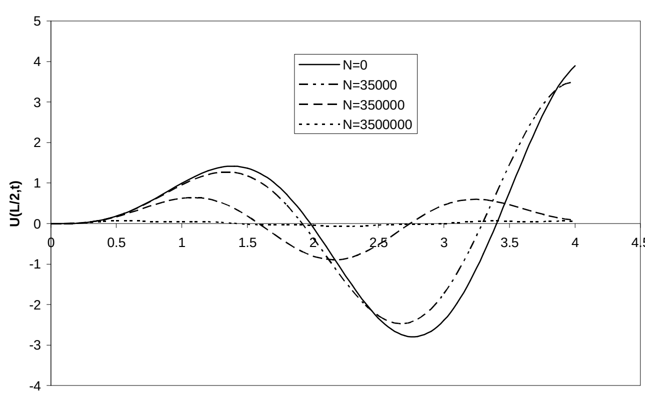


**Figure 1: The transverse displacement response for the moving force of the prestressed thin beam for various values of N and fixed foundation moduli K =** $55000N/m^3$**.**

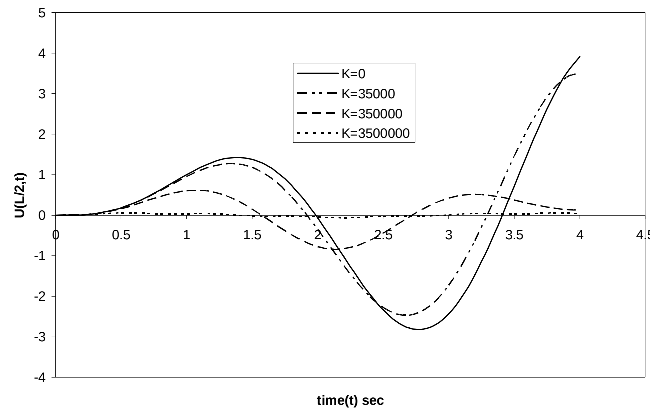


**Figure 2: The displacement profile for the moving force of the prestressed thin beam for various values of K and fixed value of axial force N = 35000*N*.**

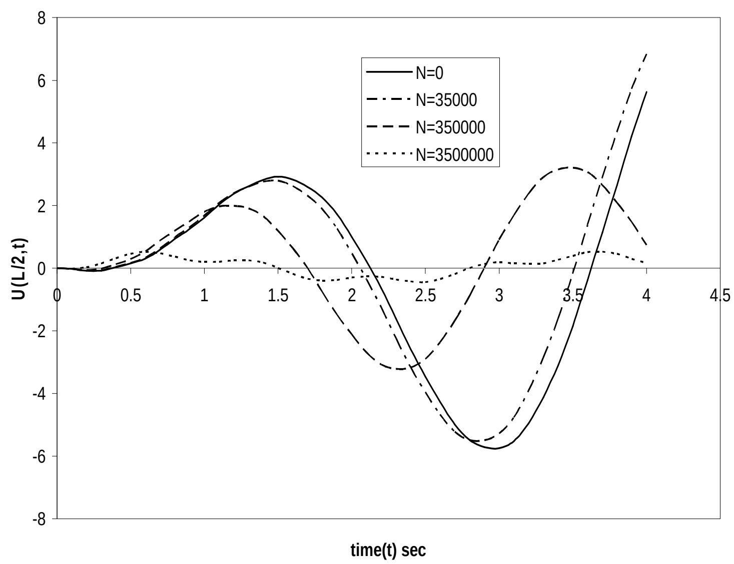


**Figure 3: The transverse displacement response for the moving mass first approximation of uniform thin beam for various values of N and fixed foundation moduli K= 55000 $N/m^3$.**

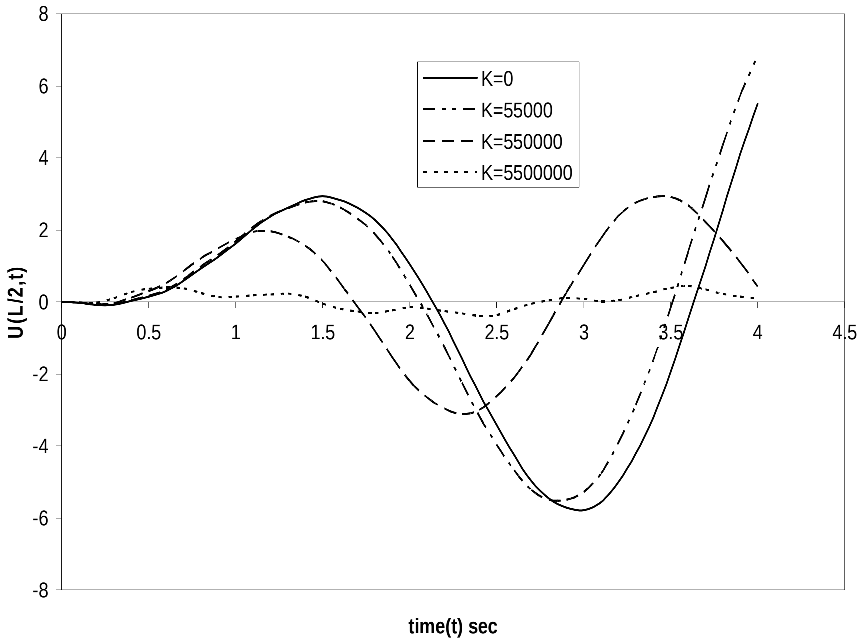


**Figure 4: The displacement profile for the moving mass first approximation of the prestressed for various values of K and fixed value of axial force N = 35000*N*.**

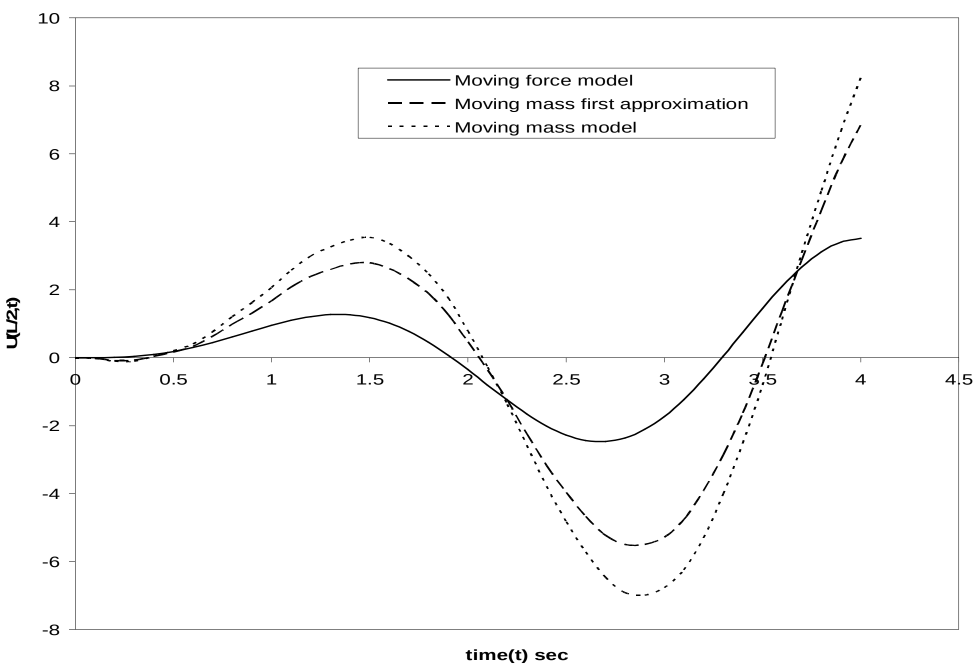


**Figure 5. Comparison of the moving force, moving mass first approximation and the entire moving mass of the prestressed thin beam for fixed value of N = 35000*N* and foundation moduli K= 55000.**

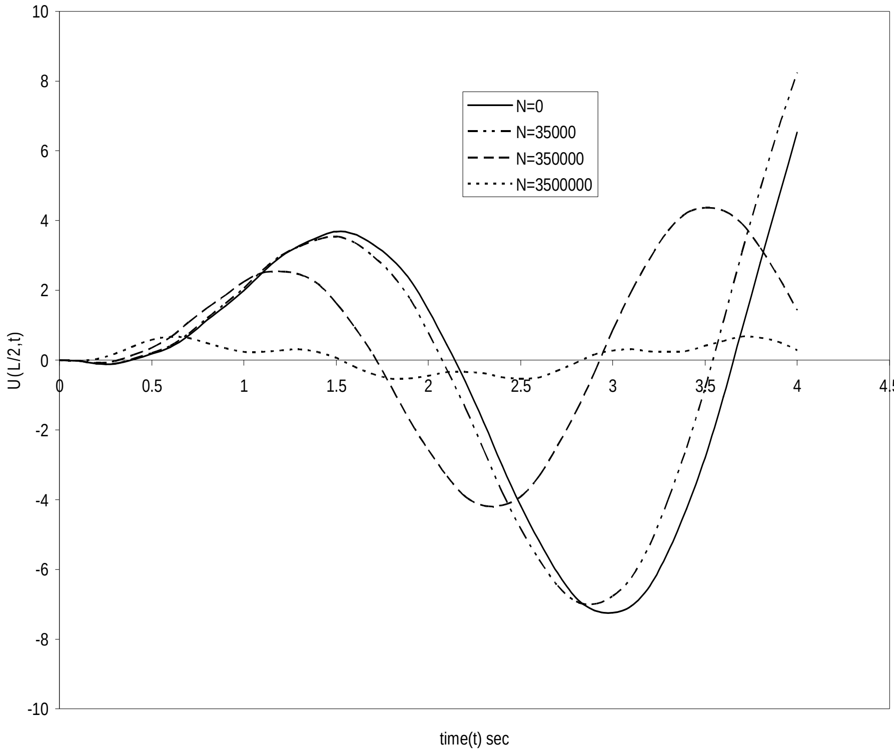


**Figure 6: The transverse displacement response for the moving mass entire of the prestressed thin beam for various values of N and fixed foundation moduli K= 55000 $N/m^3$.**

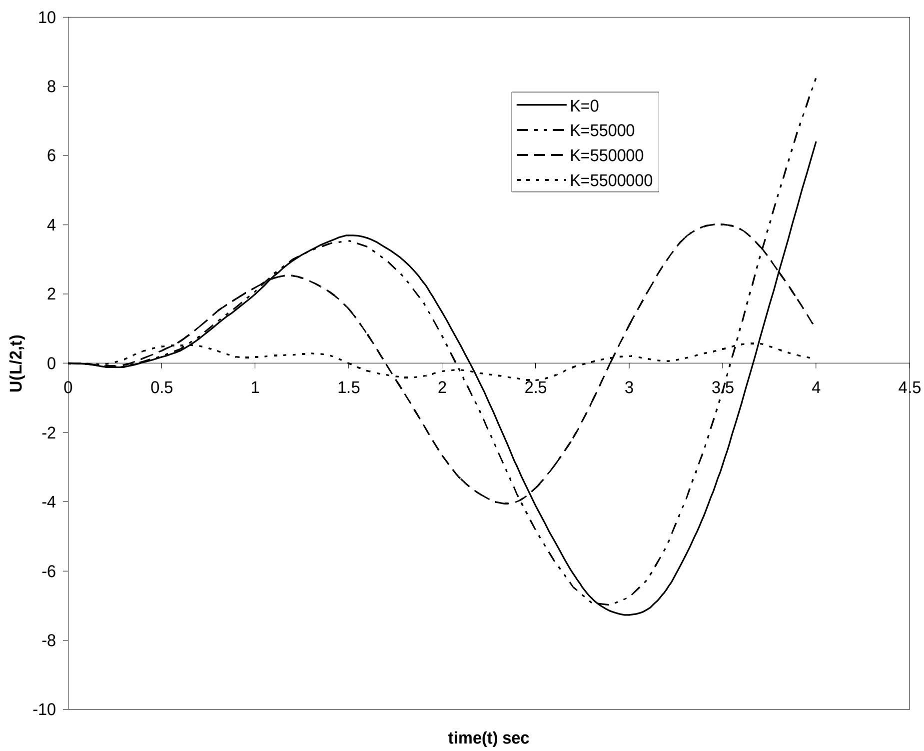


**Figure 7: The displacement profile for the moving mass entire of the prestressed thin beam for various values of foundation moduli K and fixed value of axial force N= 35000*N*.**

In Figure 3, the transverse displacement of moving mass first approximation case for Simply-Supported uniform beam traversed by a moving load is displayed. It is clearly seen from the figure that as N increases for fixed value of the foundation stiffness K the transverse displacement response of the prestressed beam decrease. In the same manner, as the foundation stiffness K increases for fixed N = 35,000*N* figure 4 shows that the transverse displacement of the thin elastic beam with finite span decrease.

Furthermore**,** figure 5 shows the transverse displacement of moving force, moving mass first approximation and moving mass cases for Simply-Supported uniform beam traversed by moving load for fixed N = 35,000*N* and K = 55,000 $N/m^3$. Clearly, the response amplitudes of moving mass first approximation and moving mass when no term is neglected are higher than that of the moving force. This result shows that it is erroneous and misleading to rely on moving force model as a save approximation to a moving mass model.

In a similar manner, the deflection profiles for the moving mass of the Simply Supported, Bernoulli-Euler beams under the actions of moving concentrated loads for various values of axial force N and for fixed K are displayed in Figure 6. It is seen from the figure that as N increases response amplitudes of the prestressed Bernoulli-Euler beam decrease. In Figure 7, it is equally shown that for various values of foundation moduli K and fixed value of N = 35, 000*N* the response amplitudes of the moving mass of the elastic beam under consideration decrease.

## 6.0 Conclusions

In this paper, analytical solution has been obtained for the dynamic response to moving concentrated load of Bernoulli-Euler beam resting on elastic foundation. The Generalized Galerkin's Method and the modified asymptotic method of Struble were used. This analytical technique has advantage over numerical method of solution in the sense that the solution obtained by it sheds more light on vital information about dynamical system.

Finally, Analytical solution and Numerical analysis in plotted curves, show that:

i resonance is reached earlier in a system traversed by both moving mass first approximation and moving mass entire than in that under the action of a moving force

ii. when the axial force N is fixed, the displacements of a uniform Bernoulli-Euler beam resting on elastic foundation and traversed by moving concentrated load decrease, as the foundation rigidity K increases.

iii. as the axial force N increases, the amplitudes of a uniform beam under the actions of moving concentrated load decrease.

iv relying on moving force model as a good approximation to moving mass model is quite misleading and tragic.

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