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Rotatory Inertia Influence on the highly prestressed orthotropic rectangular plates under travelling loads

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Abstract

The problem of assessing the rotatory inertia influence on the highly prestressed orthotropic rectangular plates is investigated in this paper. The method of Composite Expansion (MCE),a singular perturbation technique, is employed in conjunction with the method of integral transformation and Cauchy Residue theorem to obtain to $0(\varepsilon)$ a uniformly valid solution in the entire domain of definition of the rectangular plate . Analyses show that material orthotropy does not affect the leading order solution but its effects are present in the first order correction. To $O(\varepsilon)$ all the three pertinent parameters, the anisotropic prestress material orthotropy and rotatory inertial affect the response of the rectangular plate. It is also found that the critical velocities of the dynamical system increase with an increase in prestress for all values of rotatory inertia used. Thus resonance is reached ealier for lower values of prestress. Finally for high values of rotatory inertia correction factor, the critical velocity approaches a constant value indicating that resonant effect is remote for high values of rotatory inertia correction factor.

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1.0 Introduction

The study of the behaviour of solid bodies subjected to moving load has been the concern of several researchers. Among these are the work of Stanisic et al [1], Milormir et al [2], Sadiku and Lepholz [3], Oni [4], Gbadeyan and Oni [5] to mention but a few. The aforementioned researchers worked on one-dimensional dynamical beam problems. Among the earliest work on moving load plate problem is the work of Holl [6]. He solved the problem of a rectangular plate under the action of uniform moving loads. He indicated that a critical velocity existed for each vibration modes. Much later Stanistic et al [7] studied the two dimensional problems of flexural vibration of plate under the action of loads, paying more attention to moving mass. Only the inertial term that measures the effect of local acceleration in the direction of the deflection was considered. The work of Stanisic et al [7] was taken up much later by Gbadeyan and Oni [8] who studied the dynamic analysis of an elastic plate continuously supported by an elastic Pasternak foundation traversed by an arbitrary number of concentrated masses. The deflection of the plate was calculated for several values of the foundation moduli and shown graphically as a function of time.

In all the aforementioned studies, no considerations has been given to bending effects at the boundaries. In particular, when a plate structure is highly prestressed, a small parameter multiplies the highest derivative in the governing differential equation.

This class of plate dynamical problems in which a small parameter multiplies the highest derivative in the governing differential equation, is not common in literature when the plate is subjected to a moving load. However, this class of plate problems has been solved when the plate is executing free vibration or when a static load is acting on such plate using perturbation techniques. Singular perturbation has to date seen relatively little use in solid mechanics but it is nontheless been successfully used by Cole [9]. In particular, Hutter and Olunloyo [10] have also employed it in investigating rectangular membranes with small bending stiffness. Much later, Olunloyo and Hutter

[11] studied the response of thin, isotropic, prestressed rectangular plate for the case when the ratio of bending rigidity to the applied in-plane loading is small. He used the Method of Composite Expansion (MCE) to construct solutions for various boundary conditions. In a more recent paper, Gbadeyan and Oyediran [12] compared the singular perturbation techniques of MCE with that of Method of Matched Asymptotic Expansions MMAE for initially stressed thin rectangular plate. They found that the results of the MMAE agree with those obtained using the generalized MCE and specialized version of MCE when the effect of shearing deformation is $O(\varepsilon)$. Another work worthy of mention is the work of Oluloyo and Hutter [13] who investigated the dynamic response of presteressed rectangular membrane to certain external time dependent forces when the effect of bending rigidity is small using the MCE. It is remarked at this juncture that, to the best of the authors' knowledge aside, of the work of Oni [14], calculations for this class of problems for moving load plate problems do not exist in literature. However, in the work of Oni,the effect of rotatory inertia correction factor is neglected. Thus, in this work, the transverse motions of a fully clamped highly prestressed orthotropic rectangular plate incorporating the effect of rotatory inertia correction factor, under the action of moving load is considered.

2.0 Mathematical Model

The transverse displacement of highly prestressed orthotropic rectangular plate under a moving load with the effect of rotatory interia correction factor incorporated, is governed by the fourth order partial differential equation [10]

$$D_{\overline{xx}} \frac{\partial^4 U}{\partial \bar{x}^4} - 2D_{\overline{xy}} \frac{\partial^4 U}{\partial \bar{x}^2 \partial \bar{y}^2} + D_{\overline{yy}} \frac{\partial^4 U}{\partial \bar{y}^4} - N_{\overline{xx}} \frac{\partial^4 U}{\partial \bar{x}^2} - N_{\overline{yy}} \frac{\partial^4 U}{\partial \bar{y}^2}$$
$$- \overline{m} R_{ot} \frac{\partial^2}{\partial \bar{t}^2}\left(\nabla^2 U\right) + \overline{m} \frac{\partial^2 U}{\partial \bar{t}^2} = P(\bar{x}, \bar{y}; \bar{t}) \qquad (2.1)$$

Where, $D_{\overline{xx}}$ is the flexural rigidity in $\bar{x}$ – direction, $D_{\overline{yy}}$ is the flexural rigidity in $\bar{y}$ -direction $D_{\overline{xy}}$ is the effective torsional rigidity , $N_{\overline{xx}}$ is the axial prestress in $\bar{x}$ -direction, $N_{\overline{yy}}$ is the axial presstress in $\bar{y}$ -direction , $\bar{x}, \bar{y}$ are the spatial coordinates, $\bar{t}$ is the time coordinate, R_{ot} is the rotatory inertia , $\overline{m}$ is the mass of plate per unit area, U is the deflection of the plate, P(x, y; $\bar{t}$) is the applied moving load , ∇^2 U is the Laplacian operation of U, Since the plate is fully clamped , the boundary conditions of the plate are taken to be hinged, thus the deflection and slope vanish identically. Thus,

$$\left.\begin{matrix} \bar{x} = 0, & 0 \le \bar{y} \le b \\ \bar{x} = L, & 0 \le \bar{y} \le b \end{matrix}\right\} U(\bar{x}, \bar{y}; \bar{t}) = 0 \quad , \quad \frac{\partial U(\bar{x}, \bar{y}; \bar{t})}{\partial x} = 0$$
$$\left.\begin{matrix} Y = 0, & 0 \le x \le L \\ \bar{x} = b, & 0 \le x \le L \end{matrix}\right\} U(\bar{x}, \bar{y}; \bar{t}) = 0 \quad , \quad \frac{\partial U(\bar{x}, \bar{y}; \bar{t})}{\partial y} = 0 \qquad (2.2)$$

For simplicity, the initial conditions are taken to be

$$U(\bar{x}, \bar{y}; o) = 0 \quad , \quad \frac{\partial U(\bar{x}, \bar{y}; o)}{\partial t} = 0 \qquad (2.3)$$

2.1 Non-Dimensionalized Form

For the purpose of solution, it is pertinent to present equation (2.1) in a non-dimensionalized form. This is d one by substituting $U = \overline{\chi} L,\ \bar{x} = xL,\ \bar{y} = yL,\ \bar{t} = t/w$ into equation (2.1). After some simplification and rearrangement, one obtains

$$\varepsilon^2\left[\frac{\partial^4\overline{\chi}}{\partial x^4}+2\alpha_1^2\frac{\partial^4\overline{\chi}}{\partial x^2\partial y^2}+\alpha_2^2\frac{\partial^4\overline{\chi}}{\partial y^4}\right]-\beta_1^2\frac{\partial^2\overline{\chi}}{\partial x^2}-\beta_2^2\frac{\partial^2\overline{\chi}}{\partial y^2}$$
$$-\alpha_{ot}\left[\frac{\partial^4\overline{\chi}}{\partial x^2\partial t^2}+\frac{\partial^4\overline{\chi}}{\partial y^2\partial t^2}\right]+\frac{\partial^2\overline{\chi}}{\partial t^2}=P_a(x,y;t) \quad (2.4)$$

where, ε is a small parameter multiplying the highest derivative and defined by the relation

$$\varepsilon^2=\frac{D_{xx}}{N_oL^2}<<1,\ \alpha_1^2=\frac{D_{xy}}{D_{xx}},\ \alpha_2^2=\frac{D_{yy}}{D_{xx}},\ \beta_1^2=\frac{N_{xx}}{N_o},\ \beta_2^2=\frac{N_{yy}}{N_o} \quad (2.5)$$

$\alpha_1^2,\ \alpha_2^2$ are measures of material entropy, $\beta_1^2\ ;\beta_2^2$ are measures of prestress ratio.
The boundary conditions in non-dimensionalized form are

$$\left.\begin{matrix}x=0, & 0\le y\le b\\ x=1, & 0\le y\le b\end{matrix}\right\}\overline{\chi}(x,y;t)=0 \quad ,\quad \frac{\partial\overline{\chi}(x,y;t)}{\partial x}=0$$
$$\left.\begin{matrix}y=0, & 0\le x\le 1\\ y=b, & 0\le x\le 1\end{matrix}\right\}\overline{\chi}(x,y;t)=0 \quad ,\quad \frac{\partial\overline{\chi}(x,y;t)}{\partial y}=0 \quad (2.6)$$

and the initial conditions are

$$\overline{\chi}(x,y;o)=0 \quad ,\quad \frac{\partial\overline{\chi}(x,y;o)}{\partial t}=0 \quad (2.7)$$

In this dynamical system, the moving load on the rectangular plate moves at a constant velocity c along a straight line parallel to x-axis, say, y_0. Thus, $P_a(x,y,t)$ takes the form

$$P_a(x,y,t)=Mg\delta(x-ct)\delta(y-y_0) \quad (2.8)$$

where M is the mass of the Load, g is the acceleration due to gravity and $\delta(x-ct)$ is the Dirac delta function defined as

$$\delta(x-ct)=\begin{cases}0 & ,x\neq ct\\ \infty & ,x=ct\end{cases} \quad (2.9)$$

Substituting (2.8) into (2.4) leads to

$$\varepsilon^2\left[\frac{\partial^4\overline{\chi}}{\partial x^4}+2\alpha_1^2\frac{\partial^4\overline{\chi}}{\partial x^2\partial y^2}+\alpha_2^2\frac{\partial^4\overline{\chi}}{\partial y^4}\right]-\beta_1^2\frac{\partial^2\overline{\chi}}{\partial x^2}-\beta_2^2\frac{\partial^2\overline{\chi}}{\partial \overline{y}^2}$$
$$-\alpha_{ot}\left[\frac{\partial^4\overline{\chi}}{\partial x^2\partial t^2}+\alpha_2^2\frac{\partial^4\overline{\chi}}{\partial y^2\partial t^2}\right]+\frac{\partial^2\overline{\chi}}{\partial t^2}\ =Mg\delta(x-ct)\delta(y-y_o) \quad (2.10)$$

Equations (2.10), (2.6), (2.7) define completely the normalised equation of a prestressed orthotropic rectangular plate occupying the domain $D_o[0\le x\le 1,\ \ 0\le y\le b]$ and hinged along the four edges x = 0, x = 1, y = 0, y = b.

2.2 Operational simplification

Evidently, a small parameter multiplies the highest derivatives in (2.10). Thus, according to Cole [9], the problem is amenable to singular perturbation techniques. To this end, in the first instance, equation (2.10) is simplified by subjecting it to the Laplace transform defined by

$$\chi=\int_0^\infty\overline{\chi}e^{-st}dt \quad (2.11)$$

in conjunction with the initial conditions defined in (2.7). After some simplifications and rearrangements, equation (2.6) becomes

$$\varepsilon^2\left[\frac{\partial^4\chi}{\partial x^4}+2\alpha_1^2\frac{\partial^4\chi}{\partial x^2\partial y^2}+\alpha_2^2\frac{\partial^4\chi}{\partial y^4}\right]-\left(\beta_1^2+\alpha_{ot}s^2\right)\frac{\partial^2\chi}{\partial x^2}$$

$$-\left(\beta_2^2+\alpha_{ot}s^2\right)\frac{\partial^2\chi}{\partial y^2}+s^2\chi=\frac{p}{c}\delta(y-y_o)e^{-sx/c} \tag{2.12}$$

Subject to the boundary conditions

$$\left.\begin{matrix} x=0, & 0\le y\le b \\ x=1, & 0\le y\le b\end{matrix}\right\}\chi(x,y)=0; \quad \frac{\partial\chi}{\partial x}(x,y)=0$$

$$\left.\begin{matrix} y=0, & 0\le x\le 1 \\ y=b, & 0\le x\le 1\end{matrix}\right\}\chi(x,y)=0; \quad \frac{\partial\chi}{\partial y}(x,y)=0 \tag{2.13}$$

2.3 Formal Expansion Procedure

In order to solve equations (2.12) and (2.13), it is observed that a small parameter multiplies the highest derivative in the governing equation. Therefore, an exact uniformly valid solution in the entire domain of definition of the problem is not possible. This is so because of the bending effects at the boundaries. Consequently, solution valid away from the boundaries breaks down near as well as at the boundaries. Thus, only approximate solutions are possible. There are two but equivalent approaches that could be used here: the method of composite expansion (MCE) and the method of matched asymptotic expansion (MMAE).

In this study, the MCE is used. The technique is more user friendly than the cumbersome matching procedure in the method of matched asymptotic expansion. This method of composite expansion assumes the construction of uniformly valid solution of equation (2.12) in the domain of definition in the form [3]

$$\chi(x,y;s)=\psi(x,y;\varepsilon)+\sum_{\sigma=1}^{4}\rho^{\sigma}e^{-\tau_\sigma/\varepsilon} \tag{2.14}$$

where

$$\tau_\sigma=\begin{cases}\beta_2\, y/\alpha_2 & , \quad \sigma=1 \\ \beta_2\,(b-y)/\alpha_2, & \quad \sigma=2 \qquad 0\le b\le 1 \\ \beta_1 x & , \quad \sigma=3 \\ \beta_1(x-1) & , \quad \sigma=4\end{cases} \tag{2.15}$$

$\psi(x,y;\varepsilon)$ is the solution of the part characterised by the original independent variable (i.e solutions valid away from the sharp-changed regions). $\rho^{\sigma}(x,y;\varepsilon)$, $\sigma=1,2,3,4$ are the solutions of the parts characterised by the magnified independent variable (i.e. solutions valid at the shape-changed regions). $e^{-\tau_\sigma/\varepsilon}=1,2,3,4$ (one for each sharp change region) are the stretching transformations. They have the effect of stretching the regions near the boundaries as ε becomes small.

Introducing (2.14) term by term into (2.12) and (2.13) and simplifying yields five differential equations for the function ψ and $\rho^{\sigma}(x,y;\varepsilon)$, $\sigma=1,2,3,4$ viz

$$\left(\beta_1^2+\alpha_{0t}-s^2\right)\psi_{xx}+\left(\beta_2^2+\alpha_{0t}s^2\right)\psi_{yy}\ -\mathrm{s}^2\psi=-\frac{P}{c}\delta(y-y_\sigma)e^{-sx/c}+\varepsilon^2\psi_{xxxx}$$

$$+2\alpha_1^2\varepsilon^2\psi_{xxyy}+\alpha_2^2\varepsilon\psi_{yyyy} \tag{2.16}$$

$$-\frac{2}{\alpha_2}\left[\beta_2-\frac{\alpha_{ot}s^2}{\beta_2}\right]\rho_y^\sigma=(-1)^{\sigma+1}\varepsilon\left[\left(5-\frac{\alpha_{ot}s^2}{\beta_2^2}\right)\rho_{yy}^\sigma+\left(\frac{2\alpha_1^2}{\alpha_2^2}-\frac{\beta_1^2}{\beta_2^2}-\frac{\alpha_{ot}s^2}{\beta_2^2}\right)\rho_{yy}^\sigma+\frac{1}{\beta_2^2}s^2\rho^\sigma\right]$$

$$-\frac{4}{\beta}\varepsilon^2\left[\alpha_2\rho_{yyy}^\sigma-\frac{\alpha_1^2}{\alpha_2}\rho_{xxy}^\sigma\right]+(-1)^{\sigma+1}\frac{\varepsilon^3}{\beta_2^2}\left[\rho_{xxxx}^\sigma+2\alpha_1^2\rho_{xxyy}^\sigma+\alpha_2^2\rho_{yyyy}^\sigma\right],\ \sigma=1,2 \tag{2.17}$$

$$2\left[\beta_1-\frac{\alpha_{ot}s^2}{\beta_1}\right]\rho_x^\varepsilon=(-1)^{\varepsilon+1}\varepsilon\left[\left(5-\frac{\alpha_{ot}s^2}{\beta_1^2}\right)\rho_{xx}^\varepsilon+\left(2\alpha_1^2-\frac{\beta_2^2}{\beta_1^2}-\frac{\alpha_{ot}s^2}{\beta_1^2}\right)\rho_{yy}^\varepsilon+\frac{1}{\beta_1^2}s^2\rho^\varepsilon\right]$$

$$\frac{4}{\beta_1}\varepsilon^2\left[\rho_{xx}^\sigma-\alpha_1^2\rho_{yyxx}^\varepsilon\right]+(-1)^{\varepsilon+1}\frac{\varepsilon^3}{\beta_1^2}\left[\rho_{xxxx}^\varepsilon+2\alpha_1^2\rho_{yyxx}^\varepsilon+\alpha_2^2\rho_{yyyy}^\varepsilon\right],\quad \varepsilon=3,4 \tag{2.18}$$

Subject to the boundary conditions

$$\psi(x,o;\varepsilon)+\rho^1(x,o;\varepsilon)=0\quad;\ \psi_y(x,o;\varepsilon)+\rho_y^1(x,o;\varepsilon)-\frac{\beta_2}{\alpha_2\varepsilon}\rho^1(x,o;\varepsilon)=0$$

$$\varphi(x,b;\varepsilon)+\rho^2(x,b;\varepsilon)=0\quad;\ \varphi_y(x,b;\varepsilon)+\rho_y^2(x,b;\varepsilon)-\frac{\beta_2}{\alpha_2\varepsilon}\rho^2(x,b;\varepsilon)=0$$

$$\varphi(o,y;\varepsilon)+\rho^3(o,y;\varepsilon)=0\quad;\ \varphi_x(o,y;\varepsilon)+\rho_x^3(0,y;\varepsilon)-\frac{\beta_1}{\varepsilon}\rho^3(o,y;\varepsilon)=0$$

$$\psi(1,y;\varepsilon)+\rho^4(1,y;\varepsilon)=0\quad;\ \psi_x(1,y;\varepsilon)+\rho_x^4(1,y;\varepsilon)+\frac{\beta_1}{\varepsilon}\rho^4(1,y;\varepsilon)=0 \tag{2.19}$$

2.4 Perturbation scheme

In order to obtain ψ and ρ^σ $(\sigma=1,2,3,4)$, the following perturbation schemes are introduced

$$\psi(x,y;\varepsilon)=\psi_0(x,y)+\varepsilon\psi_1(x,y)+\varepsilon^2\psi_2(x,y)+\ldots$$

$$\rho^\sigma(x,y;\varepsilon)=\rho_0^\sigma(x,y)+\varepsilon\rho_1^\sigma(x,y)+\varepsilon^2\rho_2^\sigma(x,y)+\ldots \tag{2.20}$$

$$\sigma=1,2,3,4$$

When these expansions are substituted into (2.16), (2.17) and (2.18) and terms of equal power of ε are collected, the following recurrence relations emerge:

$$\left(\beta_1^2+\alpha_{ot}s^2\right)\psi_{v,xx}^{(x,y)}+\left(\beta_2^2+\alpha_{ot}s^2\right)\psi_{v,yy}^{(x,y)}-s^2\psi_v(x,y)=\begin{cases}-\frac{p}{c}\delta(y-y_o)e^{-\frac{s}{c}} & ,v=0\\ 0 & ,v=1\\ \nabla^4\psi_{v-2} & ,v\geq 2\end{cases} \tag{2.21}$$

where

$$\nabla^4\psi_{v-2} = \psi_{v-2,xxxx} + 2\alpha_1^2\psi_{v-2,xxyy} + \alpha_2^2\psi_{v-2,yyyy}$$

$$\frac{2}{\alpha_2}\left(\beta_2 - \frac{\alpha_{ot}s^2}{\beta_2}\right)\rho_{v,y}^{\sigma} \equiv (-1)^{\sigma+1}\left[\left(5 - \frac{\alpha_{ot}s^2}{\beta_2^2}\right)\rho_{v-1,yy}^{\sigma}\right.$$

$$\left. + \left(\frac{2\alpha_1^2}{\alpha_2^2} - \frac{\beta_1^2}{\beta_2^2} - \frac{\alpha_{ot}s^2}{\beta_2^2}\right)\rho_{v-1,xx}^{\sigma}\right] + \frac{1}{\beta_2^2}s^2\rho_{v-1}^{\sigma} - \frac{4}{\beta_2}\left[\alpha_2\rho_{v-2,yyy}^{\sigma} - \frac{\alpha_1^2}{\alpha_2}\rho_{v-2,xxy}^{\sigma}\right]$$

$$+ (-1)^{\sigma+1}\frac{1}{\beta_2^2}\left[\rho_{v-3,xxxx}^{\sigma} + 2\alpha_1^2\rho_{v-3,xxyy}^{\sigma} + \alpha_2^2\rho_{v-3,yyyy}^{\sigma}\right]$$

$$\text{for } v \ge 0,\ \sigma = 1, 2 \qquad (2.22)$$

$$2\left(\beta_1 - \frac{\alpha_{ot}s^2}{\beta_1}\right)\rho_{v,x}^{\varepsilon} = (-1)^{\varepsilon+1}\left[\left(5 - \frac{\alpha_{ot}s^2}{\beta_1^2}\right)\rho_{v-1,xx}^{\varepsilon} + \left(2\alpha_1^2 - \frac{\beta_2^2}{\beta_1^2} - \frac{\alpha_{ot}s^2}{\beta_1^2}\right)\rho_{v-1,yy}^{\varepsilon}\right.$$

$$\left. + \frac{1}{\beta_1^2}s^2\rho_{v-1}^{\varepsilon}\right] - \frac{4}{\beta_1}\left[\rho_{v-2,xxx}^{\varepsilon} - \alpha_1^2\rho_{v-2,yyx}^{\varepsilon}\right]$$

$$+ (-1)^{\varepsilon+1}\frac{1}{\beta_1^2}\left[\rho_{v-3,xxxx}^{\varepsilon} + 2\alpha_1^2\rho_{v-3,yyxx}^{\varepsilon} + \alpha_2^2\rho_{v-2,yyyy}^{\varepsilon}\right],$$

$$\text{for } v \ge 0,\ \varepsilon = 3,4 \qquad (2.23)$$

Similarly, substituting the perturbation schemes of (2.20) into the boundary conditions (2.19), one obtains

$$\psi_v(x,0) + \rho_v^1(x,0) = 0; \quad \psi_{v-1,y}(x,0) + \rho_{v-1,y}^1(x,0) - \frac{\beta_2}{\alpha_2}\rho_v^1(x,0) = 0$$

$$\psi_v(x,b) + \rho_v^2(x,b) = 0; \quad \psi_{v-1,y}(x,b) + \rho_{v-1,y}^2(x,b) + \frac{\beta_2}{\alpha_2}\rho_v^2(x,b) = 0$$

$$\psi_v(0,y) + \rho_v^3(0,y) = 0; \quad \psi_{v-1,x}(0,y) + \rho_{v-1,x}^3(0,y) - \beta_1\rho_v^3(0,y) = 0 \qquad (2.24)$$

$$\psi_v(1,y) + \rho_v^4(1,y) = 0 \quad ; \quad \psi_{v-1,x}(1,y) + \rho_{v-1,x}^4(1,y) + \beta_1\rho_v^4(1,y) = 0$$

In (2.21), (2.22), (2.23) and (2.24) and everywhere henceforth, the convention that functions with negative orders vanishes shall be adopted.

3.0 Solution procedure

In what follows, the solutions of equations (2.21),(2.22) and (2.23) for the functions ψ_v and ρ_v^{σ} $(\sigma = 1,2,3,4); v = 0,1,2,3,...$ subject to the boundary conditions (2.24) are sought using the various transformation techniques.

3.1 Leading order solution

Here solutions for $\psi_o(x,y)$ and $\rho_o^{\sigma}(x,y),\ \sigma = 1,2,3,4$ are sought.

3.2 Solutions for $\rho_o,\ \sigma = 1,2,3,4$

Substituting v = 0 into (2.22), (2.23) and (2.24), the governing differential equations for ρ_o^{σ} are obtained as

$$\frac{2}{\alpha_2}\left[\beta_2 - \frac{\alpha_{ot}s^2}{\beta_2}\right]\rho_{o,y}^{\sigma},\ \sigma = 1,2 \qquad 2\left[\beta_1 - \frac{\alpha_{ot}s^2}{\beta_1}\right]\rho_{o,x}^{\varepsilon} = 0,\ \sigma = 3,4 \qquad (3.1)$$

Similarly, the boundary conditions become

$$\rho_o^1(x,o) =_o^2 (x,o) = \rho_o^3(o,y) = \rho_o^4(1,y) = 0 \qquad (3.2)$$

Integrating (2.17) and using the boundary conditions (2.18) yields

$$\rho_o^{\sigma}(x,y;s) = 0\ ,\ \sigma = 1,2,3,4 \qquad (3.3)$$

Taking the Laplace inversion of (2.19), one obtains

$$\rho_o^{\sigma}(x,y;t) = 0\ ,\ \sigma = 1,2,3,4 \qquad (3.4)$$

3.3 **Solution for ψ_o**

Setting v=0 in (2.21) and (2.24), and substituting (3.2) therein yield the governing equation for ψ_o as

$$\left(\beta_1^2 + \alpha_{ot}s^2\right)\psi_{o,xx} + \left(\beta_2^2 + \alpha_{ot}s^2\right)\psi_{o,yy} - s^2\psi_o = -\frac{p}{c}\delta(y - y_o)e^{-sx/c} \qquad (3.5)$$

and the boundary conditions as

$$\psi_o(x,o) = \psi_o(x,b) = \psi_o(o,y) = \psi_o(1,y) = 0 \qquad (3.6)$$

Equation (3.5) is solved for ψ_o by introducing the finite Fourier sine transform defined as

$\varphi_S(n,y) = \int_o^1 \psi_S(x,y)\sin n\pi x dx$, with inverse

$$\psi_S(x,y) = 2\sum_{n=0}^{\infty}\psi_S(n,y)\sin n\pi x \qquad (3.7)$$

and $\tilde{\psi}_s(m,x) = \int_o^b \psi_s(x,y)\sin\frac{m\pi}{b}ydy$, with inverse

$$\psi_S(x,y) = \frac{2}{b}\sum_{m=0}^{\infty}\tilde{\psi}(m,x)\sin\frac{m\pi}{b}y \qquad (3.8)$$

so that the transform of (3.5) with respect to x is

$$\tilde{\psi}_{o,yy}(n,y) + \gamma^2\psi_o(n,y) = T,\delta(y - y_o),\quad \gamma^2 = -\left[\frac{n^2\pi^2\beta_1^2 + \left(n^2\pi^2\alpha_{ot} + 1\right)s^2}{\beta_2^2 + \alpha_{ot}s^2}\right] \qquad (3.9)$$

$$\text{and}\quad T_1 = \frac{n\pi pc\left[(-1)^n e^{-s/c} - 1\right]}{\left(\beta_2^2 + \alpha_{ot}s^2\right)\left(s^2 + n^2\pi^2 o^2\right)} \qquad (3.10)$$

while the transform of (3.5) with respect to y is $\tilde{\psi}_{o,xx}(m,x) + \beta^2\tilde{\psi}_o(m,x) = T_2 e^{-sx/c}$,

$$\text{where}\qquad \beta^2 = -\left[\frac{m^2\pi^2\beta_2^2 + \left(m^2\pi^2\alpha_{ot} + b^2\right)}{b^2\left(\beta_1^2 + \alpha_{ot}s^2\right)}\right] \qquad (3.11)$$

and $$T_2 = -\frac{P\sin\frac{m\pi}{b}y_o}{c\left(\beta_1^2+\alpha_{ot}s^2\right)} \quad (3.12)$$

Solving the resulting differential equations (3.9) and (3.11) yields

$$\tilde{\psi}_o(n,x) = 0 \quad (3.13)$$

and

$$\tilde{\psi}_o(m,x) = \frac{Pc\sin\frac{m\pi}{b}y_o}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\left[\cos\beta x - \cot\beta\sin\beta x - e^{-sx/c} + e^{-s/c}\frac{\sin\beta x}{\sin\beta}\right] \quad (3.14)$$

The inversion of (3.13) and (3.14) gives the general solution of equation (3.5) as

$$\tilde{\psi}_o(x,y) = \frac{2Pc\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y}{b\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\left[\cos\beta x - \cot\beta\sin\beta x - e^{-sx/c} + e^{-s/c}\frac{\sin\beta x}{\sin\beta}\right] \quad (3.15)$$

It remains to obtain the Laplace inversion of equation (3.15). Evidently, this is not straight forward as β and γ are complex expressions. Thus, equation (3.15) can be rewritten as

$$\psi_o(x,y) = \frac{2Pc}{b}\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y\left[\frac{\cosh\beta^c x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)} - \frac{\coth\beta^c x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\right.$$
$$\left. - \frac{e^{-s/c}\sinh\beta^c x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sinh\beta} - \frac{e^{-sx/c}}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\right] \quad (3.16)$$

where

$$\beta^c = \left[\frac{m^2\pi^2\beta_2^2 + \left(m^2\pi^2\alpha_{ot}+b^2\right)s^2}{b_2\left(\beta_1^2+\alpha_{ot}s^2\right)}\right]^{1/2} \quad (3.17)$$

The Laplace inversion of (3.16) is defined [11] as

$$\tilde{\psi}_o(x,y;t) = \frac{2Pc}{b}\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y\left[F_1(x,y;t)+F_2(x,y;t)+F_3(x,y;t)+F_4(x,y;t)\right] \quad (3.18)$$

where

$$F_1(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\cosh\beta^c x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (3.19)$$

$$F_2(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\coth\beta^c\sinh\beta^c x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (3.20)$$

$$F_3(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\left(t-x/c\right)s}}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (3.21)$$

$$F_4(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\left(t-1/c\right)s}\sinh\beta^c x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (3.22)$$

In order to evaluate the above integrals, the residue theorem is employed. The singularities in the integrals are poles. The denominators of the integrands of $F_1(x,y;t),\ F_2(x,y;t)\ \ and\ \ F_3(x,y;t)$ have simple poles at

$$s=\pm\Omega_1 \text{ and } s=\pm\Omega_2 \tag{3.23}$$

where

$$\Omega_1=\sqrt{\frac{1}{2b^2\alpha_{ot}}\left\{\left[\left(b^2\beta_1^2-c^2m^2\pi^2\alpha_{ot}-b^2c^2\right)^2+4c^2m^2\pi^2b^2\beta_2^2\alpha_{ot}\right]^{1/2}-\left(b^2\beta_1^2-c^2m^2\pi^2\alpha_{ot}-b^2c^2\right)\right\}} \tag{3.24}$$

$$\Omega_2=\sqrt{-\frac{1}{2b^2\alpha_{ot}}\left\{\left[\left(b^2\beta_1^2-c^2m^2\pi^2\alpha_{ot}-b^2c^2\right)^2+4c^2m^2\pi^2b^2\beta_2^2\alpha_{ot}\right]^{1/2}+\left(b^2\beta_1^2-c^2m^2\pi^2\alpha_{ot}-b^2c^2\right)\right\}} \tag{3.25}$$

Thus, it is straight forward to show that

$$F_1(x,y;t)=\frac{\left(e^{t\Omega_1}-e^{-t\Omega_1}\right)\cosh\Omega_4 x}{\alpha_{ot}\Omega_1\Omega_3}-\frac{\left(e^{t\Omega_2}-e^{-t\Omega_2}\right)\cosh\Omega_5 x}{\alpha_{ot}\Omega_2\Omega_3} \tag{3.26}$$

$$F_2(x,y;t)=\frac{\left(e^{t\Omega_1}-e^{-t\Omega_1}\right)\coth\Omega_4\sinh\Omega_4 x}{\alpha_{ot}\Omega_1\Omega_3}-\frac{\left(e^{t\Omega_2}-e^{-t\Omega_2}\right)\coth\Omega_5\sinh\Omega_5 x}{\alpha_{ot}\Omega_2\Omega_3} \tag{3.27}$$

$$F_3(x,y;t)=\frac{\left(e^{\left(t-x/c\right)\Omega_1}-e^{-\left(t-x/c\right)\Omega_1}\right)}{\alpha_{ot}\Omega_1\Omega_3}-\frac{\left(e^{\left(t-x/c\right)\Omega_2}-e^{-\left(t-x/c\right)\Omega_2}\right)}{\alpha_{ot}\Omega_2\Omega_3} \tag{3.28}$$

where

$$\Omega_3=2\left[\Omega_1^2-\Omega_2^2\right] \tag{3.29}$$

$$\Omega_4=\left[\frac{m^2\pi^2\beta_2^2+\left(m^2\pi^2\alpha_{ot}+b^2\right)\Omega_1^2}{b^2\left(\beta_1^2+\alpha_{ot}\Omega_1^2\right)}\right]^{1/2},\ \Omega_5=\left[\frac{m^2\pi^2\beta_2^2+\left(m^2\pi^2\alpha_{ot}+b^2\right)\Omega_2^2}{b^2\left(\beta_1^2+\alpha_{ot}\Omega_2^2\right)}\right]^{1/2} \tag{3.30}$$

Furthermore, to evaluate $F_4(x,y;t)$, its integrand is rewritten as

$$\varphi_a=\frac{e^{\left(t-1/c\right)s}\sinh\beta^c x}{\left(s^2-\Omega_1^2\right)\left(s^2-\Omega_2^2\right)\sinh\beta^c} \tag{3.31}$$

Thus, the simple poles are respectively $s=\pm\Omega_1$ and $s=\pm\Omega_2$. In order to obtain poles emanating from $Sinh\beta^c$, it is set to zero,i.e

$$Sinh\beta^c=0 \tag{3.32}$$

which implies $$\beta^c=iv\pi,\ v=1,2,3,4 \tag{3.33}$$

Thus, $s=\pm\Omega_6$, where

$$\Omega_6=\pm\left(\frac{v^2\pi^2b^2\beta_1^2+m^2\pi^2\beta_2^2}{v^2\pi^2b^2\alpha_{ot}+m^2\pi^2\alpha_{ot}+b^2}\right)^{1/2}{}_6 \tag{3.34}$$

Thus, the contribution towards $F_4(x,y;t)$ due to simple poles at $s=\pm\Omega_1$ and $s=\pm\Omega_2$ is given by

$$F_{4a}(x,y;t)=\frac{\left[e^{\left(t-1/c\right)\Omega_1}-e^{-\left(t-1/c\right)\Omega_1}\right]\sinh\Omega_4 x}{\alpha_{ot}\Omega_1\Omega_3\sinh\Omega_4}-\frac{\left[e^{\left(t-1/c\right)\Omega_2}-e^{-\left(t-1/c\right)\Omega_2}\right]\sinh\Omega_5 x}{\alpha_{ot}\Omega_2\Omega_3\sinh\Omega_5} \tag{3.35}$$

In a similar manner, the contribution due to simple poles at $s=\pm\Omega_6$ is

$$F_{4b}(x,y;t)=\frac{(-1)^v v\pi b^2\Omega_9\left[e^{\left(t-1/c\right)\Omega_6}-e^{-\left(t-1/c\right)\Omega_6}\right]\sinh iv\pi x}{\Omega_7\Omega_8^3\left(\Omega_6^2-\Omega_1^2\right)\left(\Omega_6^2-\Omega_2^2\right)\alpha_{ot}} \tag{3.36}$$

where,

$$\Omega_7=\left(v^2\pi^2b^2\beta_2^2+m^2\pi^2\beta_2^2\right)^{1/2},\quad \Omega_8=\left(v^2\pi^2b^2\alpha_{ot}+m^2\pi^2\alpha_{ot}+b^2\right)^{1/2} \tag{3.37}$$

and

$$\Omega_9=m^2\pi^2\alpha_{ot}\left(\beta_1^2-\beta_2^2\right)+b^2\beta_1^2 \tag{3.38}$$

Therefore

$$F_4(x,y;t)=F_{4a}(x,y;t)+F_{4b}(x,y;t) \tag{3.39}$$

Substituting (3.26), (3.27), (3.28) and (3.39) into (3.18) yields

$$\tilde{\psi}_o(x,y;t)=\frac{2Pc}{b\alpha_{ot}}\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y\left\{\frac{\left(e^{t\Omega_1}-e^{-t\Omega_1}\right)\cosh\Omega_4 x}{\Omega_1\Omega_3}-\frac{\left(e^{t\Omega_2}-e^{-t\Omega_2}\right)\cosh\Omega_5 x}{\Omega_2\Omega_3}\right.$$

$$-\frac{\left(e^{\left(t-x/c\right)\Omega_1}-e^{-\left(t-x/c\right)\Omega_1}\right)}{\Omega_1\Omega_3}+\frac{\left(e^{t\Omega_1}-e^{-t\Omega_1}\right)}{\Omega_2\Omega_3}-\frac{\left(e^{\left(t-x/c\right)\Omega_1}-e^{-\left(t-x/c\right)\Omega_1}\right)\coth\Omega_4\sinh\Omega_4 x}{\Omega_1\Omega_3}$$

$$\frac{\left[e^{\left(t-\frac{1}{c}\right)\Omega_1}-e^{-\left(t-\frac{1}{c}\right)\Omega_1}\right]Sinh\Omega_4 x}{\Omega_1\Omega_3 Sinh\Omega_4}-\frac{\left[e^{\left(t-\frac{1}{c}\right)\Omega_2}-e^{-\left(t-\frac{1}{c}\right)\Omega_2}\right]Sinh\Omega_5 x}{\Omega_2\Omega_3 Sinh\Omega_5}$$

$$-\frac{\left(e^{t\Omega_2}-e^{-t\Omega_2}\right)\coth\Omega_5\sinh\Omega_5 x}{\Omega_2\Omega_3}$$

$$+\frac{(-1)^{\nu}\nu\pi b^{2}\alpha_{ot}\Omega_{9}\sinh i\nu\pi x\left(e^{\left(t-1/c\right)\Omega_{6}}-e^{-\left(t-1/c\right)\Omega_{6}}\right)}{\Omega_{7}\Omega_{8}^{3}\left(\Omega_{6}^{2}-\Omega_{1}^{2}\right)\left(\Omega_{6}^{2}-\Omega_{2}^{2}\right)}\Bigg\} \tag{3.40}$$

A combination of the results (3.4) and (3.40) yields the desired leading order solution of (2.1) namely,

$$\overline{\chi}(x,y;t)=\tilde{\psi}_{o}(x,y;t)=\overline{\chi}_{o}(x,y;t) \tag{3.41}$$

This represents the uniformly valid solution in the entire domain of definition of the given plate to the leading order. In what follows the first order solutions are calculated.

4.0 First Order correction

4.1 Solution for $\rho_1^{\sigma},\ \sigma=1,2,3,4$

The problems for $\rho_1^{\sigma},\ \sigma=1,2,3,4$ (obtained by setting ν = 1in (2.22), (2.23) and (2.24), subject to the boundary conditions (3.2) are governed by

$$\frac{2}{\alpha_2}\left[\beta_2-\frac{\alpha_{ot}s^2}{\beta_2}\right]\rho_{1,y}^{\sigma}=0,\ \sigma=1,2\ ,\ 2\left[\beta_1-\frac{\alpha_{ot}s^2}{\beta_1}\right]\rho_{1,x}^{\varepsilon},\ \ \varepsilon=3,4 \tag{4.1}$$

while the boundary conditions are

$$\rho_1^{1}(x,0)=\frac{\alpha_2}{\beta_2}\psi_{0,y}(x,0),\ \rho_1^{2}(x,b)=-\frac{\alpha_2}{\beta_2}\psi_{0,y}(x,b) \tag{4.2}$$

$$\rho_1^{3}(0,y)=\frac{1}{\beta_1}\psi_{0,x}(0,y),\ \rho_1^{4}(1,y)=-\frac{1}{\beta_1}\psi_{0,x}(1,y) \tag{4.3}$$

Solving (4.1a) and (4.1b) respectively with respect to y, x, subject to (4.2) and (4.3) yields

$$\rho_1^{\sigma}(x,y)=\frac{2m\pi\alpha_2 pc\sin\frac{m\pi}{b}y_o}{b^2\beta_2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\left[(2-\sigma)+(-1)^{m+1}(\sigma-1)\right]$$
$$\left[\cos\beta x-\cot\beta\sin\beta x-e^{-sx/c}+\frac{\sin\beta x e^{-s/c}}{\sin\beta x}\right],\qquad \text{for } \sigma=1,2 \tag{4.4}$$

and

$$\rho_1^{\varepsilon}(x,y)=\frac{2pc\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y}{b\beta_1\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\Bigg[(4-\varepsilon)+\left(-\beta\cot\beta+\frac{\beta e^{-s/c}}{\sin\beta}+\frac{s}{c}\right)$$
$$+(\varepsilon-3)\left(\beta\sin\beta+\beta\cot\beta\cos\beta-\frac{\beta\cos\beta e^{-s/c}}{\sin\beta}-\frac{s}{c}e^{-s/c}\right)\Bigg] \tag{4.5}$$
$$\varepsilon=3,4$$

It remains to obtain the Laplace inversion of (4.5). The procedure is analogous to that outlined in previous section.

It is observed that β and α are complex, as such equation (4.4) and (4.5) can be rewritten as

$$\rho_1^{\sigma}(x,y) = \frac{2m\pi\alpha_2 pc\sin\frac{m\pi}{b}y_o}{b^2\beta_2}\left[(2-\sigma)+(-1)^{m+1}(\sigma-1)\right]$$

$$\left[\frac{\cosh\beta^c x}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)} - \frac{\coth\beta^c\sinh\beta^c x}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)} - \frac{e^{-sx/c}}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)} + \frac{e^{-s/c}\sinh\beta^c x}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)\sinh\beta^2}\right] \quad (4.6)$$

and

$$\rho_1^{\varepsilon}(x,y) = \frac{2Pc}{b\beta_1}\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y\left\{(4-\varepsilon)\left[\frac{-\beta^c\coth\beta^c}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)}\right]\right.$$

$$\left.+\frac{\beta^c e^{-s/c}}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)\sinh\beta^c} + \frac{s/c}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)}\right]$$

$$+(\varepsilon-3)\left[\frac{\beta^c\sinh\beta^c}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)} + \frac{\beta^c\coth\beta^c\cosh\beta^c}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)} - \frac{s/c\, e^{-s/c}}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)}\right.$$

$$\left.\left.-\frac{\beta^c e^{-\frac{S}{c}}\coth\beta^c}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)}\right]\right\}, \qquad \varepsilon = 3,4 \quad (4.7)$$

The Laplace inversion of (4.6) is given by

$$\tilde{\rho}_1^{\sigma}(x,y;t) = \frac{2m\pi\alpha_2 pc\sin\frac{m\pi}{b}y_o}{b^2\beta_2}\left[(2-\sigma)+(-1)^{m+1}(\sigma+1)\right]\left[A_1(x,y;t)\right.$$

$$\left.+A_2(x,y;t)+A_3(x,y;t)+A_4(x,y;t)\right] \quad \sigma = 1,2 \quad (4.8)$$

where

$$A_1(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\cosh\beta^c x}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)}ds \quad (4.9)$$

$$A_2(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\coth\beta^c\sinh\beta^c x}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)}ds \quad (4.10)$$

$$A_3(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\left(\frac{t-1}{c}\right)s}\sinh\beta^c x}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)\sinh\beta^c}ds \quad (4.11)$$

$$A_4(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\left(t-x/c\right)s}}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)}ds \quad (4.12)$$

The evaluation of the above integrals is analogous to that in the previous section used to generate $F_1(x,y;t) - F_4(x,y;t)$. It is sufficient to state the following results

$$A_1(x,y;t) = \frac{\left(e^{t\Omega_1} - e^{-t\Omega_1}\right)}{\Omega_1\Omega_3\alpha_{ot}}\cosh\Omega_4 x - \frac{\left(e^{t\Omega_2} - e^{-t\Omega_2}\right)}{\Omega_2\Omega_3\alpha_{ot}}\cosh\Omega_5 x \qquad (4.13)$$

$$A_2(x,y;t) = -\frac{\left(e^{t\Omega_1} - e^{-t\Omega_1}\right)}{\Omega_1\Omega_2\alpha_{ot}}\coth\Omega_4\sinh\Omega_4 x + \frac{\left(e^{t\Omega_2} - e^{-t\Omega_2}\right)}{\Omega_2\Omega_3\alpha_{ot}}\coth\Omega_5\sinh\Omega_5 x \qquad (4.14)$$

$$A_3(x,y;t) = \frac{\left(e^{\left(t-\frac{1}{c}\right)\Omega_1} - e^{-\left(t-\frac{1}{c}\right)\Omega_1}\right)\sinh\Omega_4}{\Omega_1\Omega_3\alpha_{ot}\sinh\Omega_4} - \frac{\left(e^{-\left(t-\frac{1}{c}\right)\Omega_2} - e^{-\left(t-\frac{1}{c}\right)\Omega_2}\right)}{\Omega_2\Omega_3\alpha_{ot}\sinh\Omega_5}\sinh\Omega_5 x$$

$$+(-1)^{\nu}\nu\pi b^2\Omega_9\sinh i\nu\pi x\frac{\left(e^{\left(t-\frac{1}{c}\right)\Omega_6} - e^{-\left(t-\frac{1}{c}\right)\Omega_6}\right)}{\Omega_7\Omega_8^3\left(\Omega_6^2-\Omega_1^2\right)\left(\Omega_6^2-\Omega_2^2\right)\alpha_{ot}} \qquad (4.15)$$

$$A_4(x,y;t) = \frac{\left(e^{\left(t-\frac{x}{c}\right)\Omega_1} - e^{-\left(t-\frac{x}{c}\right)\Omega_1}\right)}{\Omega_1\Omega_3\alpha_{ot}} - \frac{\left(e^{\left(t-\frac{x}{c}\right)\Omega_2} - e^{-\left(t-\frac{x}{c}\right)\Omega_2}\right)}{\Omega_2\Omega_3\alpha_{ot}} \qquad (4.16)$$

Similarly, the inversion of (4.7) is given by

$$\tilde{\rho}_1^{\varepsilon}(x,y;t) = \frac{2pc}{b\beta_1}\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y\{(4-\varepsilon)[E_1(x,y;t)+E_2(x,y;t)+E_3(x,y;t)]+$$

$$(\varepsilon-3)[E_4(x,y;t)+E_5(x,y;t)+E_6(x,y;t)+E_7(x,y;t)]\} \qquad \varepsilon = 3,4 \qquad (4.17)$$

where,

$$E_1(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^c\coth\beta^c}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.18)$$

$$E_2(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^c e^{\left(t-\frac{1}{c}\right)s}}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sinh\beta^c}ds \qquad (4.19)$$

$$E_3(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\frac{s}{c}e^{st}}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.20)$$

$$E_4(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^c\sinh\beta^c}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.21)$$

$$E_5(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^c e^{st}\coth\beta^c\cosh\beta^c}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.22)$$

$$E_6(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^c e^{\left(t-\frac{1}{c}\right)s}\coth\beta^c}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.23)$$

$$E_7(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{s/c\, e^{(t-1/c)s}}{(\beta_1^2+\alpha_{ot}s^2)(s^2+\beta^2c^2)} ds \tag{4.24}$$

Using similar argument as in the solutions to integrals (3.19 – 3.22), one obtains

$$E_1(x,y;t) = \frac{\Omega_4\left(e^{t\Omega_1}-e^{-t\Omega_1}\right)\coth\Omega_4}{\Omega_1\Omega_3\alpha_{ot}} - \frac{\Omega_5\left(e^{\Omega_2 t}-e^{-\Omega_2 t}\right)Coth\Omega_5}{\Omega_2\Omega_3\alpha_{ot}} \tag{4.25}$$

$$E_2(x,y;t) = \frac{\Omega_4\left(e^{(t-1/c)\Omega_1}-e^{-(t-1/c)\Omega_1}\right)}{\Omega_1\Omega_3\alpha_{ot}\sinh\Omega_4} - \frac{\Omega_5\left(e^{(t-1/c)\Omega_2}-e^{-(t-1/c)\Omega_2}\right)}{\Omega_2\Omega_3\alpha_{ot}\sinh\Omega_5} \tag{2.52b}$$

$$+\frac{(-1)^v iv^2\pi^2b^2\Omega_9\left(e^{(t-1/c)\Omega_6}-e^{-(t-1/c)\Omega_6}\right)}{\Omega_7\Omega_8^3\left(\Omega_6^2-\Omega_1^2\right)\left(\Omega_6^2-\Omega_2^2\right)\alpha_{ot}}$$

$$E_3(x,y;t) = \frac{e^{t\Omega_1}-e^{-t\Omega_1}}{c\Omega_3\alpha_{ot}} + \frac{e^{t\Omega_2}-e^{-t\Omega_2}}{c\Omega_3\alpha_{ot}} \tag{4.26}$$

$$E_4(x,y;t) = \frac{\Omega_4\left(e^{t\Omega_1}-e^{-t\Omega_1}\right)\sinh\Omega_4}{\Omega_1\Omega_3\alpha_{ot}} - \frac{\Omega_5\left(e^{t\Omega_2}-e^{-t\Omega_2}\right)\sinh\Omega_5}{\Omega_2\Omega_3\alpha_{ot}} \tag{4.27}$$

$$E_5(x,y;t) = \frac{\left(e^{t\Omega_1}-e^{-t\Omega_1}\right)\Omega_4\cot\Omega_4\cosh\Omega_4}{\Omega_1\Omega_3\alpha_{ot}} - \frac{\left(e^{t\Omega_2}-e^{-t\Omega_2}\right)\Omega_5\coth\Omega_5 Cosh\Omega_5}{\Omega_2\Omega_3\alpha_{ot}} \tag{4.28}$$

$$E_6(x,y;t) = -\frac{\Omega_4\left(e^{(t-1/c)\Omega_1}-e^{-(t-1/c)\Omega_1}\right)\coth\Omega_4}{\Omega_1\Omega_3\alpha_{ot}} + \frac{\Omega_5\left(e^{(t-1/c)\Omega_2}-e^{-(t-1/c)\Omega_2}\right)\coth\Omega_5}{\Omega_2\Omega_3\alpha_{ot}} \tag{4.29}$$

$$E_7(x,y;t) = -\frac{\left(e^{(t-1/c)\Omega_1}-e^{-(t-1/c)\Omega_1}\right)}{c\Omega_3\alpha_{ot}} - \frac{\left(e^{(t-1/c)\Omega_2}-e^{-(t-1/c)\Omega_2}\right)}{c\Omega_3\alpha_{ot}} \tag{4.30}$$

4.2 **Solution For** ψ_1

The governing differential equation for ψ_1 [obtained by setting v = 1 in (2.21)] is

$$\left(\beta_1^2+\alpha_{ot}s^2\right)\psi_{1,xx} + \left(\beta_2^2+\alpha_{ot}s^2\right)\psi_{1,yy} - s^2\psi_1 = 0 \tag{4.31}$$

subject to the boundary conditions

$$\psi_1(x,o) = \frac{-2m\pi\alpha_2 pc\sin\frac{m\pi}{b}y_o}{b^2\beta_2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\left[\cos\beta x - \cot\beta\sin\beta x + \frac{\sin\beta x}{\sin\beta}e^{-s/c} - e^{sx/c}\right]$$

$$= \Delta_1 \tag{4.32}$$

$$\psi_1(x,b) = (-1)^{m+1}\Delta_1 \tag{4.33}$$

$$\psi_1(o,y)=\frac{-2pc\sin\frac{m\pi}{b}y_o}{b\beta_1\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\left[-\beta\cot\beta+\frac{\beta e^{-s/c}}{\sin\beta}+\frac{s}{c}\right]\sin\frac{m\pi}{b}y \qquad (4.34)$$

$$\psi_1(1,y)=\frac{-2pc\sin\frac{m\pi}{b}y_o}{b\beta_1\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)} \qquad (4.35)$$

The solution of (4.31) is obtained by taking the finite Fourier sine transform of the differential equation first with respect to x, then y to obtain

$$\psi_{1,yy}(n,y)+\gamma^2\psi_1(n,y)=T_3\sin\frac{m\pi}{b}y \qquad (4.36)$$

where

$$T_3=-2n\pi pc\sin\frac{m\pi}{b}y_o\left[(-1)^{m+1}\left(\beta\sin\beta+\beta\cot\beta\cos\beta-\beta\cot\beta e^{-s/c}-\frac{s}{c}e^{-s/c}\right)\right.$$
$$\left.-\beta cot\beta+\frac{\beta e^{-s/c}}{\sin\beta}+\frac{s}{c}\right] \qquad (4.37)$$

and

$$\psi_{1,xx}(m,x)+\beta^2\psi_1(m,x)=T_4\left[\cos\beta-\cot\beta\sin\beta x+\frac{\sin\beta x}{\sin\beta}e^{-s/c}-e^{-sx/c}\right] \qquad (4.38)$$

where

$$T_4=\frac{4m^2\pi^2\alpha_2\left(\beta_2^2+\alpha_{ot}s^2\right)pc\sin\frac{m\pi}{b}y_o}{b^3\beta_2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)} \qquad (4.39)$$

Solving the resulting differential equations (4.36) and (4.38) yield

$$\psi_1(n,y)=\frac{m\pi\alpha_2 Pc\sin\frac{m\pi}{b}y_o}{b^2\beta_2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sin n\pi x}\left[-\cos\beta x+\cot\beta\sin\beta x-\frac{\sin\beta x}{\sin\beta}e^{-s/c}+e^{-sx/c}\right]$$

$$\left[\cos\gamma y+\left((-1)^{m+1}-\cos\gamma b\right)\frac{\sin\gamma y}{\sin\gamma b}\right] - \frac{2n\pi bpc\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y}{\beta_1\left(b^2\gamma^2-m^2\pi^2\right)\left(s^2+\beta^2c^2\right)}$$

$$\left[(-1)^{n+1}\left(\beta\sin+\beta\cot\beta\cos\beta-\beta\cot\beta e^{-s/c}-\frac{s}{c}e^{-s/c}\right)-\beta\cot\beta+\frac{\beta e^{-s/c}}{\sin\beta}+\frac{s}{c}\right] \qquad (4.40)$$

$$\psi_1(m,x)=\left\{\frac{Pc\sin\frac{m\pi}{b}y_o}{\beta_1\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\left[\beta\cot\beta-\frac{\beta e^{-s/c}}{\sin\beta}-\frac{s}{c}\right]\right.$$

$$\left.-\frac{4m^2\pi^2\alpha_2\left(\beta_2^2+\alpha_{ot}s^2\right)Pc\sin\frac{m\pi}{b}y_o}{b^3\beta_2\beta^2\left(\beta_1^2+\alpha_{ot}s^2\right)s}\right\}\cos\beta x$$

$$+\left\{\frac{Pc\sin\frac{m\pi}{b}y_o}{\beta_1\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\left[\beta-\frac{\frac{s}{c}e^{s/c}}{\sin\beta}+\frac{s}{c}\cot\beta\right]+\frac{4m^2\pi^2\alpha_2\left(\beta_2^2+\alpha_{ot}s^2\right)Pc\sin\frac{m\pi}{b}y_o}{b^3\beta_2\beta\left(\beta_1^2+\alpha_{ot}s^2\right)s^2}\right.$$

$$\left.\left[\cot\beta+\frac{e^{-s/c}}{\sin\beta}\right]\right\}\sin\beta x-\frac{4m^2\pi^2\alpha_2\left(\beta_2^2+\alpha_{ot}s^2\right)pc\sin\frac{m\pi}{b}y_o}{b^3\beta_2\beta^2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\left[\cos\beta x-cot\beta\sin\beta x\right.$$

$$\left.+\frac{\sin\beta x}{\sin\beta}e^{-s/c}\right]-\frac{4m^2\pi^2\alpha_2\left(\beta_2^2+\alpha_{ot}s^2\right)Pc\sin\frac{m\pi}{b}y_o}{b^3\beta_2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)s^2}\left(c^2e^{-sx/c}\right) \tag{4.41}$$

The inverse of (4.40) and (4.41) gives the general solution of equation (4.31) as

$$\psi(x,y;s)=\frac{2m\pi\alpha_2pc\sin\frac{m\pi}{b}y_o}{b^2\beta_2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\left[-\cos\beta x\cos\gamma y-(-1)^{m+1}\frac{\sin\gamma y\cos\beta x}{\sin\gamma b}\right.$$

$$+\cot\gamma b\sin\gamma y\cos\beta x+\cot\beta\sin\beta x\cos\gamma y+[-1]^{m+1}\frac{\cot\beta\sin\beta x\sin\gamma y}{\sin\gamma b}$$

$$-\cot\beta\sin\beta x\cot\gamma b\sin\gamma y-\frac{\sin\beta x\cos\gamma y e^{-s/c}}{\sin\beta}-(-1)^{m+1}\frac{\sin\beta x\sin\gamma y e^{-s/c}}{\sin\beta\sin\gamma b}$$

$$\left.+\frac{\sin\beta x\cot\gamma b\sin\gamma y e^{-s/c}}{\sin\beta}+\cos\gamma y e^{-sx/c}+\frac{(-1)^{m+1}\sin\gamma y e^{-sx/c}}{\sin\gamma b}-\cot\gamma b\sin\gamma y e^{-sx/c}\right]$$

$$-\frac{2n\pi bPc\sin\frac{m\pi}{b}y\sin n\pi x}{\beta_1\left(b^2\gamma^2-m^2\pi^2\right)\left(s^2+\beta^2c^2\right)}\left[(-1)^{n+1}\beta\sin\beta+(-1)^{m+1}\beta\cot\beta\cos\beta\right.$$

$$\left.-(-1)^{n+1}\beta\cot\beta e^{-s/c}-(-1)^{n+1}\frac{s}{c}e^{-s/c}-\beta\cot\beta+\frac{\beta e^{-s/c}}{\sin\beta}+\frac{s}{c}\right]$$

$$+\frac{2pc\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y}{b\beta_1\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}\Big[\beta\cot\beta Cos\beta x-\beta e^{-s/c}\cos\beta x$$

$$-\frac{s}{c}\cos\beta x+\beta\sin\beta x-\frac{s}{c}e^{-s/c}\sin\beta x+\frac{s}{c}\cot\beta\sin\beta x\Big]$$

$$+\frac{8m^2\pi^2\alpha_2pc\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y}{b^4\beta_2\left(\beta_1^2+\alpha_{ot}s^2\right)}\left(\beta_2^2+\alpha_{ot}s^2\right)\Bigg[\frac{\cos\beta x}{\beta^2s^2}+\frac{\cot\beta\sin\beta x}{\beta^2s^2}+\frac{e^{-s/c}\sin\beta x}{\beta^2s^2\sin\beta}$$

$$-\frac{\cos\beta x}{\beta^2\left(s^2+\beta^2c^2\right)}+\frac{\cot\beta\sin\beta x}{\beta^2\left(s^2+\beta^2c^2\right)}-\frac{\sin\beta xe^{-s/c}}{\beta^2\left(s^2+\beta^2c^2\right)\sin\beta}-\frac{c^2e^{-sx/c}}{s^2\left(s^2+\beta^2c^2\right)}\Bigg] \qquad (4.43)$$

At this juncture, when it is noted that β and γ are complex expressions, the Laplace inversion of (4.43) is defined as

$$\begin{aligned}\overline{\psi}(x,y;t)=&P_{a_1}\left[G_1(x,y;t)+G_2(x,y;t)+\ldots+G_{12}(x,y;t)\right]\\&-P_{a_2}\left[G_{13}(x,y;t)+G_{14}(x,y;t)+\ldots+G_{19}(x,y;t)\right]\\&+P_{a_3}\left[G_{20}(x,y;t)+G_{21}(x,y;t)+\ldots+G_{25}(x,y;t)\right]\\&+P_{a_4}\left[G_{26}(x,y;t)+G_{27}(x,y;t)+\ldots+G_{32}(x,y;t)\right]\end{aligned} \qquad (4.44)$$

where

$$P_{a_1}=\frac{2m\pi\alpha_2P_oc\sin\frac{m\pi}{b}y_o}{b^2\beta_2}\ ,\ P_{a_2}=-\frac{2n\pi bP_oc\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y\sin n\pi x}{\beta_1} \qquad (4.46)$$

$$P_{a_3}=2p_oc\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y\ ,\ P_{a_4}=\frac{8m^2\pi^2\alpha_2p_oc\sin\frac{m\pi}{b}y_o\sin\frac{m\pi}{b}y}{b^4\beta_2} \qquad (4.47)$$

$$G_1(x,y;t)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\cosh\beta^cx\cosh\gamma^cy}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds\ , \qquad (4.48)$$

$$G_2(x,y;t)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^{m+1}e^{st}\sinh\gamma^cy\cosh\beta^cx}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sinh\gamma^cb}ds \qquad (4.49)$$

$$G_3(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\coth\gamma^cb\sinh\gamma^cy\cosh\beta^cx}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.50)$$

$$G_4(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\coth\beta^c\sinh\beta^cx\cosh\gamma^cy}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.51)$$

$$G_5(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^{m+1}e^{st}\coth\beta^c\sinh\beta^cx\sinh\gamma^cy}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sinh\gamma^cb}ds \qquad (4.52)$$

$$G_6(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}\coth\beta^c \sinh\beta^c x \coth\gamma^c b \sinh\gamma^c y}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (4.53)$$

$$G_7(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\left(t-1/c\right)s}\sinh\beta^c x \cosh\gamma^c y}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sinh\beta^c}ds \quad (4.54)$$

$$G_8(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^{m+1}e^{\left(t-1/c\right)s}\sinh\beta^c x \sinh\gamma^c y}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sinh\beta^c \sinh\gamma^c b}ds \quad (4.55)$$

$$G_9(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\left(t-1/c\right)s}\sinh\beta^c x \coth\gamma^c b \sinh\gamma^c y}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sinh\beta^c}ds \quad (4.56)$$

$$G_{10}(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\left(t-x/c\right)s}\cosh\gamma^c y}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (4.57)$$

$$G_{11}(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^{m+1}e^{\left(t-x/c\right)s}\sinh\gamma^c y}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sinh\gamma^c b}ds \quad (4.58)$$

$$G_{12}(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\left(t-x/c\right)s}\coth\gamma^c b \sinh\gamma^c y}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (4.59)$$

$$G_{13}(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^{n+1}\beta^c e^{st}\sinh\beta^c}{\left(b^2\gamma^2-m^2\pi^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (4.60)$$

$$G_{14}(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^{m+1}\beta^c e^{st}\coth\beta^c \cosh\beta^c}{\left(b^2\gamma^2-m^2\pi^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (4.61)$$

$$G_{15}(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^{n+1}\beta^c e^{\left(t-1/c\right)s}\coth\beta^c}{\left(b^2\gamma^2-m^2\pi^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (4.62)$$

$$G_{16}(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{(-1)^{n+1}\frac{s}{c}e^{\left(t-1/c\right)s}}{\left(b^2\gamma^2-m^2\pi^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (4.63)$$

$$G_{17}(x,y;t) = -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^c e^{st}\coth\beta^c}{\left(b^2\gamma^2-m^2\pi^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (4.64)$$

$$G_{18}(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^c e^{\left(t-1/c\right)s}}{\left(b^2\gamma^2-m^2\pi^2\right)\left(s^2+\beta^2c^2\right)\sinh\beta^c}ds \quad (4.65)$$

$$G_{19}(x,y;t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{s/c\, e^{st}}{\left(b^2\gamma^2-m^2\pi^2\right)\left(s^2+\beta^2c^2\right)}ds \quad (4.66)$$

$$G_{20}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^{c}e^{st}\coth\beta^{c}\cosh\beta^{c}x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.67)$$

$$G_{21}(x,y;t)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^{c}e^{\left(t-1/c\right)s}\cosh\beta^{c}x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.68)$$

$$G_{22}(x,y;t)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{s/c\, e^{st}\cosh\beta^{c}x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.69)$$

$$G_{23}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\beta^{c}e^{st}\sinh\beta^{c}x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.70)$$

$$G_{24}(x,y;t)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{s/c\, e^{\left(t-1/c\right)s}\sinh\beta^{c}x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.71)$$

$$G_{25}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{s/c\, e^{st}\coth\beta^{c}\sinh\beta^{c}x}{\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.72)$$

$$G_{26}(x,y;t)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\left(\beta_2^2+\alpha_{ot}s^2\right)e^{st}\cosh\beta^{c}x}{\beta^2s^2\left(\beta_1^2+\alpha_{ot}s^2\right)}ds \qquad (4.73)$$

$$G_{27}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\left(\beta_2^2+\alpha_{ot}s^2\right)e^{st}\coth\beta^{c}\sinh\beta^{c}x}{\beta^2s^2\left(\beta_1^2+\alpha_{ot}s^2\right)}ds \qquad (4.74)$$

$$G_{28}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\left(\beta_2^2+\alpha_{ot}s^2\right)e^{\left(t-1/c\right)s}\sinh\beta^{c}x}{\beta^2s^2\left(\beta_1^2+\alpha_{ot}s^2\right)\sinh\beta^{c}}ds \qquad (4.75)$$

$$G_{29}(x,y;t)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\left(\beta_2^2+\alpha_{ot}s^2\right)e^{st}\cosh\beta^{c}x}{\beta^2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.76)$$

$$G_{30}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\left(\beta_2^2+\alpha_{ot}s^2\right)e^{st}\coth\beta^{c}\sinh\beta^{c}x}{\beta^2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.77)$$

$$G_{31}(x,y;t)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\left(\beta_2^2+\alpha_{ot}s^2\right)e^{\left(t-1/c\right)s}\sinh\beta^{c}x}{\beta^2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)\sinh\beta^{c}}ds \qquad (4.78)$$

$$G_{32}(x,y;t)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{c^2\left(\beta_2^2+\alpha_{ot}s^2\right)e^{\left(t-x/c\right)s}}{s^2\left(\beta_1^2+\alpha_{ot}s^2\right)\left(s^2+\beta^2c^2\right)}ds \qquad (4.79)$$

Next, is the evaluation of integrals $G_1(x,y;t)-G_{32}(x,y;t)$. To do this, the procedure outlined previously for $F_1(x,y;t)-F_4(x,y;t)$ is followed. The results are somewhat lenghy for presentation in this paper. Substitution of integrals $G_1(x,y;t)$ - $G_{32}(x,y;t)$ into equation (4.44) gives the complete inversion of $\tilde{\psi}_1(x,y;t)$.Thus, the relation

$$\tilde{\chi}_1(x,y;t)=\tilde{\psi}_1(x,y;t)+\sum_{\sigma=1}^{4}\rho_1^{\sigma}(x,y;t) \tag{4.80}$$

expresses the first order correction to the uniformly valid analytical solution in the entire domain of the plate. From the perturbation scheme, a uniformly valid solution in the entire domain of definition of the plate problem is given by

$$\tilde{\chi}(x,y;t)=\tilde{\chi}_o(x,y;t)+\varepsilon\overline{\chi}_1(x,y;t) \tag{4.81}$$

where χ_o is the leading order solution and χ_1 is the first order correction. These are given espectively as (2.43) and (4.44). Thus, the substitution of (2.43) and (4.44) into (4.81) gives the required solution.

5.0 Remarks on Theory

Equations (2.43) and (4.44) are respectively the leading order and the first order solutions of our problem. Equation (4.81) combines the leading order and the first order solutions to form the composite solution which is uniformly valid in the entire domain of the highly prestressed rectangular plate. It is observed from equation (2.43) that the measures of material orthotrophy, α_1 and α_2, do not affect the leading order composite solution. Furthermore, it is found that the anisotropic prestress, material orthotrophy and rotatory inertia affect the response to O($\boldsymbol{\varepsilon}$) of the rectangular plate.

In the system considered in this paper, damping was not considered. Thus, it is pertinent to examine the phenomenon of resonance. The displacement response of a highly prestressed rectangular plate under the actions of moving loads may increase without bound. This is an interesting aspect in a dynamical system such as this.

From equation (4.81) it is evident that the prestressed plate traversed by a moving load will encounter resonance effect when:

$$\frac{1}{2b^2\alpha_{ot}}\left\{\left[\left(b^2\beta_1^2-C^2m^2\pi^2\alpha_{ot}-b^2c^2\right)^2+4c^2m^2\pi^2b^2\beta_2^2\alpha_{ot}\right]^{1/2}+\left[b^2\beta_1^2-c^2m^2\pi^2\alpha_{ot}-b^2c^2\right]\right\}=\frac{v^2\pi^2b^2\beta_1^2+m^2\pi^2\beta_2^2}{v^2\pi^2b^2\alpha_{ot}+m^2\pi^2\alpha_{ot}+b^2} \tag{5.1}$$

other conditions when the system reaches a state of resonance are

$$\frac{1}{2b^2\alpha_{ot}}\left\{\left[\left(b^2\beta_1^2-c^2m^2\pi^2\alpha_{ot}-b^2c^2\right)^2+4c^2m^2\pi^2b^2\beta_2^2\alpha_{ot}\right]^{1/2}+\left[b^2\beta_1^2-c^2m^2\pi^2\alpha_{ot}-b^2c^2\right]\right\}=\frac{v^2\pi^2\beta_1^2+n^2\pi^2b^2\beta_1^2}{v^2\pi^2b^2\alpha_{ot}+n^2\pi^2b^2\alpha_{ot}+b^2} \tag{5.2}$$

and

$$\frac{1}{2b^2\alpha_{ot}}\left\{\left[\left(b^2\beta_1^2-c^2m^2\pi^2\alpha_{ot}-b^2c^2\right)^2+4c^2m^2\pi^2b^2\beta_2^2\alpha_{ot}\right]^{1/2}\right.$$

$$+\left[b^2\beta_1^2 - c^2m^2\pi^2\alpha_{ot} - b^2c^2\right]\Big\} = \frac{n^2\pi^2b^2\beta_1^2 + m^2\pi^2\beta_2^2}{n^2\pi^2b^2\alpha_{ot} + m^2\pi^2\alpha_{ot} + b^2} \qquad (5.3)$$

The three resonant states arise from, among others, integrals F_4, G_2 and G_{13}.

Equation (5.1) to (5.3) show that the state of resonance of the rectangular plate is not affected by material orthotropy but is dependent upon prestress. It is also evident that to any order of calculation, resonance conditions are affected by rotatory inertia correction factor.

At this juncture, one seeks the critical velocities at which these resonance conditions occur. It is then straight forward to examine the effects of the various parameters, such as rotatory inertia correction factor and anisotropic prestress, on our dynamical system. From (5.1) to (5.3), the critical velocities at the respective states of resonance are:

$$C_{\gamma 31} = \left\{ \frac{b^2\left[\alpha_{ot}\left(v^2\pi^2b^2\beta_1^2 + m^2\pi^2\beta_2^2\right)^2 - \beta_1^2\left(v^2\pi^2b^2\beta_1^2 + m^2\pi^2\beta_2^2\right)H_1(v,m,\pi)\right]}{m^2\pi^2\beta_2^2H_1^2(v,m,\pi) - \left(m^2\pi^2\alpha_{ot} + b^2\right)\left(v^2\pi^2b^2\beta_1^2 + m^2\pi^2\beta_2^2\right)H_1(v,m,\pi)} \right\}^{1/2} \qquad (5.4)$$

$$C_{\gamma 2} = \left\{ \frac{\left(v^2\pi^2b^2\beta_2^2 + n^2\pi^2b^4\beta_1^2\right)\left[v^2\pi^2\alpha_{ot}\left(\beta_2^2 - \beta_1^2\right) - b^2\beta_1^2\right]}{H_2(v,n,\pi)\left[m^2n^2\pi^4b^2\alpha_{ot}\left(\beta_2^2 - \beta_1^2\right) + \pi^2b^2\beta_2^2\left(n^2 - v^2\right) - n^2\pi^2b^4\beta_1^2\right]} \right\}^{1/2} \qquad (5.5)$$

$$C_{\gamma 3} = \left\{ \frac{b^2\left[\alpha_{ot}\left(n^2\pi^2b^2\beta_1^2 + m^2\pi^2\beta_2^2\right)^2 - \beta_1^2\left(n^2\pi^2b^2\beta_1^2 + m^2\pi^2\beta_2^2\right)H_3(m,n,\pi)\right]}{m^2\pi^2\beta_2^2H_3^2(m,n,\pi) - \left(m^2\pi^2\alpha_{ot} + b^2\right)\left(n^2\pi^2b^2\beta_1^2 + m^2\pi^2\beta_2^2\right)H_3(m,n,\pi)} \right\}^{1/2} \qquad (5.6)$$

$$H_1(v,m,\pi) = v^2\pi^2b^2\alpha_{ot} + m^2\pi^2\alpha_{ot} + b^2\ ,\ H_2(v,n,\pi) = v^2\pi^2\alpha_{ot} + n^2\pi^2b^2\alpha_{ot} + b^2$$

$$H_3(m,n,\pi) = n^2\pi^2b^2\alpha_{ot} + m^2\pi^2\alpha_{ot} + b^2 \qquad (5.7)$$

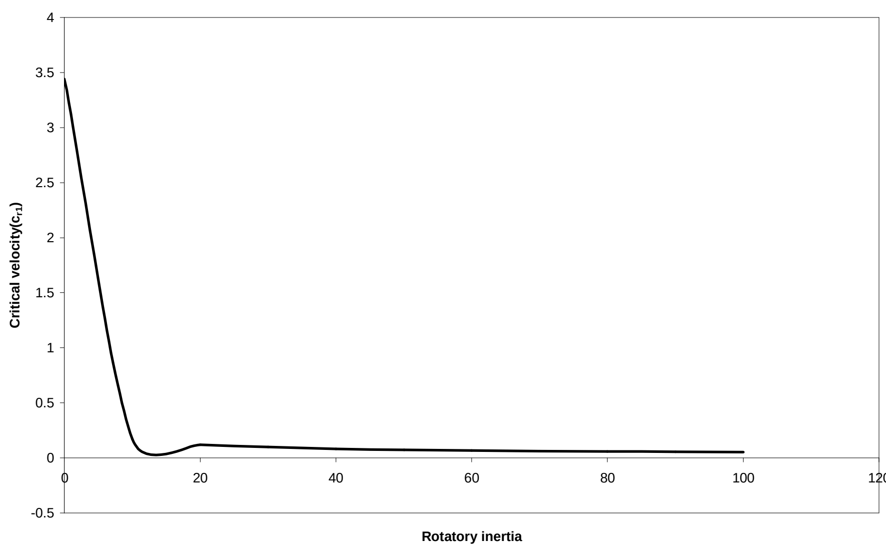

Fig 6.1. The graph of Crirical velocity against Rotatory inertia

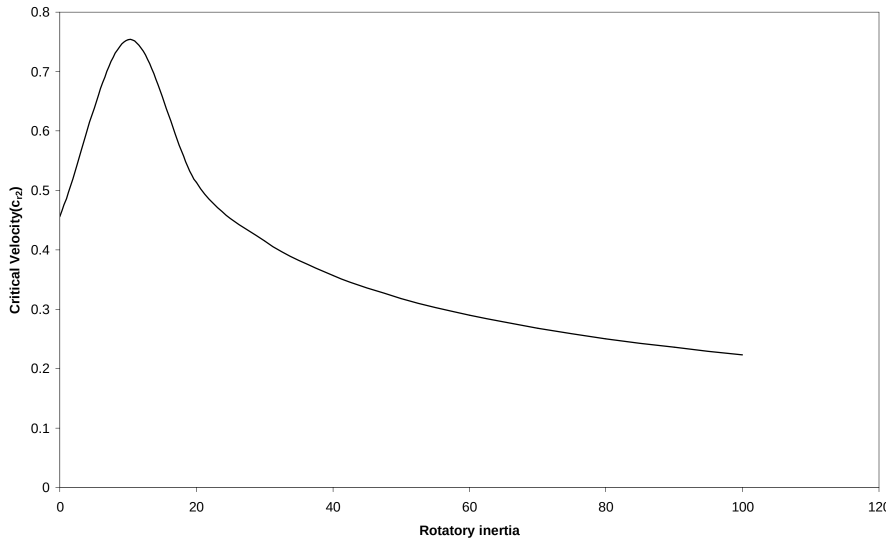

Fig 6.2. The graph of Critical velocity against Rotatory inertia

Thus, three distinct critical velocities exist in the dynamical system. $C_{\gamma_{3n}}$ = c at which resonance occurs. In what follows, the effects of various pertinent parameters on the critical velocities are analysed.

6.0 Numerical Calculations

This section presents calculations of practical interest in Applied Mmathematics and Engineering design. A rectangular plate of length Lx = 1.0m and width 0.5m is considered. Other values used for the analysis in this section are b = 0.5m, v = 3, $\pi = 22/7$. The values of the prestress ratios in x – direction β_1^2 range between 0.5 and 1.9. For fixed prestress ratio, the values of critical velocities are plotted against various values of rotatory inertia correction factor. Also for fixed rotatory inertia correction factor, the critical velocity is plotted against various values of prestress ratio in the *x* – direction.

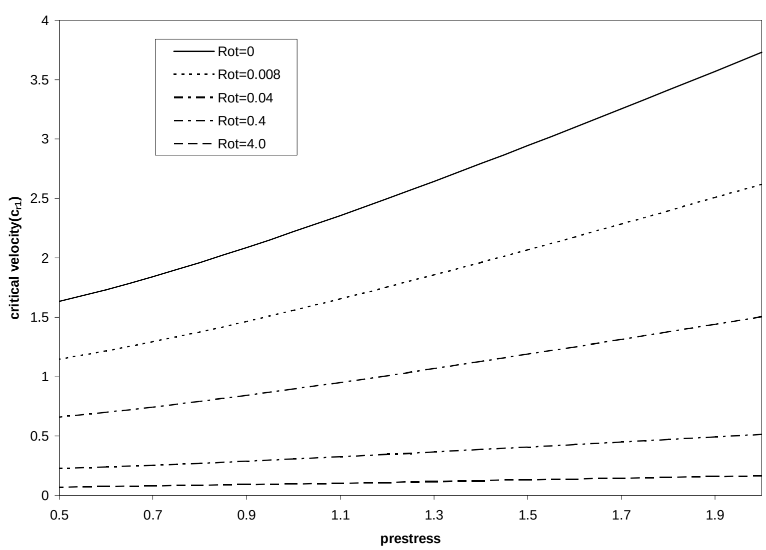

Figure 6.3: The graph of Critical velocity against Prestress (Rot = α_{ot})

Figure 6.1 displays the graph of critical velocity, C_{r_1}, against various values of rotatory inertia correction factor. Evidently, lower values of rotatory inertia show variation hence the possibility of resonance. However, as the value of rotatory inertia increases, critical speed approaches more or less constant value. Thus, design incorporating high value of rotatory inertia is more stable and reliable. In a similar manner, in figure 6.2 when $0 \le \alpha_{ot} < 17$ as α_{ot} increases, the critical velocity, C_{r_2}, increases. Thus, in this range, resonance is reached earlier for lower values of rotatory inertia correction factor. However, as the value of α_{ot} increases, C_{r_2} also approaches a constant value. Hence as the C_{r_1}, for high values of rotatory inertia, structural design is more stable and reliable.The graph of C_{r_1} against the prestress for various values of ratotory inertia is shown in figure 6.3. Evidently, the critical speed increases with prestress for all the values of ratotory inertia used.Thus resonance is reached earlier for low values of prestress than for high values of prestress. Thus, for high values of prestress our design is more stable and the risk of resonance is very remote.

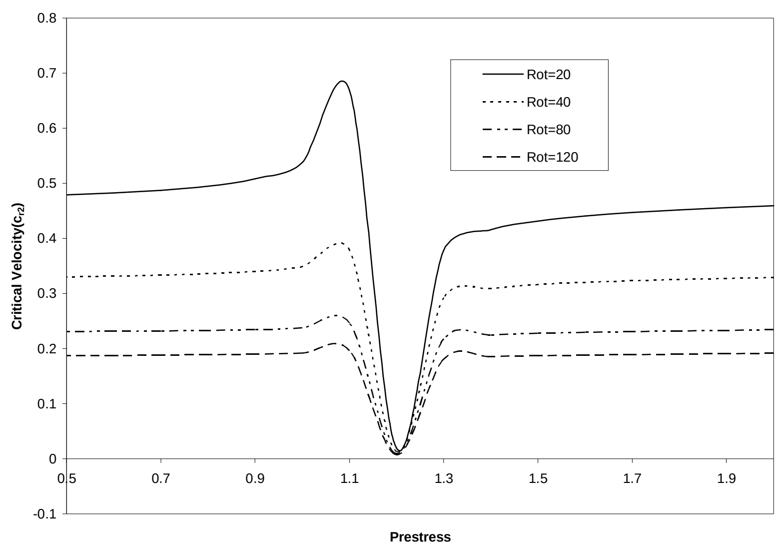

Figure 6.4: The graph of Critical velocity against Prestress (Rot = α_{ot})

Figure 6.4 shows the plotted curves of Cr_2 against prestress for the various values of rotatory interia. The graph shows that as prestress is increased, the critical speed increases as obtained in the graph of C_{r_1} against prestress. Thus the risk of resonance is remote as prestress is increased for any choice of value of rotatory inertia. However, as prestress is increased, the critical velocity approaches a constant value. Thus for high value of prestress, the stability and reliability of structural design is ensured.

The critical speed C_{r_3} behaves exactly the same way as C_{r_1}.Results and analysis similar to those of Figure 6.1 are obtained.

7.0 Conclusion

This study concerns the transverse motions of highly prestressed orthotropic rectangular plate under a travelling load. The problem is governed by a fourth order non-homogenous partial differential equation. For the purpose of solution, the equation is presented in a non-dimensionalized form. As a result of this, a small parameter ε multiplies the highest derivative in the governing differential equation. For an analytical solution to this equation, a singular perturbation technique, namely, the Method of Composite Expansion (MCE) is used in conjunction with the methods of Integral transformations and the Cauchy residue theorem to obtain a uniformly valid solution in the entire domain of definition of the rectangular plate problem. This solution is analysed and three distinct resonance conditions are obtained in the dynamical system. Numerical analysis is carried out and the study exhibits the following results:

(1) The measures of material orthotropy α_1 and α_2 do not affect the leading order composite solutions. However, the effects are present in the first order correction.

(2) The anisotropic prestress, material orthotropy and rotatory inertia affect the response to $0(\varepsilon)$ of the rectangular plate.

(3) The critical velocities of the dynamical system increase with an increase in prestress for all the values of rotatory inertia used. Thus, resonance is reached earlier for lower values of presterss than for higher values of prestress.

(4) For high values of rotatory inertia correction factor, the critical velocity approaches a constant value indicating that resonant effect is remote for high values of rotatory inertia correction factor.

(5)

(6) There may be more than one resonance condition in a dynamical system such as this which involves plate flexure under moving loads.

Finally,this work has exhibited the use of a valuable method for the solution of this class of dynamical problems.

Reference

[1] Stanisic, M. M., Euler, J. A. and Mongomery, S. T.: On a theory concerning the dynamic behaviour of structures carrying moving masses. Ing. Achieve 43, 295 – 305. (1974).

[2] Milormir et. al.: On the response of beams to an arbitrary number of concentrated moving masses. Journal of the Franklin Institute 287 (2), 115-123. (1969).

[3] Sadiku, S. and Leipholz, H. H. E.: On the dynamics of elastic systems with moving concentrated masses Ing. Archive 57, 223 – 242. (1981).

[4] Oni, S. T.: On the dynamic response of elastic structures to moving multi-mass systems. Ph.D Thesis, University of Ilorin. (1991).

[5] Gbadeyan, J. A. and Oni S. T. : Dynamic Behaviour of beams and rectangular plates under moving loads. Journal of Sound and Vibration 182(5) 677 – 695. (1995).

[6] Holl D. K. :Dynamic loads on thin plates on elastic foundations. Proc. of symposium in applied Mathematics, Vol. 13, New York, Mc-Graw Hill. (1950).

[7] Stanisic, M. M., Euler, J. A. and Mongomery, S. T : On the response of the plate to a multimasses moving system. Acta Mechanica 5, 377 – 53. (1968).

[8] Gbadeyan, J. A. and Oni, S. T. : Dynamic response to moving concentrated masses of elastic plates on a non-winkler elastic foundation 154, 343 – 358. (1992).

[9] Cole, J. D, Perturbation Methods of Applied Methamatics. Blaisdell. (1968).

[10] Hutter, K. and Olunloyo V. O. S. : Vibration of an anisotropically prestressed thick rectangular membrane with small bending rigidity, Acta Mechanica 20, 1 – 22. (1974).

[11] Olunloyo V. O. S. and Hutter, K. : The response of an anisotropically prestressed thick rectangular membrane to dynamic loading. Acta mechanica 28, 293 – 311. (1977).

[12] Gbadeyan, J. A. and Oyediran A. A. : A comparison of two singular perturbation techniques for initially stressed thin rectangular plate. Journal of sound and vibration 124 (3), 517 – 528. (1988).

[13] Hutter, K. and Olunloyo, V. O. S.: The transient and steady state response of a thick membrane to static and dynamic loading. Zamm 54, 795 – 806. (1974).

[14] Oni S. T.: Dynamic response to a moving load of a fully clamped prestressed orthotropic rectangular plate. J. of modelling, measurement and control 53 (3), 12 – 38. (1994).