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Dynamic behaviour of non-uniform Bernoulli-Euler beams subjected to concentrated loads travelling at varying velocities.

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Abstract

This paper investigates the dynamics behaviour of non-uniform Bernoulli-Euler beams subjected to concentrated loads ravelling at variable velocities. The solution technique is based on the Generalized Galerkin Method and the use of the generating function of the Bessel function type. The results show that, for all the illustrative examples considered, for the same natural frequency, the critical speed for the system consisting of a non-uniform beam traversed by a force moving at a non-uniform velocity is greater than that of the corresponding moving mass problem. It was also found that, for fixed axial force, an increase in foundation moduli reduces the response amplitudes of the dynamical system. Furthermore, it was shown that the transverse-displacement amplitude of a clamped-clamped non-uniform Bernoulli-Euler beam traversed by a load moving at variable velocities is lower than that of the cantilever. The response amplitude of the same dynamical systems which is simply supported is higher than those which consist of clamped-clamped or clamped-free (Cantilever) end conditions. Finally, an increase in the values of foundation moduli and axial force reduces the critical speed for all variants of the boundary conditions

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1.0 Introduction

Studies in structural dynamics dealing with moving loads on bridges are enormous and have been enriched in the last few decades by the development of high-speed railway networks in the developed countries. Similarly, there exist remarkable advances in various branches of transport. These advances are characterized by increasingly higher speeds and weights of vehicles. Hence, structures and media over or in which the vehicles move have been subjected to vibrations and dynamic stresses far larger than ever before. Thus, there is the need for continuous study of the behaviour of bodies subjected to moving loads. Such a study will, for instance, provide a safer and more economic design of structures on which the loads move.

In most of the studies available in literature, such as the work of Sadiku and Leipholz [1], Oni [2], Gbadeyan and Oni [3], Huang and Thambiratrian [4] , Lee and Ng [5], Adams [6], Chen and Li [7], Savin [8], Rao [9], Shadnam et al [10], the scope has been limited [11] to structural members having uniform cross-sections whether the inertia of the moving load is considered or not. The speeds at which these loads travel have also been idealized to be uniform. Nonetheless, for practical purposes, these are not so. In reality the cross-sections of structural members such as bridge, girders, hull of ships, concrete slabs etc are often non-prismatic and the velocity of the loads which move over these elastic solid bodies are often non-uniform.

Among recent studies where non-uniform structural members have been subjected to heavy masses is the attempt of Oni [12] who investigated the response of a non-uniform beam resting on an elastic foundation to several moving masses. The deflection of the non-uniform beam was calculated for several values of foundation moduli and shown graphically as a function of time. He found that the response amplitudes of both moving force and moving mass problems decrease with increasing foundation constant. However, his method of solution was limited to simply supported end conditions. A more elegant method was presented by Oni and Awodola [13] to assess the vibration under a moving load of a non-uniform Rayleigh beam on variable elastic foundation. The technique is based on the Generalized Galerkin's Method and Struble's asymptotic technique. An important feature of this technique is that it can handle this class of problem for all variants of classical boundary conditions. Nonetheless, the load speed here was assumed to be uniform.

Thus, this work is concerned with the dynamic behaviour of non-uniform Bernoulli-Euler beams subjected to concentrated loads travelling at variable velocities. The main objective of this study is to obtain an analytical solution to this problem. Numerical analysis will be carried out and results in plotted curves will be presented.

2.0 Formulation of the Problem

A non-uniform beam is considered. A relatively large mass M with considerable inertia is assumed to strike at time t = 0 and travel across it at a variable velocity such that the motion of the contact point of the moving mass is given by $X_p = f(t)$. The beam's properties such as moment of inertia I(x) and the mass per unit length of the beam $\mu(x)$ vary along the span L of the beam. The equation of motion with damping neglected, is given by the fourth order partial differential equation [11,14].

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right] + \mu(x)\frac{\partial^2 V(x,t)}{\partial t^2} - N\frac{\partial^2 V(x,t)}{\partial x^2} + K(x)V(x,t)$$
$$+ M\delta[x-(x_0+\gamma Sin\beta t)]\left[\frac{\partial^2 V(x,t)}{\partial t^2} + 2\frac{df(t)}{dt}\frac{\partial^2 V(x,t)}{\partial x\partial t} + \left(\frac{df(t)}{dt}\right)\frac{\partial^2 V(x,t)}{\partial x^2}\right. \qquad (2.1)$$
$$\left. + \frac{d^2 f(t)}{dt^2}\frac{\partial V(x,t)}{\partial x}\right] = P\delta[x-(x_0+\gamma Sin\beta t)]$$

where x is the spatial coordinate, t is the time, $V(x,t)$ is the Transverse Displacement of the beam, E is the Young's Modulus, $I(x)$ is the variable Moment of inertia, $\mu(x)$ is the variable mass per unit length of the beam, N is the axial force, K is the elastic foundation constant and
$P = Mg$ (where g is the acceleration due to gravity). The boundary conditions of the structure under consideration will be stated later under illustrative examples. The initial conditions without any loss of generality are taken as $V(x,0) = 0 = \dfrac{\partial V(x,0)}{\partial t}$ (2.2)

To get $\dfrac{df}{dt}$ which is the velocity of the moving load, we adopt the example in [14] and set the distance function as

$$f(t) = (x_o + \gamma Sin\beta t)\,. \qquad (2.3)$$

where x_0 is the equilibrium position of the longitudinally oscillating load, γ is the longitudinal amplitude of oscillation of the load and β is the longitudinal frequency of the load. We also adopt the example in [13] and take $I(x)$ and $\mu(x)$ to be of the form

$$I(x) = I_0\left(1 + Sin\frac{\pi x}{L}\right)^3 \qquad (2.4)$$

and

$$\mu(x) = \mu_0\left(1 + Sin\frac{\pi x}{L}\right) \qquad (2.5)$$

Substituting equations (2.4) and (2.5) in equation (2.1) one obtains

$$\frac{1}{4}EI_0\frac{\partial^2}{\partial x^2}\left[\left(10+15Sin\frac{\pi x}{L}-6Cos\frac{2\pi x}{L}-Sin\frac{3\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial x^2}\right]$$

$$+\mu_0\left(1+Sin\frac{\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial x^2}-N\frac{\partial^2 V(x,t)}{\partial x^2}+K^0V(x,t) \tag{2.6}$$

$$+M\delta[x-(x_0+\gamma Sin\beta t)]\left[\frac{\partial^2 V(x,t)}{\partial t^2}+2\gamma\beta Cos\beta t\frac{\partial^2 V(x,t)}{\partial x\partial t}+(\gamma\beta Cos\beta t)^2\frac{\partial^2 V(x,t)}{\partial x^2}\right.$$

$$\left.-\gamma\beta^2 Sin\beta t\frac{\partial V(x,t)}{\partial x}\right]=P\delta[x-(x_0+\gamma Sin\beta t)]$$

3.0 Solution Procedures

It is evident that an exact closed form solution of the partial differential equation (2.6) is impossible. Thus, we generalize the Galerkin's Method described in [12] and make use of this to reduce the partial differential equation to a sequence of ordinary differential equations. Thus, a solution of the form

$$V_n(x,t)=\sum_{m=1}^{n}Y_m(t)U_m(x) \tag{3.1}$$

is sought where $U_m(x)$ is chosen such that pertinent boundary conditions are satisfied. Equation (3.1) when substituted into the equation (2.6) yields

$$\sum_{M=1}^{N}\left\{\frac{1}{4}EI_0\frac{\partial^2}{\partial x^2}\left[10U''_m(x)+15Sin\frac{\pi x}{L}U''_m(x)-6Cos\frac{2\pi x}{L}U''_m(x)-Sin\frac{3\pi x}{L}U''_m(x)\right]Y_m(t)\right.$$

$$-NU''_m(x)Y_m(t)+K^0U_m(x)Y_m(t)+\mu_0\left[U_m(x)+U_m(x)Sin\frac{\pi x}{L}\right]\ddot{Y}_m(t) \tag{3.2}$$

$$+M\delta[x-(x_0+\gamma Sin\beta t)][U_m(x)\ddot{Y}_m(t)++2\gamma\beta Cos\beta tU'_m(x)\dot{Y}_m(t)$$

$$\left.+(\gamma\beta Cos\beta t)^2U''_m(x)Y_m(t)-\gamma\beta^2 Sin\beta tU'_m(x)Y_m(t)]\right\}-P\delta[x-(x_0+\gamma Sin\beta t)]=0$$

An appropriate selection of functions for beam problems are beam mode shapes. Thus, the m^{th} normal mode of vibration of a uniform beam [11]

$$U_m(x)=\sin\frac{\lambda_m x}{L}+A_m\cos\frac{\lambda_m x}{L}+B_m\sinh\frac{\lambda_m x}{L}+C_m\cosh\frac{\lambda_m x}{L} \tag{3.3}$$

is chosen such that the pertinent boundary conditions are satisfied. In (3.3), λ_m is the mode number. A_m, B_m, C_m are constants which are obtained by substituting (3.3) into the appropriate boundary conditions. (see Section 6).

4.0 Operational Simplification.

By applying the Generalized Galerkin's Method (GGM) of (3.1), equation (3.2) can be written as

$$\sum_{m=1}^{n}\{(\Phi_0+\Phi_1)\ddot{Y}_m(t)+(G_1[\Phi_2+\Phi_3+\Phi_4]-G_2\Phi_5+G_3\Phi_6)Y_m(t)+\Delta_A(t) \tag{4.1}$$

$$+\Delta_B(t)+\Delta_C(t)-\Delta_D(t)\}-U_K^*(x)=0$$

where

$$G_1 = \frac{EI_0}{4\mu_0},\ G_2 = \frac{N}{\mu_0},\ G_3 = \frac{K}{\mu_0} \tag{4.2}$$

$$\Phi_0 = \int_0^L U_m(x)U_k(x)dx, \qquad \Phi_1 = \int_0^L Sin\frac{\pi x}{L}U_m(x)U_k(x)dx \tag{4.3}$$

$$\Phi_2 = \int_0^L \left(10 + 15Sin\frac{\pi x}{L} - 6Cos\frac{2\pi x}{L} - Sin\frac{3\pi x}{L}\right)U_m^{iv}(x)U_k(x)dx \tag{4.4}$$

$$\Phi_3 = \int_0^L \left(\frac{30\pi}{L}Cos\frac{\pi x}{L} + \frac{24\pi}{L}Sin\frac{2\pi x}{L} - \frac{6\pi}{L}Cos\frac{3\pi x}{L}\right)U_m'''(x)U_k(x)dx \tag{4.5}$$

$$\Phi_4 = \int_0^L \left(\frac{9\pi^2}{L^2}Sin\frac{3\pi x}{L} + \frac{24\pi^2}{L^2}Cos\frac{2\pi x}{L} - \frac{15\pi^2}{L^2}Sin\frac{\pi x}{L}\right)U_m''(x)U_k(x)dx \tag{4.6}$$

$$\Phi_5 = \int_0^L U_m'''(x)U_k(x)dx, \qquad \Phi_6 = \int_0^L U_m(x)U_k(x)dx \tag{4.7}$$

$$\Delta_A(t) = \int_0^L \frac{M}{\mu_0}\delta[x - (x_0 + \gamma Sin\beta t)]U_m(x)U_k(x)dx \tag{4.8}$$

$$\Delta_B(t) = \int_0^L \frac{2M\gamma\beta Cos\beta t}{\mu_0}\delta[x - (x_0 + \gamma Sin\beta t)]U_m'(x)U_k(x)dx \tag{4.9}$$

$$\Delta_C(t) = \int_0^L \frac{M(\gamma\beta Cos\beta t)^2}{\mu_0}\delta[x - (x_0 + \gamma Sin\beta t)]U_m''(x)U_k(x)dx \tag{4.10}$$

and

$$\Delta_D(t) = \int_0^L \frac{M\gamma\beta^2 Sin\beta t}{\mu}\delta[x - (x_0 + \gamma Sin\beta t)]U_m'(x)U_k(x)dx \tag{4.11}$$

In order to evaluate the integrals (4.1) to (4.11), use is made of the property of the Dirac Delta function as an even function to express it in Fourier cosine series namely:

$$\delta[x - (x_0 + \gamma \sin\beta t)] = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty} \cos\frac{n\pi}{L}(x_0 + \gamma\sin\beta t)Cos\frac{n\pi x}{L} \tag{4.12}$$

In view of (3.1), using equation (4.12) in equation (4.1), after some simplifications and rearrangements one obtains

$$\sum_{m=1}^{n}\Bigg\{\ddot{Y}_m(t)+\frac{\Omega_1(k,m)}{\Omega_0(k,m)}Y_m(t)+\frac{\varepsilon_1}{\Omega_0(k,m)}\Bigg[H_b^0(k,m)\ddot{Y}_m(t)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0$$

$$+\gamma Sin\beta t)H_c^0(k,m,n)\ddot{Y}_m(t)+2\gamma\beta H_d^0(k,m)Cos\beta t\,\dot{Y}_m(t)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}\;Cos\frac{n\pi}{L}(x_0$$

$$+\gamma Sin\beta t)H_e^0(k,m,n)\dot{Y}_m(t)+(\gamma\beta Cos\beta t)^2H_f^0(k,m)Y_m(t)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}\;Cos\frac{n\pi}{L}(x_0 \tag{4.13}$$

$$+\gamma Sin\beta t)H_g^0(k,m,n)Y_m(t)-\gamma\beta^2H_i^0(k,m)Sin\beta t\,Y_m(t)-2\gamma\beta^2Sin\beta t\sum_{n=1}^{\infty}\;Cos\frac{n\pi}{L}(x_0)$$

$$+\gamma Sin\beta t)H_j^0(k,m,n)Y_m(t)\Bigg]\Bigg\}=\frac{P}{\Omega_0(k,m)\mu_0}\left[Sin\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)+A_kCos\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)\right.$$

$$\left.+B_kSinh\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)+C_kCosh\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)\right]$$

where

$$\Omega_0(k,m)=\Phi_0+\Phi_1\,,\;\;\Omega_1(k,m)=\frac{EI_0(\Phi_2+\Phi_3+\Phi_4)}{4\mu_0}-\frac{\Phi_5}{\mu_0}+\frac{\Phi_6}{\mu_0} \tag{4.14}$$

and

$$\varepsilon_1=\frac{M}{\mu_0L} \tag{4.15}$$

Equation (4.13) is the transformed equation governing the problem of a uniform Bernoulli-Euler beam on a constant elastic foundation. This coupled non-homogeneous Second order ordinary differential equation holds for all variants of the classical boundary conditions.

In what follows, two special cases of equation (4.11) are considered.

5.0 Solution of the Transformed Equation

Two special cases of the above equation (4.13) are considered in this section. These cases are termed: (i) the moving force problem (ii) the moving mass problem.

(i) The moving force

Setting $\varepsilon_1=0$ in the transformed equation (4.13) one obtains

$$\ddot{Y}_m(t)+\frac{\Omega_1(k,m)}{\Omega_0(k,m)}Y_m(t)=\frac{P}{\Omega_0(k,m)\mu_0}\left[Sin\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)+A_kCos\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)\right.$$
$$\left.+B_kSinh\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)+C_kCosh\frac{\lambda_k}{L}(x_0+\gamma Sin\beta t)\right] \tag{5.1}$$

This is the classical case of a moving force problem associated with the system. It is an approximate model which assumes the inertia to be effect of the moving mass as negligible. A further rearrangement of equation (5.1) yields

$$\ddot{Y}_m(t)+\omega_{aj}^2Y_m(t)=\frac{P}{\Omega_0(k,m)\mu_0}\left[a_0Sin(GSin\beta t)+a_1Cos(GSin\beta t)\right.$$
$$\left.+a_2Cosh(GSin\beta t)+a_3Sinh(GSin\beta t)\right] \tag{5.2}$$

where $$\omega_{aj}^2 = \frac{\Omega_1(k,m)}{\Omega_0(k,m)}, \qquad a_0 = \left(Cos\frac{\lambda_k x_0}{L} - A_k Sin\frac{\lambda_k x_0}{L} \right), \tag{5.3}$$

$$a_1 = \left(Sin\frac{\lambda_k x_0}{L} + A_k Cos\frac{\lambda_k x_0}{L} \right), \qquad a_2 = \left(B_k Sinh\frac{\lambda_k x_0}{L} + C_k Cos\frac{\lambda_k x_0}{L} \right), \tag{5.4}$$

$$a_3 = \left(B_k Cosh\frac{\lambda_k x_0}{L} + C_k Sinh\frac{\lambda_k x_0}{L} \right) \text{ and } G = \frac{\gamma\lambda_k}{L} \tag{5.5}$$

The general solution of equation (5.2) is given by

$$\overline{V}(m,t) = C_1 Cos\omega_{aj} t + C_2 Sin\omega_{aj} t + P_1(t) Cos\omega_{aj} t + P_2(t) Sin\omega_{aj} t \tag{5.6}$$

where

$$P_1(t) = -\frac{Q}{\omega_{aj}} \int \{ a_0 Sin(GSin\beta t) + a_1 Cos(GSin\beta t) + S_1 e^{GSin\beta t} + S_2 e^{-GSin\beta t} Sin\omega_{aj} t\, dt \}, \tag{5.7}$$

$$P_2(t) = \frac{Q}{\omega_{aj}} \int \left\{ a_0 Sin(GSin\beta t) + a_1 Cos(GSin\beta t) + S_1 e^{GSin\beta t} + S_2 e^{-GSin\beta t} \right\} Cos\omega_{aj} t\, dt, \tag{5.8}$$

$$S_1 = \frac{a_2 + a_3}{2}, \quad S_2 = \frac{a_2 - a_3}{2}, \tag{5.9}$$

and C_1 and C_2 are constants to be determined by the initial conditions.
In order to evaluate integrals (5.7) and (5.8), use is made of the following Bessel relations

(i) $$Cos(ZSin\theta) = J_0(Z) + 2\sum_{k=1}^{\infty} J_{2k}(Z) Cos(2k\theta) \tag{5.10}$$

(ii) $$Sin(ZSin\theta) = 2\sum_{k=0}^{\infty} J_{2k+1}(Z) Sin(2k+1)\theta \tag{5.11}$$

(iii) $$Cos(ZCos\theta) = J_0(Z) + 2\sum_{k=1}^{\infty} (-1)^k J_{2k}(Z) Cos(2k\theta) \tag{5.12}$$

(iv) $$Sin(ZCos\theta) = 2\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(Z) Cos(2k+1)\theta \tag{5.13}$$

Other useful relations are

(i) $$e^{\frac{z}{2}(t+\frac{1}{t})} = \sum_{k=-\infty}^{\infty} t^k I_{2k}(z), \quad t \neq 0 \tag{5.14}$$

(ii) $$e^{ZCos\theta} = I_0(Z) + 2\sum_{k=1}^{\infty} I_{2k}(Z) Cos(2k\theta) \tag{5.15}$$

(iii) $$e^{ZSin\theta} = I_0(Z) + 2\sum_{k=0}^{\infty} (-)^k I_{2k+1}(Z) Sin(2k+1)\theta$$

$$+2\sum_{k=1}^{\infty}(-1)^k I_{2k}(Z)Cos(2k\theta) \qquad (5.16)$$

where $$J_k(Z)=\sum_{m=0}^{\infty}(-1)^m\left(\frac{Z}{m}\right)^{k+2m}\frac{1}{m!(k+1)!} \qquad (5.17)$$

and $$I_k(Z)=\sum_{m=0}^{\infty}\frac{\left(\frac{Z}{2}\right)^{(k+2m)}}{m!(k+m)!} \qquad (5.18)$$

is the modified Bessel function of the first kind of order K. If we put k = 0, the particular case of $I_k(Z)$ is obtained as

$$I_0(Z)=\sum_{m=0}^{\infty}\frac{\left(Z/2\right)^{2m}}{(m!)^2}=1+\left(\frac{Z}{2}\right)^2+\frac{\left(\frac{Z}{2}\right)^4}{(2!)^2}+\frac{\left(\frac{Z}{2}\right)^6}{(3!)^2}+\cdots \qquad (5.19)$$

In view of Bessel relations (5.10) to (5.16), equations (5.7) and (5.8) become

$$P_1(t)=-\frac{P_m}{\omega_{aj}}\int\left\{a_0\left[2\sum_{k=0}^{\infty}J_{2k+1}(G)Sin(2k+1)\beta t\right]\right.$$
$$+a_1\left[J_0(G)+2\sum_{k=1}^{\infty}J_{2k}(G)Cos2k\beta t\right]$$
$$+S_1\left[I_0(G)+2\sum_{k=1}^{\infty}(-1)^k I_{2k+1}(G)Sin(2k+1)\beta tt\right. \qquad (5.20)$$
$$+2\sum_{k=1}^{\infty}(-1)^k I_{2k}(G)Cos2k\beta$$
$$+S_2 I_0(-G)+2\sum_{k=1}^{\infty}(-1)^k I_{2k+1}(-G)Sin(2k+1)\beta t$$
$$\left.+2\sum_{k=1}^{\infty}(-1)^k I_{2k}(-G)Cos2k\beta t]\right\}Sin\omega_{aj}t\,dt$$

and

$$P_2(t) = \frac{P_m}{\omega_{aj}} \int \Bigg\{ a_0 \left[2\sum_{k=0}^{\infty} J_{2k+1}(G) Sin(2k+1)\beta t \right]$$

$$+ a_1 \left[J_0(G) + 2\sum_{k=1}^{\infty} J_{2k}(G) Cos 2k\beta t \right]$$

$$+ S_1 \Big[I_0(G) + 2\sum_{k=1}^{\infty} (-1)^k I_{2k+1}(G) Sin(2k+1)\beta t \qquad (5.21)$$

$$+ 2\sum_{k=1}^{\infty} (-1)^k I_{2k}(G) Cos 2k\beta t \Big]$$

$$+ S\big[{}_2 I_0(-G) + 2\sum_{k=1}^{\infty} (-1)^k I_{2k+1}(-G) Sin(2k+1)\beta t$$

$$+ 2\sum_{k=1}^{\infty} (-1)^k I_{2k}(-G) Cos 2k\beta t \big] \Bigg\} Cos\,\omega_{aj} t\, dt$$

where
$$P_m = \frac{P}{\Omega_0(k,m)\mu_0} \qquad (5.22)$$

Evaluating (5.20) and (5.21) above after some simplifications and rearrangements yield

$$P_1(t) = -\frac{P_m}{\omega_{aj}} \Bigg\{ a_0 \sum_{k=0}^{\infty} J_{2k+1}(G) \left[\frac{Sinb_4 t}{b_4} - \frac{Sinb_3 t}{b_3} \right]$$

$$- a_1 \left[J_0(G) \frac{Cosb_0 t}{b_0} + \sum_{k=1}^{\infty} J_{2k}(G) \left(\frac{Cosb_1 t}{b_1} + \frac{Cosb_2 t}{b_2} \right) \right]$$

$$+ S_1 \Bigg[-I_0(G) \frac{Cosb_0 t}{b_0} + \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G) \left[\frac{Sinb_4 t}{b_4} - \frac{Sinb_3 t}{b_3} \right]$$

$$- \sum_{k=1}^{\infty} (-1)^k I_{2k}(G) \left(\frac{Cosb_1 t}{b_1} + \frac{Cosb_2 t}{b_2} \right) \Bigg] \qquad (5.23)$$

$$+ S_2 \Bigg[-I_0(-G) \frac{Cosb_0 t}{b_0} + \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G) \left[\frac{Sinb_4 t}{b_4} - \frac{Sinb_3 t}{b_3} \right]$$

$$- \sum_{k=1}^{\infty} (-1)^k I_{2k}(-G) \left(\frac{Cosb_1 t}{b_1} + \frac{Cosb_2 t}{b_2} \right) \Bigg] \Bigg\}$$

and

$$P_2(t) = \frac{P_m}{\omega_{aj}}\left\{a_0 \sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{Cosb_4 t}{b_4} - \frac{Cosb_3 t}{b_3}\right]\right.$$

$$-a_1\left[J_0(G)\frac{Sinb_0 t}{b_0} + \sum_{k=1}^{\infty} J_{2k}(G)\left(\frac{Sinb_1 t}{b_1} + \frac{Sinb_2 t}{b_2}\right)\right]$$

$$+S_1\left[I_0(G)\frac{Sinb_0 t}{b_0} + \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(G)\left[\frac{Cosb_4 t}{b_4} - \frac{Cosb_3 t}{b_3}\right]\right. \tag{5.24}$$

$$\left.+\sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left(\frac{Sinb_1 t}{b_1} + \frac{Sinb_2 t}{b_2}\right)\right]$$

$$+S_2\left[I_0(-G)\frac{Sinb_0 t}{b_0} + \sum_{k=0}^{\infty} (-1)^k I_{2k+1}(-G)\left[\frac{Cosb_4 t}{b_4} - \frac{Cosb_3 t}{b_3}\right]\right.$$

$$\left.\left.+\sum_{k=1}^{\infty} (-1)^k I_{2k}(-G)\left(\frac{Sinb_1 t}{b_1} + \frac{\sin b_2 t}{b_2}\right)\right]\right\}$$

where

$$b_0 = \omega_{aj}, \qquad b_1 = \omega_{aj} + 2k\beta, \qquad b_2 = \omega_{aj} - 2k\beta \tag{5.25}$$

$$b_3 = \omega_{aj} + (2k+1)\beta, \; b_4 = \omega_{aj} - (2k+1)\beta \tag{5.26}$$

Substituting (5.23) and (5.24) into equation (5.6) yields

$$\overline{V}(m,t) = C_1 Cos\omega_{aj} t + C_2 Sin\omega_{aj} t$$

$$+\frac{P_m}{\omega_{aj}}\left\{a_0 \sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{Sin(\omega_{aj} - b_4)t}{b_4} - \frac{Sin(\omega_{aj} - b_3)t}{b_3}\right]\right.$$

$$+a_1 J_0(G)\left[\frac{Cos(\omega_{aj} - b_0)t}{b_0}\right] + a_1 \sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\omega_{aj} - b_1)t}{b_1}\right.$$

$$\left.+\frac{Cos(\omega_{aj} - b_2)t}{b_2}\right] + S_1 I_0(G)\left[\frac{Cos(\omega_{aj} - b_0)t}{b_0}\right]$$

$$+S_1 \sum_{k=1}^{\infty} (-1)^k I_{2k}(G)\left[\frac{Cos(\omega_{aj} - b_1)t}{b_1} + \frac{Cos(\omega_{aj} - b_2)t}{b_2}\right]$$

$$+S_1\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(G)\left[\frac{Sin(\omega_{aj}-b_4)t}{b_4}-\frac{Sin(\omega_{aj}-b_3)t}{b_3}\right]$$

$$+S_2 I_0(-G)\left[\frac{Cos(\omega_{aj}-b_0)t}{b_0}\right]+S_2\sum_{k=1}^{\infty}(-1)^k I_{2k}(-G)\left[\frac{Cos(\omega_{aj}-b_1)t}{b_1}\right. \tag{5.27}$$

$$\left.+\frac{Cos(\omega_{aj}-b_2)t}{b_2}\right]+S_2\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(-G)\left[\frac{Sin(\omega_{aj}-b_4)t}{b_4}-\frac{Sin(\omega_{aj}-b_3)t}{b_3}\right]\Bigg\}$$

Applying the initial conditions (2.2), one obtains

$$C_1=-\frac{P_m}{\omega_{aj}}\left\{a_1\frac{J_0(G)}{b_0}+a_1\sum_{k=1}^{\infty}J_{2k}(G)\left[\frac{1}{b_1}+\frac{1}{b_2}\right]\right.$$

$$+S_1\frac{I_0(G)}{b_0}+S_1\sum_{k=1}^{\infty}(-1)^k I_{2k}(G)\left[\frac{1}{b_1}+\frac{1}{b_2}\right] \tag{5.28}$$

$$\left.+S_2\frac{I_0(-G)}{b_0}+S_2\sum_{k=1}^{\infty}(-1)^k I_{2k}(-G)\left[\frac{1}{b_1}+\frac{1}{b_2}\right]\right\}$$

and

$$C_2=-\frac{P_m}{\omega_{aj}^2}\left\{a_0\sum_{k=0}^{\infty}J_{2k+1}(G)\left[\frac{(\gamma_{aj}-b_4)}{b_4}+\frac{(b_3-\gamma_{aj})}{b_3}\right]\right.$$

$$+S_1\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(G)\left[\frac{(\gamma_{aj}-b_4)}{b_4}+\frac{(b_3-\gamma_{aj})}{b_3}\right] \tag{5.29}$$

$$\left.+S_2\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(-G)\left[\frac{(\gamma_{aj}-b_4)}{b_4}+\frac{(b_3-\gamma_{aj})}{b_3}\right]\right\}$$

Substituting equations (5.28) and (5.29) into equation (5.27), simplifying and inverting yield

$$V_n(x,t)=\sum_{m=1}^{n}\frac{P_m}{\omega_{aj}^2}\left(\omega_{aj}a_1 J_0(G)\left[\frac{Cos(\omega_{aj}-b_0)t-Cos\,\omega_{aj}t}{b_0}\right]\right.$$

$$+a_1\omega_{aj}\sum_{k=1}^{\infty}J_{2k}(G)\left[\frac{Cos(\omega_{aj}-b_1)t-Cos\,\omega_{aj}t}{b_1}+\frac{Cos(\omega_{aj}-b_2)t-Cos\,\omega_{aj}t}{b_2}\right]$$

$$+a_0\sum_{k=0}^{\infty}J_{2k+1}(G)\left[\frac{\omega_{aj}Sin(\omega_{aj}-b_4)t-(\omega_{aj}-b_4)Sin\,\omega_{aj}t}{b_4}\right.$$

$$\left.-\frac{\omega_{aj}Sin(\omega_{aj}-b_3)t-(\omega_{aj}-b_3)Sin\,\omega_{aj}t}{b_3}\right]+S_1\omega_{aj}I_0(G)\left[\frac{Cos(\omega_{aj}-b_0)t-Cos\,\omega_{aj}t}{b_0}\right]$$

$$+S_1\omega_{aj}\sum_{k=1}^{\infty}(-1)^k I_{2k}(G)\left[\frac{Cos(\omega_{aj}-b_1)t-Cos\omega_{aj}t}{b_1}+\frac{Cos(\omega_{aj}-b_2)t-Cos\omega_{aj}t}{b_2}\right]$$

$$+S_1\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(G)\left[\frac{\omega_{aj}Sin(\omega_{aj}-b_4)t-(\omega_{aj}-b_4)Sin\omega_{aj}t}{b_4}\right.$$

$$\left.-\frac{\omega_{aj}Sin(\omega_{aj}-b_3)t-(\omega_{aj}-b_3)Sin\omega_{aj}t}{b_3}\right]+S_2\omega_{aj}I_0(-G)\left[\frac{Cos(\omega_{aj}-b_0)t-Cos\omega_{aj}t}{b_0}\right]$$

$$+S_2\omega_{aj}\sum_{k=1}^{\infty}(-1)^k I_{2k}(-G)\left[\frac{Cos(\omega_{aj}-b_1)t-Cos\omega_{aj}t}{b_1}+\frac{Cos(\omega_{aj}-b_2)t-Cos\omega_{aj}t}{b_2}\right]$$

$$+S_2\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(-G)\left[\frac{\omega_{aj}Sin(\omega_{aj}-b_4)t-(\omega_{aj}-b_4)Sin\omega_{aj}t}{b_4}\right.$$

$$\left.\left.-\frac{\omega_{aj}Sin(\omega_{aj}-b_3)t-(\omega_{aj}-b_3)Sin\omega_{aj}t}{b_3}\right]\right)\times\left(Sin\frac{\lambda_m x}{L}+A_mCos\frac{\lambda_m x}{L}\right.$$

$$\left.+B_mSinh\frac{\lambda_m x}{L}+C_mCosh\frac{\lambda_m x}{L}\right) \tag{5.30}$$

Equation (5.30) represents the transverse displacement response to a moving force moving at variable velocity of a non-uniform Bernoulli-Euler beam resting on an elastic foundation and having arbitrary end support conditions.

(ii) **The moving mass problem**

If the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving mass is not negligible. Thus, $\varepsilon_1 \neq 0$ and one is required to solve the entire equation (4.13). This is termed the moving mass problem. To this end, equation (4.13) is rearranged to take the form

$$\ddot{Y}_m(t)+\omega_{aj}^2Y_m(t)+\varepsilon_1\left\{H_2(k,m)\ddot{Y}_m(t)+2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\ddot{Y}_m(t)\right.$$

$$+2\gamma\beta H_4(k,m)Cos\beta t\,\dot{Y}_m(t)+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_5(k,m,n)\dot{Y}_m(t)$$

$$+(\gamma\beta Cos\beta t)^2H_6(k,m)Y_m(t)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_7(k,m,n)Y_m(t) \tag{5.31}$$

$$\left.-\gamma\beta^2H_8(k,m)Sin\beta t\,Y_m(t)-2\gamma\beta^2Sin\beta t\sum_{n=1}^{\infty}Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_9(k,m,n)Y_m(t)\right\}$$

$$=\frac{\varepsilon_1Lg}{\Omega_0(k,m)\mu_0}\left[a_0Sin(GSin\beta t)+a_1Cos(GSin\beta t)+a_2Cosh(GSin\beta t)+a_3Sinh(GSin\beta t)\right]$$

where $H_2(k,m)=\dfrac{H_b^0(k,m)}{\Omega_0(k,m)}$ $\qquad H_3(k,m,n)=\dfrac{H_c^0(k,m,n)}{\Omega_0(k,m)}$

$$H_4(k,m)=\frac{H_d^0(k,m)}{\Omega_0(k,m)} \qquad H_5(k,m,n)=\frac{H_e^0(k,m,n)}{\Omega_0(k,m)}$$

$$H_6(k,m) = \frac{H_f^0(k,m)}{\Omega_0(k,m)} \qquad H_7(k,m,n) = \frac{H_g^0(k,m,n)}{\Omega_0(k,m)}$$

$$H_8(k,m) = \frac{H_i^0(k,m)}{\Omega_0(k,m)} \qquad H_9(k,m,n) = \frac{H_j^0(k,m,n)}{\Omega_0(k,m)} \tag{5.32}$$

Furthermore, equation (5.31) is simplified and rearranged to take the form

$$\ddot{Y}_m(t) + \frac{\varepsilon_1\left[2\gamma\beta H_4(k,m)Cos\beta t + 4\gamma\beta Cos\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_5(k,m,n)\right]}{\left[1+\varepsilon_1\left(H_2(k,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]}\dot{Y}_m(t)$$

$$+\frac{\left[\omega_{aj}^2+\varepsilon_1\left\{(\gamma\beta Cos\beta t)^2 H_6(k,m)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_7(k,m,n)\right\}\right]}{\left[1+\varepsilon_1\left(H_2(k,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]}Y_m(t)$$

$$-\frac{\varepsilon_1\left[\gamma\beta^2 H_8(k,m)Sin\beta t+2\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_9(k,m,n)\right]}{\left[1+\varepsilon_1\left(H_2(k,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]}Y_m(t)$$

$$=\frac{\varepsilon_1 Lg}{\Omega_0(k,m)}\frac{\left[a_0 Sin(GSin\beta t)+a_1 Cos(GSin\beta t)+a_2 Cosh(GSin\beta t)+a_3 Sinh(GSin\beta t)\right]}{\left[1+\varepsilon_1\left(H_2(k,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]} \tag{5.33}$$

Evidently, unlike in the case of the moving force problem an exact analytical solution to equation (5.31) is not possible. Though the equation yields readily to numerical technique, an analytical approximate method is desirable as solutions so obtained often shed light on vital information about the vibrating system. To this end, we are going to use a modification of the asymptotic method due to Struble's. By this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the effect of axial force N. An equivalent free system operator defined by the modified frequency then replaces equation (5.33). Thus, we set the right-hand-side of (5.33) to zero and consider a parameter η <1 for any arbitrary ratio ε_1, defined as

$$\eta = \frac{\varepsilon_1}{1+\varepsilon_1} \tag{5.34}$$

so that
$$\varepsilon_1 = \eta + O(\eta^2) \tag{5.35}$$

$$\frac{1}{\left[1+\eta\left(H_2(k,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right]}$$
$$=\left[1-\eta\left(H_2(k,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)+O(\eta^2)\right]+\cdots \tag{5.36}$$

where $$\left|\eta\left(H_2(k,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right|<1 \tag{5.37}$$

Substituting equation (5.35) and (5.36) into the homogeneous part of equation (5.33) one obtains

$$\begin{aligned}
&\ddot{Y}_m(t)+\eta\left[2\gamma\beta H_4(k,m)Cos\beta t+4\gamma\beta Cos\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma\beta Sin\beta t)H_5(k,m,n)\right]\dot{Y}_m(t)\\
&+\left[\omega_{aj}^2-\eta\omega_{aj}^2\left(H_2(k,m)+2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_3(k,m,n)\right)\right.\\
&\left.+\eta\left\{(\gamma\beta Cos\beta t)^2H_6(k,m)+2(\gamma\beta Cos\beta t)^2\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_7(k,m,n)\right\}\right]Y_m(t)\\
&-\eta\left[\gamma\beta^2H_8(k,m)Sin\beta t+2\gamma\beta^2 Sin\beta t\sum_{n=1}^{\infty} Cos\frac{n\pi}{L}(x_0+\gamma Sin\beta t)H_9(k,m,n)\right]Y_m(t)\\
&=\frac{\eta Lg}{\Omega_0(k,m)}\left[a_0Sin(GSin\beta t)+a_1Cos(GSin\beta t)+a_2Cosh(GSin\beta t)+a_3Sinh(GSin\beta t\right]
\end{aligned} \tag{5.38}$$

to $O(\eta)$ only. When η is set to zero in equation (5.38) a situation corresponding to the case in which the axial force effect is regarded as negligible is obtained, then the solution of (5.38) becomes

$$\overline{V}_{nf}(m,t)=C_{nf}Cos\left[\omega_{nf}t-\psi_{nf}\right] \tag{5.39}$$

where C_{nf}, ω_{nf} *and* ψ_{nf} are constants. Furthermore as η<1 Struble's technique requires that the asymptotic solutions of the homogeneous part of the equation (5.31) be of the form

$$\overline{V}(m,t)=A(m,t)Cos\left[\omega_{nf}t-\phi(m,t)\right]+\eta\Phi_1+O(\eta^2) \tag{5.40}$$

where $\Lambda(m,t)$ and $\phi(m,t)$ are slowly varying functions of time.

To obtain the modified frequency, equation (5.40) and its derivatives are substituted into equation (5.38) and taking in account the following trigonometric identities

$$Sin\left[\omega_{aj}t-\phi(m,t)\right]Cos\beta t=\frac{1}{2}\left\{Sin\left[\omega_{aj}t-\phi(m,t)+\beta t\right]+Sin\left[\omega_{aj}t-\phi(m,t)-\beta t\right]\right\} \tag{5.41a}$$

$$Cos\left[\omega_{aj}t-\phi(m,t)\right]Sin\beta t=\frac{1}{2}\left\{Sin\left[\omega_{aj}t-\phi(m,t)+\beta t\right]-Sin\left[\omega_{aj}t-\phi(m,t)-\beta t\right]\right\} \tag{5.41b}$$

$$\begin{aligned}Sin\left[\omega_{aj}t-\phi(m,t)\right]Cos2k\beta t=\frac{1}{2}&\left\{Sin\left[\omega_{aj}t-\phi(m,t)+2k\beta t\right]\right.\\&\left.+Sin\left[\omega_{aj}t-\phi(m,t)-2k\beta t\right]\right\}\end{aligned} \tag{5.41c}$$

$$\begin{aligned}Sin\left[\omega_{aj}t-\phi(m,t)\right]Sin(2k+1)=\frac{1}{2}&\left\{Cos\left[\omega_{aj}t-\phi(m,t)-(2k+1)\beta t\right]\right.\\&\left.-Cos\left[\omega_{aj}t-\phi(m,t)+(2k+1)\beta t\right]\right\}\end{aligned} \tag{5.41d}$$

$$\begin{aligned}Cos\left[\omega_{aj}t-\phi(m,t)\right]Sin(2k+1)\beta t=\frac{1}{2}&\left\{Sin\left[\omega_{aj}t-\phi(m,t)+(2k+1)\beta t\right]\right.\\&\left.-Sin\left[\omega_{aj}t-\phi(m,t)-(2k+1)\beta t\right]\right\}\end{aligned} \tag{5.41e}$$

$$Cos\left[\omega_{aj}t-\phi(m,t)\right]Cos2k\beta t=\frac{1}{2}\left\{Cos\left[\omega_{aj}t-\phi(m,t)+2k\beta t\right]\right.$$
$$\left.+Cos\left[\omega_{aj}t-\phi(m,t)-2k\beta t\right]\right\} \tag{5.41f}$$

one obtains.

$$-2\omega_{aj}\dot{A}(m,t)Sin\left[\omega_{aj}t-\phi(m,t)\right]+2\omega_{aj}A(m,t)\dot{\phi}(m,t)Cos\left[\omega_{aj}t-\phi(m,t)\right]$$
$$-\eta\omega_{aj}^2A(m,t)H_2(k,m)Cos\left[\omega_{aj}t-\phi(m,t)\right]$$
$$-2\eta\omega_{aj}^2A(m,t)\sum_{n=1}^{\infty}H_3(k,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1)Cos\left[\omega_{aj}t-\phi(m,t)\right] \tag{5.42}$$
$$+\frac{1}{2}\eta A(m,t)(\gamma\beta)^2H_6(k,m)Cos\left[\omega_{aj}t-\phi(m,t)\right]$$
$$+\eta A(m,t)(\gamma\beta)^2\sum_{n=1}^{\infty}H_7(k,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1)Cos\left[\omega_{aj}t-\phi(m,t)\right]=0$$

retaining terms to $O(\eta)$ only. The variational equations are obtained by equating the coefficients of $Sin\left[\omega_{nf}\,t-\phi(m,t)\right]$ and $Cos\left[\omega_{nf}\,t-\phi(m,t)\right]$ on both sides of the equation (5.42). Thus,

$$-2\omega_{aj}\dot{A}(m,t)=0 \tag{5.43}$$

and

$$2\omega_{aj}A(m,t)\dot{\phi}(m,t)-\eta\omega_{aj}^2A(m,t)H_2(k,m)$$
$$-2\eta\omega_{aj}^2A(m,t)\sum_{n=1}^{\infty}H_3(k,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1)+\frac{1}{2}\eta A(m,t)(\gamma\beta)^2H_6(k,m) \tag{5.44}$$
$$+\eta A(m,t)(\gamma\beta)^2\sum_{n=1}^{\infty}H_7(k,m,n)Cos\frac{n\pi x_0}{L}J_0(G_1)=0$$

Solving equations (5.43) and (5.44) respectively gives

$$A(m,t)=C_0^* \tag{5.45}$$

where C_0^* is a constant and

$$Q(m,t)=\frac{\eta}{2}\left[\omega_{aj}\left(H_2(k,m)+R_a(k,m,n)\right)\right.$$
$$\left.-\left((\gamma\beta)^2\frac{\{H_6(k,m)+2R_b(k,m,n)\}}{2\omega_{aj}}\right)\right]t+\psi_m \tag{5.46}$$

to $O(\eta)$ only. Where ψ_m is a constant. Therefore, when the inertia effect of the moving mass is considered, the first approximation to the homogeneous system is

$$Y_m(t)=C_0^*Cos\left[\omega_{aj}t-\Phi_m\right] \tag{5.47}$$

where

$$\omega_{bj} = \omega_{aj}\left\{1-\frac{\eta}{2}\left[\left(H_2(k,m)+R_a(k,m,n)\right)-(\gamma\beta)^2\frac{\left[H_6(k,m)+2R_b(k,m,n)\right]}{2\omega_{aj}^2}\right]\right\} \tag{5.48}$$

represents the modified natural frequency due to the presence of the moving mass. It is observed that when $\eta = 0$, we recover the frequency of the moving force problem when the inertia effect of the moving mass is neglected. Thus, to solve the non-homogeneous equation (5.31), the differential operator which acts on $\overline{V}(m,t)$ and $\overline{V}(k,t)$ is replaced by the equivalent free System operator defined by the modified frequency ω_{aj}. Using equation (5.48) the homogeneous part of equation (5.13) can be written as

$$\frac{d^2Y_m(t)}{dt^2}+\omega_{bj}^2 Y_m(t)=0 \tag{5.49}$$

Thus, the entire equation (4.13), becomes

$$\begin{aligned}&\frac{d^2Y_m(t)}{dt^2}+\omega_{bj}^2 Y_m(t)\\ &=\frac{\eta Lg}{\Omega_0(k,m)}\left[a_0Sin\left(GSin\beta t\right)+a_1Cos\left(GSin\beta t\right)+a_2Cosh\left(GSin\beta t\right)+a_3Sinh\left(GSin\beta t\right)\right]\end{aligned} \tag{5.50}$$

retaining $O(\lambda)$ only. This is analogous to equation (5.2). Thus, using similar argument as in moving force problem, $\overline{V}(m,t)$ can be obtained and when inverted gives

$$V_m(x,t) = \sum_{m=1}^{\infty}\frac{Q_m}{\omega_{bj}^2}\left(\omega_{bj}a_1 J_0(G)\left[\frac{Cos(\omega_{bj}-b_0)t - Cos\,\omega_{bj}\,t}{b_0}\right]\right.$$

$$+ a_1\omega_{bj}\sum_{k=1}^{\infty} J_{2k}(G)\left[\frac{Cos(\omega_{bj}-b_1)t - Cos\,\omega_{bj}\,t}{b_1} + \frac{Cos(\omega_{bj}-b_2)t - Cos\,\omega_{bj}\,t}{b_2}\right]$$

$$+ a_0\sum_{k=0}^{\infty} J_{2k+1}(G)\left[\frac{\omega_{bj}Sin(\omega_{bj}-b_4)t - (\omega_{bj}-b_4)Sin\,\omega_{bj}t}{b_4}\right.$$

$$\left. - \frac{\omega_{bj}Sin(\omega_{bj}-b_3)t - (\omega_{bj}-b_3)Sin\,\omega_{bj}t}{b_3}\right] + S_1\omega_{bj}I_0(G)\left[\frac{Cos(\omega_{bj}-b_0)t - Cos\,\omega_{bj}\,t}{b_0}\right]$$

$$+ S_1\omega_{bj}\sum_{k=1}^{\infty}(-1)^k I_{2k}(G)\left[\frac{Cos(\omega_{bj}-b_1)t - Cos\,\omega_{bj}\,t}{b_1} + \frac{Cos(\omega_{bj}-b_2)t - Cos\,\omega_{bj}\,t}{b_2}\right]$$

$$+ S_1\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(G)\left[\frac{\omega_{bj}Sin(\omega_{bj}-b_4)t - (\omega_{bj}-b_4)Sin\,\omega_{bj}t}{b_4}\right.$$

$$\left. - \frac{\omega_{bj}Sin(\omega_{bj}-b_3)t - (\omega_{bj}-b_3)Sin\,\omega_{bj}t}{b_3}\right] + S_2\omega_{bj}I_0(-G)\left[\frac{Cos(\omega_{bj}-b_0)t - Cos\,\omega_{bj}\,t}{b_0}\right]$$

$$+ S_2\omega_{bj}\sum_{k=1}^{\infty}(-1)^k I_{2k}(-G)\left[\frac{Cos(\omega_{bj}-b_1)t - Cos\,\omega_{bj}\,t}{b_1} + \frac{Cos(\omega_{bj}-b_2)t - Cos\,\omega_{bj}\,t}{b_2}\right]$$

$$+ S_2\sum_{k=0}^{\infty}(-1)^k I_{2k+1}(-G)\left[\frac{\omega_{bj}Sin(\omega_{bj}-b_4)t - (\omega_{bj}-b_4)Sin\,\omega_{bj}t}{b_4}\right.$$

$$\left.\left. - \frac{\omega_{bj}Sin(\omega_{bj}-b_3)t - (\omega_{bj}-b_3)Sin\,\omega_{bj}t}{b_3}\right]\right) \times \left(Sin\frac{\lambda_m x}{L} + A_m Cos\frac{\lambda_m x}{L}\right.$$

$$\left. + B_m Sinh\frac{\lambda_m x}{L} + C_m Cosh\frac{\lambda_m x}{L}\right) \qquad (5.51)$$

Equation (5.51) represents the transverse-displacement response to a moving mass moving at variable velocities of a non-uniform Bernoulli-Euler beam resting on elastic foundation when the boundary conditions are arbitrary.

6.0 Illustrative Examples

In order to illustrate our results in the foregoing analysis, in what follows, we provide some examples;

(a) Simply Supported boundary conditions.
(b) Clamped-Clamped boundary conditions
(c) The Cantilever.

6.1 Simply Supported Boundary Conditions

In this case, the displacement and the bending moment vanish. Thus

$$V(0,t)=0=V(L,t), \qquad \frac{\partial^2 V(0,t)}{\partial x^2}=0=\frac{\partial^2 V(L,t)}{\partial x^2} \tag{6.1}$$

Hence for normal modes

$$U_m(0)=0=U_m(L), \qquad \frac{\partial^2 U_m(0)}{\partial x^2}=0=\frac{\partial^2 U_m(L)}{\partial x^2} \tag{6.2}$$

which implies that

$$U_k(0)=0=U_k(L), \qquad \frac{\partial^2 U_k(0)}{\partial x^2}=0=\frac{\partial^2 U_k(L)}{\partial x^2} \tag{6.3}$$

Applying (6.2) and (6.3), one obtains

$$A_m = A_k = 0; \quad B_m = B_k = 0; \quad C_m = C_k = 0 \tag{6.4}$$

$$\lambda_m = m\pi \qquad and \qquad \lambda_k = k\pi \tag{6.5}$$

Thus, substituting equations (6.4) and (6.5) into equation (4.13) and rearranging, the moving force problem reduces to the non-homogeneous second order ordinary differential equation given by

$$\ddot{Y}_m(t)+\frac{\Delta_1(k,m)}{\Delta_0(k,m)}Y_m(t)=\frac{P}{\Delta_0(k,m)\mu_0}Sin\frac{k\pi}{L}\left(x_0+\gamma Sin\beta t\right) \tag{6.6}$$

where

$$\begin{aligned}\Delta_1(k,m)=\frac{EI_0}{4\mu_0}\Bigg[&\frac{5m^4\pi^4}{L^3}-\frac{60m^5\pi^3k}{L^3\left[(1-k)^2-m^2\right]\left[(1+k)^2-m^2\right]}-\frac{6m^4\pi^4}{L^4}\left(\frac{\alpha_1}{2}-\frac{\alpha_2}{2}\right)\\
&+\frac{12m^5\pi^3k}{L^3\left[(3-k)^2-m^2\right]\left[(3+k)^2-m^2\right]}+\frac{60\pi^3m^3k(m^2+1-k^2)}{L^3\left[(1+m)^2-k^2\right]\left[(1-m)^2-k^2\right]}\\
&-\frac{24\pi^4m^3}{L^4}\left(\frac{\alpha_3}{2}-\frac{\alpha_4}{2}\right)-\frac{12\pi^3m^3k(m^2+9-k^2)}{L^3\left[(3+m)^2-k^2\right]\left[(3-m)^2-k^2\right]}\\
&+\frac{108\pi^3m^3k}{L^3\left[(3-k)^2-m^2\right]\left[(3+k)^2-m^2\right]}-\frac{24m^2\pi^2}{L^4}\left(\frac{\alpha_1}{2}-\frac{\alpha_2}{2}\right)\\
&-\frac{60m^3\pi^3k}{L^3\left[(1-k)^2-m^2\right]\left[(1+k)^2-m^2\right]}\Bigg]+\frac{Nm^2\pi^2}{2\mu_0L}+\frac{K^0L}{2\mu_0}\end{aligned} \tag{6.7}$$

and

$$\Delta_0(k,m)=\frac{L}{2}-\frac{4mkL}{\pi\left[(1-k)^2-m^2\right]\left[(1+k)^2-m^2\right]} \tag{6.8}$$

Equation (6.6) can be rewritten as

$$\ddot{Y}_m(t)+\omega_{mf}^2Y_m(t)=P_{mf}\left[SinF^0Cos\left(G^0Sin\beta t\right)+CosF^0Sin\left(G^0Sin\beta t\right)\right] \tag{6.9}$$

where

$$\omega_{mf}^2=\frac{\Delta_1(k,m)}{\Delta_0(k.m)},\; P_{mf}=\frac{P}{\Delta_0(k,m)\mu_0},\; F^0=\frac{k\pi x_0}{L} \text{ and } G^0=\frac{k\pi\gamma}{L} \tag{6.10}$$

Equation (6.6) when solved in conjunction with the initial conditions, one obtains an expression for $\overline{V}(m,t)$ which on inversion yields

$$V_n(x,t) = \sum_{m=1}^{n} \frac{P_{mf}}{\omega_{mf}^2} \left\{ \omega_{mf} \frac{SinF^0 J_0(G^0)}{b_0} \left[Cos(\omega_{mf} - b_0)t - Cos\omega_{mf} t \right] \right.$$

$$+ \omega_{mf} SinF^0 \sum_{k=1}^{\infty} J_{2k}(G^0) \left[\frac{Cos(\omega_{mf} - b_1)t - Cos\omega_{mf} t}{b_1} + \frac{Cos(\omega_{mf} - b_2)t - Cos\omega_{mf} t}{b_2} \right] \quad (6.11)$$

$$+ CosF^0 \sum_{k=0}^{\infty} J_{2k+1}(G^0) \left[\frac{\omega_{mf} Sin(\omega_{mf} - b_4)t - (\omega_{mf} - b_4) Sin\omega_{mf} t}{b_4} \right.$$

$$\left. \left. - \frac{\omega_{mf} Sin(\omega_{mf} - b_3)t - (\omega_{mf} - b_3) Sin\omega_{mf} t}{b_3} \right] \right\} \times Sin \frac{m\pi x}{L}$$

Equation (6.11) represents the transverse-displacement response to a moving force moving at a variable velocity of a simply supported non-uniform Bernoulli-Euler beam resting on elastic foundation. Substituting equations (6.4) and (6.5) into equation (5.51), rearranging and following arguments similar to those in previous section, Struble's technique is used to obtain

$$\omega_{mm} = \omega_{mf} \left\{ 1 - \frac{\eta_0}{2} \left[\left(R_1(k,m) + R_2(k,m) B_0^* \right) + \frac{\left(R_3(k,m) + R_4(k,m) B_0^* \right)}{2\omega_{mf}^2} \right] \right\} \quad (6.12)$$

to order $O(\eta_0)$ only as the modified natural frequency of the free system due to the presence of the moving mass of this model.
where

$$R_1(k,m) = \frac{L}{2\Delta_0(k,m)}, \quad R_2(k,m) = 2R_1(k,m), \quad R_3 = \frac{\gamma\beta m\pi}{2\Delta_0(k,m)}, \quad R_4 = 2R_3(k,m) \quad (6.13)$$

and

$$B_0^* = SinFJ_0(G) SinF^0 J_0(G^0) + 2SinFSinF^0 \sum_{k=1}^{\infty} J_{2k}(G) \sum_{k=1}^{\infty} J_{2k}(G^0)$$
$$+ 2CosFCosF^0 \sum_{k=0}^{\infty} J_{2k+1}(G) \sum_{k=0}^{\infty} J_{2k+1}(G^0) \quad (6.14)$$

neglecting higher order terms of λ. Thus, the moving mass problem reduces to

$$\frac{d^2 Y_m(t)}{dt^2} + \omega_{mm}^2 Y_m(t) = \frac{\varepsilon_1 Lg}{\Delta_0(k,m)} Sin \frac{k\pi}{L} \left(x_0 + \gamma Sin\beta t \right) \quad (6.15)$$

which when solved in conjunction with the initial conditions yields expression for $\overline{V}(m,t)$ and on inversion becomes

$$V_n(x,t)=\sum_{m=1}^{n}\frac{\varepsilon_1 Lg}{\Delta_0(k,m)\omega_{mm}^2}\left\{\omega_{mm}\frac{SinF^0 J_0(G^0)}{b_0}\left[Cos(\omega_{mm}-b_0)t-Cos\omega_{mm}t\right]\right.$$

$$+\omega_{mm}SinF^0\sum_{k=1}^{\infty}J_{2k}(G^0)\left[\frac{Cos(\omega_{mm}-b_1)t-Cos\omega_{mm}t}{b_1}\right.$$

$$\left.+\frac{Cos(\omega_{mm}-b_2)t-Cos\omega_{mm}t}{b_2}\right]+CosF^0\sum_{k=0}^{\infty}J_{2k+1}(G^0)\times \tag{6.16}$$

$$\left[\frac{\omega_{mm}Sin(\omega_{mm}-b_4)t-(\omega_{mm}-b_4)Sin\omega_{mm}t}{b_4}\right.$$

$$\left.\left.-\frac{\omega_{mm}Sin(\omega_{mm}-b_3)t-(\omega_{mm}-b_3)Sin\omega_{mm}t}{b_3}\right]\right\}\times Sin\frac{m\pi x}{L}$$

This represents the transverse-displacement response to a concentrated mass moving with variable velocity of a simply supported non-uniform Bernoulli-Euler beam resting on elastic foundation.

6.2 Clamped-Clamped End Conditions

At a clamped end, both deflection and slope vanish. Thus,

$$V(0,t)=0=V(L,t) \text{ and } \frac{\partial V(0,t)}{\partial x}=0=\frac{\partial V(L,t)}{\partial x} \tag{6.17}$$

Hence for normal modes

$$U_m(0)=0=U_m(L) \text{ and } \frac{\partial U_m(0)}{\partial x}=0=\frac{\partial U_m(L)}{\partial x} \tag{6.18}$$

which implies that

$$U_k(0)=0=U_k(L) \text{ and } \frac{\partial U_k(0)}{\partial x}=0=\frac{\partial U_k(L)}{\partial x} \tag{6.19}$$

Thus, it can be shown that

$$A_m=\frac{Sinh\lambda_m-Sin\lambda_m}{Cos\lambda_m-Cosh\lambda_m}=\frac{Cos\lambda_m-Cosh\lambda_m}{Sin\lambda_m+Sinh\lambda_m}=-C_m \text{ and } B_m=-1 \tag{6.20}$$

In view of (6.20), the frequency equation is given as

$$Cos\lambda_m Cosh\lambda_m=1 \tag{6.21}$$

It follows from equation (6.21), that

$$\lambda_1=4.73004,\ \lambda_2=7.85320,\ \lambda_3=10.99561 \tag{6.22}$$

Expression for $A_k, B_k\ \ , C_k$ and the corresponding frequency equation are obtained by a simple interchange of m and k in (6.19) and (6.20). Thus, the general solutions of the associated moving force and moving mass problems are obtained by substituting relevant results in equations (6.20)- (6.22) into (5.30) and (5.51)

6.3. The Cantilever

In this illustrative example, cantilever with free right-hand end and clamped at the left hand end is considered. Accordingly, the boundary conditions are

$$V(0,t)=0=\frac{\partial V(0,t)}{\partial x} \text{ and } \frac{\partial^2 V(L,t)}{\partial x^2}=0=\frac{\partial^3 V(L,t)}{\partial x^3} \tag{6.23}$$

and hence also

$$U_m(0) = 0 = \frac{dU_m(0)}{dx} \text{ and } \frac{d^2U_m(L)}{dx^2} = 0 = \frac{d^3U_m(L)}{dx^3} \qquad (6.24)$$

which implies that

$$U_k(0) = 0 = \frac{dU_k(0)}{dx} \text{ and } \frac{d^2U_k(L)}{dx^2} = 0 = \frac{d^3U_k(L)}{dx^3} \qquad (6.25)$$

Using (6.23) in (6.24), it can be shown that

$$A_m = \frac{Sin\lambda_m - Sinh\lambda_m}{Cos\lambda_m - Cosh\lambda_m} = \frac{Cos\lambda_m - Cosh\lambda_m}{Sinh\lambda_m + Sin\lambda_m} = -C_m \text{ and } B_m = -1 \qquad (6.26)$$

at end x = 0 and at end x = L

$$A_m = \frac{-Sin\lambda_m - Sinh\lambda_m}{Cos\lambda_m + Cosh\lambda_m} = \frac{-Cos\lambda_m - Cosh\lambda_m}{Sinh\lambda_m - Sin\lambda_m} = -C_m \text{ and } B_m = -1 \qquad (6.27)$$

and the frequency equation for both end conditions is

$$Cos\lambda_m Cosh\lambda_m = -1 \qquad (6.28)$$

and we have that

$$\lambda_1 = 1.875,\ \lambda_2 = 4.694,\ \lambda_3 = 7.855 \qquad (6.29)$$

Using (6.26), (6.27) and (6.28) in equations (5.30) and (5.51), one obtains the transverse displacement response respectively to a moving force and a moving mass of a cantilever Bernoulli-Euler beam resting on elastic foundation.

7.0 Remarks on Analytical Solutions

The response amplitude of a dynamical system such as this may grow without bound. Conditions under which this happens are termed resonance conditions. Equation (6.9) clearly shows that the Simply Supported elastic beam resting on elastic foundation and traversed by moving force experiences resonance effect whenever

$$\omega_{mf} = 2k\beta \text{ and } \omega_{mf} = (2k+1)\beta \qquad (7.1)$$

while equation (5.52) shows that the same beam under the action of a moving mass reaches a state of resonance when

$$\omega_{mm} = 2k\beta \text{ and } \omega_{mm} = (2k+1)\beta \qquad (7.2)$$

From equation (3.11),

$$\omega_{mm} = \omega_{mf}\left\{1 - \frac{\lambda}{2}\left[\left(2 - J_0(2G)Cos2F\right) + \frac{\left(2R_m - R_m J_0(G)Cos2F\right)}{2\omega_{mf}^2}\right]\right\} \qquad (7.3)$$

which implies

$$\omega_{mf} = \frac{2k\beta}{1 - \frac{\lambda}{2}\left[\left(2 - J_0(2G)Cos2F\right) + \frac{\left(2R_m - R_m J_0(G)Cos2F\right)}{2\omega_{mf}^2}\right]} \qquad (7.4)$$

It is therefore evident, that for the same natural frequency, the critical speed for the system consisting of a Simply Supported non-uniform beam resting on an elastic foundation and traversed by a force moving at a non-uniform velocity is greater than that of the moving mass problem. Thus, for the same natural frequency of a non-uniform beam, resonance is reached earlier in the moving mass system than in the moving force system.

For other classical boundary conditions other than Simply Supported end conditions, equation (5.45) clearly shows that the non-uniform beam resting on an elastic foundation and traversed by a force moving with variable velocity reaches a state of resonance whenever

$$\omega_{aj} = 2k\beta \quad \text{and} \quad \omega_{aj} = (2k+1)\beta \tag{7.5}$$

while equation (5.52) shows that the same non-uniform beam under the action of a moving mass experiences resonance effect whenever

$$\omega_{bj} = 2k\beta \text{ and } \omega_{bj} = (2k+1)\beta \tag{7.6}$$

From equation (2.174)

$$\omega_{bj} = \omega_{aj}\left\{1-\frac{\lambda}{2}\left[\left(H_b(m,m)+R_1(m,m,n)\right)+\frac{(\gamma\beta)^2\left(H_f(m,m)+2R_2(m,m,n)\right)}{2\omega_{aj}^2}\right]\right\} \tag{7.7}$$

This implies that

$$\omega_{aj} = \frac{2k\beta}{1-\frac{\lambda}{2}\left[\left(H_b(m,m)+R_1(m,m,n)\right)+\frac{(\gamma\beta)^2\left(H_f(m,m)+2R_2(m,m,n)\right)}{2\omega_{aj}^2}\right]} \tag{7.8}$$

Evidently, from equation (7.6) and (7.8), the same results and analysis obtained in the case of a Simply Supported non-uniform Bernoulli-Euler beam are obtained for all other examples of classical boundary conditions.

8.0 Numerical Calculations and Analysis.

In order to illustrate the theory in this paper numerically, it is assumed that the non-uniform elastic beam of length 12.2m is at rest and the equilibrium position of the longitudinal oscillating load $x_0 = \frac{1}{20}$ other data are as follows: $\gamma = 2\times10^{-4}m$, $\beta = \frac{3\pi}{4}$, $\frac{EI}{\mu} = 2200m^4/s^2$ and the ratio of the mass of the load to the mass of the beam is 0.25. The values of axial force N and subgrade K, are between 0 and 20,000,000 and 0 and $400{,}000N/m^3$ respectively.

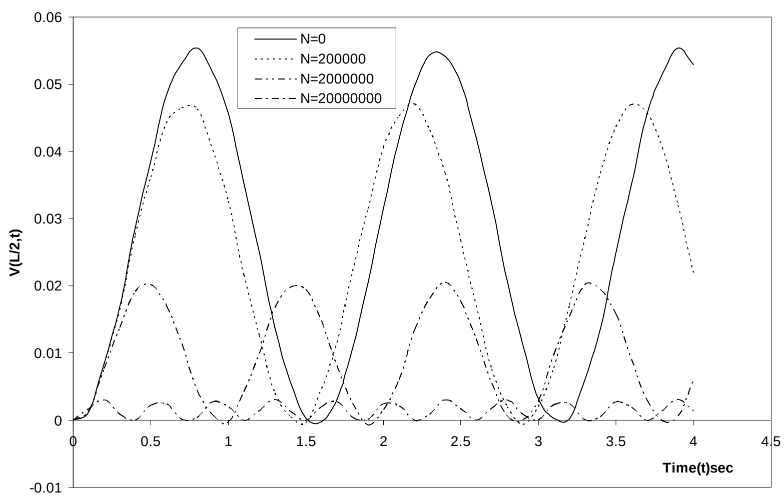

Fig 8.1: Transverse displacement of the simply supported non-uniform beam under the action of concentrated masses moving at variable velocities for various values of axial force N for fixed K (40000)

Figures 8.1, 8.3, 8.5 depict respectively the deflection profiles for Simply Supported, Clamped-Clamped and Cantilever non-uniform beams under the actions of concentrated loads travelling at varying velocities for various values of axial force N and for fixed K (40,000). The figures show that as N increases, response amplitudes of the non-uniform beam decrease. In a similar

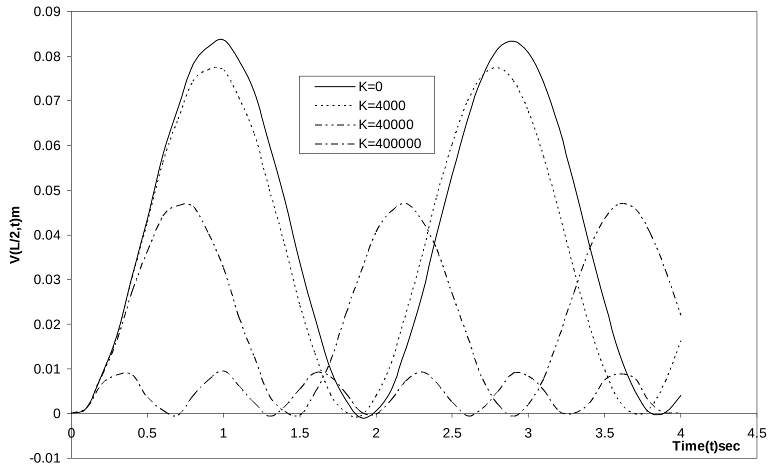

Fig 8.2: Deflection profile of the simply supported non-uniform beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli K and for fixed N (200000)

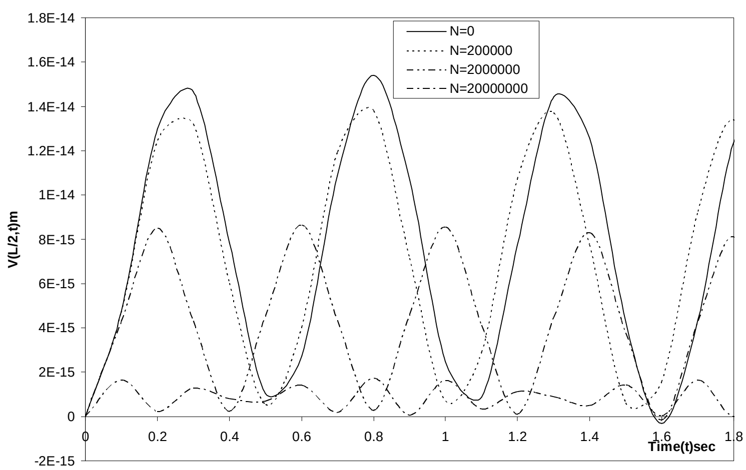

Fig. 8.3: Transverse displacement of the clamped-clamped non-uniform beam under the action of concentrated masses moving at variable velocities for various values of axial force N for fixed K (40000)

manner, as the foundation moduli K increase, for fixed value of N, the displacement response of Simply-Supported, Clamped-Clamped and Cantilever non-uniform beams under heavy masses moving at varying velocities decrease as shown in figures 8.2, 8.4, and 8.6. In figure 8.7, the transverse displacement of moving force and moving mass cases for Simply-Supported non-uniform beam traversed by a load moving at varying velocities for fixed N = 200,000 and K = 40,000 is displayed. Clearly, the response amplitudes of moving mass are higher than that of the moving force.

The same result is obtained for other illustrative boundary conditions of Clamped-Clamped and Cantilever in Figures 8.8 and 8.9 for the same beam model, as in paper [15]. In general, higher values of axial force N and foundation modulus K are required for a more noticeable effect on the response amplitudes of the beam in the case of other boundary conditions than those of Simply-Supported end conditions.

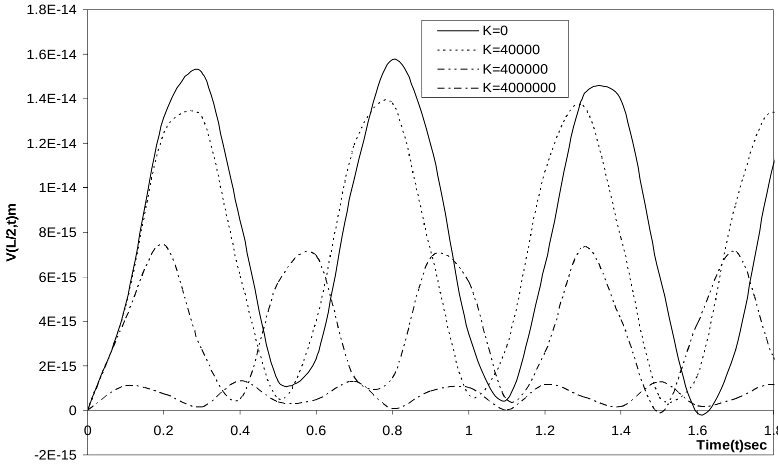

Fig. 8.4: Deflection profile of the clamped-clamped non-uniform beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli K for fixed N (200000)

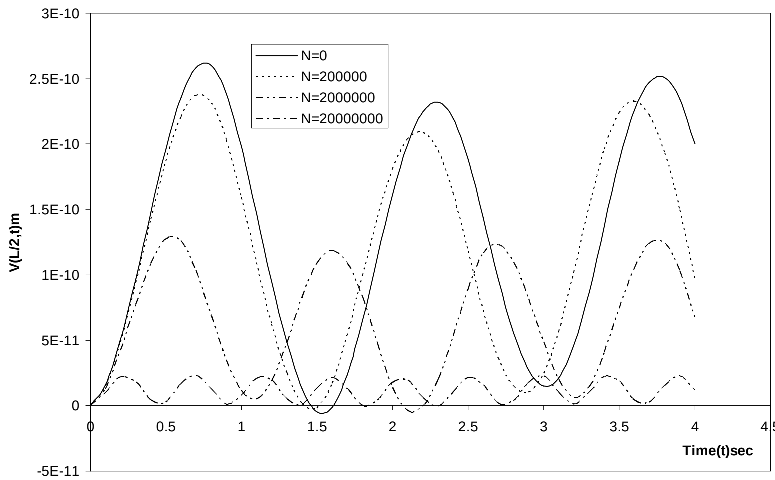

Fig. 8.5: Transverse displacement of the clamped-free non-uniform beam under the action of concentrated masses moving at variable velocities for various values of axial force N for fixed K (40000)

9.0 **Conclusions.**

In this study, analytical solution has been obtained for the dynamic behaviour of non-uniform Bernoulli-Euler beams subjected to concentrated masses travelling at varying velocities. The method proposed is very versatile and is capable of tackling this class of problem for any of the classical boundary conditions often encountered in structural design. It has enormous advantages over the numerical techniques as solutions obtained by it shed light on vital information about the vibrating system. The effects of various parameters, such as inertia, foundation moduli and axial force on the

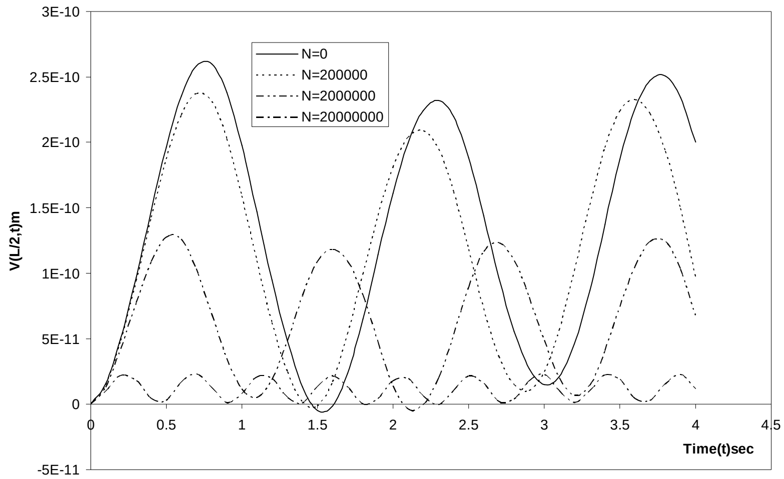

Fig. 8.5: Transverse displacement of the clamped-free non-uniform beam under the action of concentrated masses moving at variable velocities for various values of axial force N for fixed K (40000)

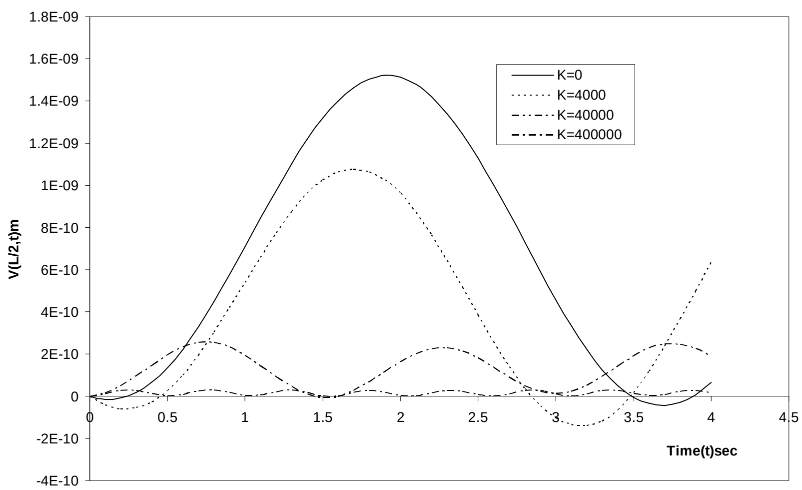

Fig. 8.6: Deflection profile of the clamped-free non uniform beam under the action of concentrated masses moving at variable velocities for various values of foundation moduli K for fixed N (20000)

dynamical system are investigated. It is found that, generally, as foundation moduli K and axial force N are increased, the response amplitudes of the vibrating system decrease. Also, in all the illustrative examples considered, for the same natural frequency, the critical speed for moving mass problem is smaller than that of the moving force problem. Hence, resonance is reached earlier in the moving mass problem. Thus, accurate evaluation of the moving mass problem is desirable as approximation by the moving force solution is highly misleading.

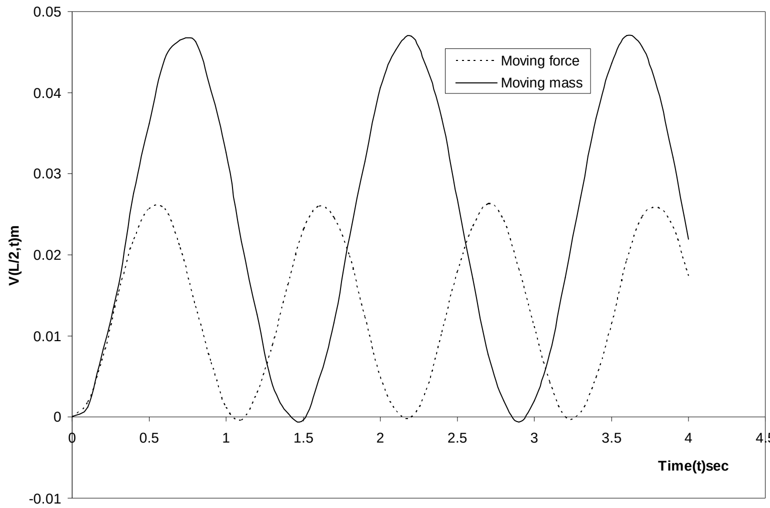

Fig 8.7: Comparison of the displacement of moving force and moving mass cases for simply supported non-uniform beam for N=200000 and K=40000

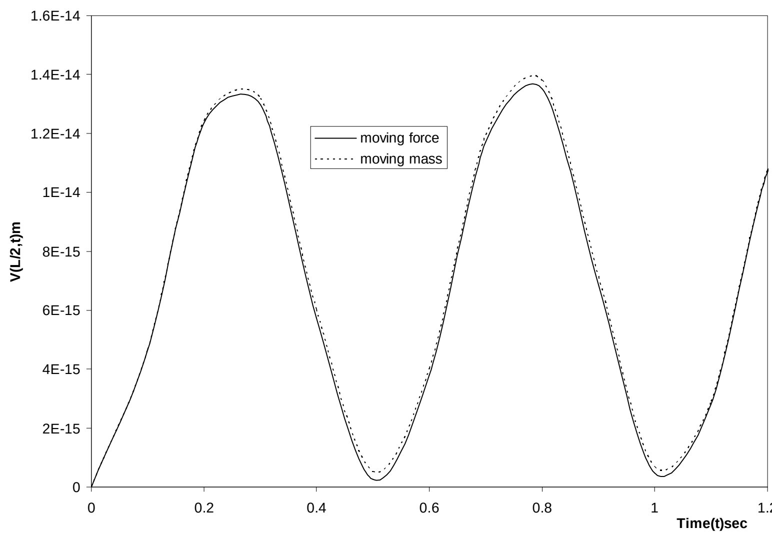

Fig. 8.8: Comparison of the displacement response of moving force and moving mass cases for clamped-clamped non-uniform beam for N=200000 and K=40000

References

[1] Sadiku, S. and Leipholz, H. H. E, 1981. On the Dynamics of Elastic Systems with Moving Concentrated Masses Ing. Archiv. 57: 223-242

[2] Oni, S. T., 2000. Flexural Vibrations under Moving Loads of Isotropic Rectangular Plates on a Non-Winkler Elastic Foundation. Journal of the Nigerian Society of Engineers 35(1): 18- 27,40-41

[3] Gbadeyan, J. A. and Oni, S. T., 1995. Dynamic Behaviour of Beams and Rectangular Plates under Moving Loads. Journal of sound and vibration 182(5): 677-695

[4] Huang, M. H. and Thambiratnam, D. P., 2000 Deflection response of plate on Winkler foundation to moving accelerated loads. Engineering Structures 23: 1134-1141

[5] Lee, H. P. and Ng T. Y., 1996, Transverse vibration of a plate moving over multiple points supports. Applied Acoustics 47(4): 291-301

[6] Adams, GG., 1995. Critical Speeds and the Response of a Tensioned Beam on an Elastic Foundation to Repetitive Moving Loads: Int. J. Mech. Science 37(7): 773-781

[7] Chen, Y. H. and Li, C. Y., 2000, Dynamic response of elevated high speed railway. American Society of Civil Engineers, Journal of Bridge Engineering 5: 124-130

[8] Savin E, 2001. Dynamic Amplification Factor and Response Spectrum for the Evaluation of Vibrations of Beams under Successive Moving Loads. Journal of sound and vibration 248 2): 267-288

[9] Rao, G. V., 2000, Linear dynamics of an elastic beam under moving loads. ASME. Journal of Vibrations and Acoustics 122

[10] Shadnam, M. R., Rofooei, F. R., Mofid M and Mehri, B., 2002. Periodicity in the Response of Non-Linear Plate under Moving Mass. Thin-walled structures 40: 283-295

[11] Frybal, L., 1972. Vibrations of Solids and Structures under moving loads. Groningen Noordhoff

[12] Oni S. T ; Response of a non-uniform beam resting on an elastic foundation to several moving masses. Abacus, Journal of Mathematical Association of Nigeria. vol. 24, No 2, 1996.

[13] Oni, S. T., and Awodola, T. O., 2003. Vibrations under a Moving Load of a Non-Uniform Rayleigh Beams on Variable Elastic Foundation. Journal of Nigerian association of Mathematical Physics. 7: 191-206

[14] Gbadeyan, J. A. and Ayesimi, Y. M., 1990, Response of an elastic beam resting on viscoelastic foundation to a load moving at non-uniform speed. Nigeria Journal of Mathematics and Applications 3: 73-90

[15] Oni S. T., and Omolofe B., 2004, Dynamical Analysis of a Prestressed Elastic Beam with General Boundary Conditions under Loads Moving at Non-uniform Speeds. Journal of Engineering and Engineering Technology, (Accepted for publication).