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Some topolocial properties of a random subset of Lebesgue measure zero

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Abstract

The set of fast points associated with the process $Y(s) = \inf_{t \geq s} |x(t)|$ ***where x(t) is a standard Brownian motion, is considered. This random subset of exceptional time set is shown to be everywhere dense of the second category and has the cardinality of the continuum.***

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1.0 Introduction

Let (Ω, f, P) be a probability space. Let $x(t)$ denote a standard Brownian motion taking values in R^n, the Euclidean space of n-dimensions, $n \geq 3$. "Almost sure" sample path properties of Brownian motion are well known, that is, properties of $X(t, \omega)$ as a function of t which determine events of probability one in the underlying probability space Ω of the process (see for example [2,9]). For each point ω of the underlying probability space $(\Omega, f, P), x(t) = X(t, \omega)$ is a sample path.

Some authors have studied various sets connected with sample paths and in particular, the measure properties of such sets (see for example, [2,7,8,9,])

Recent interests in such sets that are neither smooth nor surface like, stem from the fact that many facets of nature can only be described with the help of such sets (see for example [5])

The main purpose of this paper is to discuss some topological properties of the set of fast points defined in the next section. The results obtained in this paper are similar to those obtained by Goldberg [3, p128-130], for a set of point at which a given real-valued function on the line is discontinuous.

2.0 Notations and Preliminary results

The celebrated law of iterated logarithm tells us that for fixed t $\geq$ 0 Brownian motion in R^n, almost surely satisfies

$$\limsup_{h \downarrow 0} \frac{|x(t+h) - x(t)|}{\left(2h \log\left|\log \frac{1}{h}\right|\right)^{1/2}} = 1 \qquad (2.1)$$

However, we know there are some values of t > 0 where (2.1) fails. We denote such exceptional time set by

$$E(1) = \left\{ t \in [0,1] : \limsup_{h \downarrow 0} \frac{|x(t+h) - x(t)|}{\left(2h \log\left|\log \frac{1}{h}\right|\right)^{1/2}} \neq 1 \right\} \qquad (2.2)$$

When the *lim sup* is greater than 1, $E(1)$ is called the set of fast points while $E(1)$ is called the set of slow points ***if lim sup*** is less than 1. The measure properties of fast points associated with Brownian motion are well known (see [8]). Instead we investigate some topological properties of fast points connected with the large values of the process *Y(s)* defined by

$$Y(s) = \inf_{t \geq s} |x(t)| \qquad (2.3)$$

It is well known that almost all paths of *x(t)* are everywhere continuous, hence *Y(s)* is both continuous and monotone.

The asymptotic size of the large values of *Y(s)* as $s \to \infty$ was evaluated in the paper with Ugbebor [6]. For a standard transient Brownian motion, define

$$Y_t(s) = \inf_{h \geq s} |x(t+h) - x(t)| \qquad (2.4)$$

For a fixed $t = t_0$, $Y_t(s)$ has the same distribution as *Y(s)*, so that the local growth behaviour at a prescribed time $t \geq t_0$ is given by

$$P\left(\limsup_{h \downarrow 0} \frac{Y_t(s)}{\varphi(h)} = 1 \right) = 1,$$

where

$$\varphi(h) = \left(2h \log\log \frac{1}{h} \right)^{\frac{1}{2}}$$

see [6], and the set $N \subset \Omega$ for which

$$\limsup_{h \downarrow 0} \frac{Y_t(s)}{\varphi(h)} \neq 1$$

is of probability zero; that is

$$P(\omega \in N) = 0 \text{ and } P(\omega \in \Omega - N) = 1.$$

But this exceptional null *N* set depends on *t*. Thus by Fubini's theorem [4, p 143] we have

***Lemma* 2.1**

The exceptional time set

$$\left\{ t \in [0,1] : l\,im\sup_{h \downarrow 0} \frac{Y_t(s)}{\varphi(h)} \neq 1 \right\}$$

has zero Lebesgue measure with probability one.

3.0 The Main result

***Definition* 3.1**

A set E is said to be an F_σ set if it is possible to find sets E_1, E_2, E_3,…,*inF* such that E = $\bigcup_{i=1}^{\infty} E_i$.

If β is a class of open sets, then $\beta_\sigma = \beta$. Define $Z_{ij} = \left\{ t : \varepsilon > 0 \text{ and } h \in \left(0, \frac{1}{j}\right) \right.$ such that $\forall u \in (t - \varepsilon, t + \varepsilon)$,

$$\sup_u \frac{Y_u(S)}{\varphi(h)} \succ 1 + \frac{1}{i} \qquad (3.1)$$

Since $Y_u(s)$ is a continuous and monotonic function, and $\varphi(h)$ is also continuous, it follows that Z_{ij} is open relative to $(t - \varepsilon, t + \varepsilon)$. By the separability of *Y(s)*, if *j* is fixed and *I* is an open interval with rational end points, then $Z_{ij} \cap I \neq \phi$ and Z_{ij} is therefore dense. By right continuity, we see that for some $\delta > 0$

$$\sup_{v} \frac{Y_u(s)}{\varphi(h)} > 1 + \frac{1}{i} \text{ for } v \in (t, t+\delta)$$

and clearly $(t, t+\delta) \subset z_{i\,j}$. Since $\sup\limits_{u} \dfrac{Y_u(s)}{\varphi(h)}$ is, for each u a non-decreasing function of h we have that

$$E(\alpha) = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} Z_{i\,j} = \left\{ t \in [0,1] : \lim_{h \downarrow 0} \sup \frac{Y_u(s)}{\varphi(h)} = \alpha \right\}, \quad \alpha \succ 1.$$

Consequently we have

***Theorem* 3.1**

The exceptional time set

$$\left\{ t \in [0,1] : \lim_{h \downarrow 0} \sup \frac{Y_t(s)}{\varphi(h)} = \alpha \right\}, \qquad \alpha \succ 1$$

is of type F_σ. Moreover

$$\lim_{h \downarrow 0} \sup \frac{Y_t(s)}{\varphi(h)} = \alpha, \quad \alpha \succ 1$$

whenever

$$t \in \mu = \bigcap_{j} Z_{i\,j}$$

so that by Baires category theorem [1] we have

***Theorem* 3.2**

$$E(\alpha) = \left\{ t \in [0,1] : l\,im \sup_{h \downarrow 0} \frac{Y_t(s)}{\varphi(h)} = \alpha \right\}, \quad \alpha \succ 1$$

is everywhere dense of second category with probability one.

Let

$$E^*(1) = \left\{ t \in [0,1] : l\,im \sup_{h \downarrow 0} \frac{Y_t(s)}{\varphi(h)} = 1 \right\}$$

so that $E(\alpha) \subset E^*(1)$. By the continuum hypothesis, we can see that $E(\alpha)$ has the power of the continuum almost surely.

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