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Coefficient inequalities and some convolution properties of a subclass of analytic functions

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Abstract

In this work the author introduced a subclass $T_n^{\alpha}(p,\beta)$, a subset of class $T_n^{\alpha}(\beta)$ introduced and investigated by Opoola [1]. ***The author derives some coefficient inequalities and convolution properties for the class. $T_n^{\alpha}(p,\beta)$ using Salagean differential operator.***

pp 49 - 56

1.0 Introduction

Let A be the class of functions regular in the unit disk $E=\{z:|z|<1\}$ and of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

Furthermore, Opoola [1] denote by $T_n^{\alpha}(\beta)$ a subclass of A consisting of functions satisfying the following conditions.

$$\text{Re}\left\{\frac{D^n f(z)^{\alpha}}{z^{\alpha}}\right\} > \beta, \quad z\in E, \quad n=0,1,2\ldots \tag{1.2}$$

$\alpha>0,\ 0\le\beta<1$ and the operator D is the same as in $B_n(\alpha)$ namely the Salagean differential operator defined as

$$D^0 f(z)=f(z),\ \ D^1 f(z)=zf'(z),\ \ D^n f(z)=z\left(D^{n-1}f(z)\right)' \tag{1.3}$$

Opoola in his remarks gave some other existing subclasses such as S_o, $B(\beta)$, $\delta(\beta)$, and $B_n(\alpha)$, by varying the parameters α, β, and *n* in (1.2) see [1] for details. Also in [1] Opoola proved the following results.

***Theorem* 1.1**

$T_n^{\alpha}(\beta)$ *C S for n* ≥ 1 *where S is the subclass of* D *consisting of univalent functions in E.*

T*heorem* 1.2

$$T_{n+1}^{\alpha}(\beta) \subset T_n^{\alpha}(\beta) \quad for \quad n\ge 1$$

***Theorem* 1.3**

$$f\in T_n^{\alpha}(\beta), \text{ then Re}\left\{\frac{f(z)^{\alpha}}{z^{\alpha}}\right\} > \beta, \quad z\in E, \quad 0\le\beta<1,\ \alpha>0\ldots$$

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=p+i}^{\infty} a_k z^k \qquad (1.4)$$

$$\left(a_k \geq 0, \quad p, i \in N = \{1, 2, .. \quad \}\right)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$ Motivated by the earlier works of Opoola [1], Owa [2], Ahuja [3], Owa and srivastava [4] the author introduced a subclass $T_n^{\alpha}(p, \beta) \subset T_n^{\alpha}(\beta)$ whose functions are of the form (1.4) and satisfy the same condition (1.2).

The following results are true for the Functions in the class $T_n^{\alpha}(p, \beta)$

***Theorem* A**

$T_n^{\alpha}(p, \beta) \subset S_p, n \geq 1$ *where* S_p *is the subclass of* $A(p)$ *consisting of univalent functions in E.*

***Theorem* B**

$T_{n+1}^{\alpha}(p, \beta) \subset T_n^{\alpha}(p, \beta)$ for $n \geq 1$, $p \in N$.

In this work the author derives the coefficient inequalities and convolution properties for the subclass $T_n^{\alpha}(p, \beta)$ using Salagean differential Operator defined by (1.3) and the method of mathematical induction.

2.0 Coefficient Inequalities

***Theorem* 2.1**

If $f(z) \in T_n^{\alpha}(p, \beta)$ *and p-valently starlike of order* β *then.*

$$\sum_{k=p+i}^{\infty} (\beta - k) a_k \leq p - \beta, \quad \left(a_k \geq 0, 0 \leq \beta < 1, \ p, i \in N = \{1, 2, 3, \ldots\ldots \}\right) \qquad (2.1)$$

Proof

By definition (1.3) it is sufficient to show that

$$\text{Re} \left\{ \frac{D^1 f(z)}{D^0 f(z)} \right\} > \beta, \quad 0 \leq \beta < 1, \quad z \in E \qquad (2.2)$$

holds true.

From (1.4) and (2.2) we have that

$$\frac{D^1 f(z)}{D^0 f(z)} = \frac{p + \sum_{k=p+i}^{\infty} k a_k z^{k-p}}{1 + \sum_{k=p+i}^{\infty} a_k z^{k-p}} \qquad (2.3)$$

and from (2.2) and (2.3) we have

$$\operatorname{Re}\frac{D^1 f(z)}{D^0 f(z)} = \left\{\frac{p+\sum_{k=p+i}^{\infty} k\,a_k\,z^{k-p}}{1+\sum_{k=p+i}^{\infty} a_k\,z^{k-p}}\right\} > \beta \tag{2.4}$$

for all $z \in E$. Choose the values of z on the real line $z = re^{i0}$ $(0 \le r < 1)$, then (2.4) implies

$$\operatorname{Re}\left\{\frac{p+\sum_{k=p+i}^{\infty} k\,a_k\,z^{k-p}}{1+\sum_{k=p+i}^{\infty} a_k\,z^{k-p}}\right\} = \left\{\frac{p+\sum_{k=p+i}^{\infty} k\,a_k\,r^{k-p}}{1+\sum_{k=p+i}^{\infty} a_k\,r^{k-p}}\right\} > \beta \tag{2.5}$$

Since $p+\sum_{k=p+i}^{\infty} k\,a_k\,r^{k-p} > 0$, we have

$$p+\sum_{k=p+i}^{\infty} k\,a_k\,r^{k-p} > \beta\left(p+\sum_{k=p+i}^{\infty} k\,a_k\,r^{k-p}\right) \tag{2.6}$$

By letting $r \to 1$ through half line $z = re^{i0}$ $(0 \le r < 1)$ in (2.6), we have $\sum_{k=p+i}^{\infty} k\,a_k - \beta \sum_{k=p+i}^{\infty} a_k \ge \beta - p$, which finally yields $\sum_{k=p+i}^{\infty} (\beta - k)a_k \le p - \beta$ which concludes the proof of Theorem 2.1

***Theorem* 2.2**

If $f(z) \in T_n^{\alpha}(p,\beta)$ *and p-valently convex of order* β *then*

$$\sum_{k=p+i}^{\infty} (\beta - k) \le p(p-\beta) \tag{2.7}$$

$$\left(a_k \ge 0,\ \ 0 \le \beta < 1,\ \ p, i \in N = \{1, 2, \ldots..\}\right)$$

Proof

It is sufficient to show that $\operatorname{Re}\left\{\frac{D^2\ f(z)}{D^1 f(z)}\right\} > \beta, \ \ z \in E \quad 0 \le \beta < 1$ (2.8)

From (1.4) and (2.7) we have

$$\operatorname{Re}\left(\frac{D^2 f(z)}{D^{1f}(z)}\right) = \operatorname{Re}\left\{\frac{p^2+\sum_{k=p+i}^{\infty} k^2\,a_k\,z^{k-p}}{1+\sum_{k=p+i}^{\infty} k\,a_k\,z^{k-p}}\right\} \tag{2.9}$$

By (2.8) and (2.9) we have

$$\operatorname{Re}\left(\frac{D' f(z)}{D^0 f(z)}\right) = \operatorname{Re}\left\{\frac{p^2 + \sum_{k=p+i}^{\infty} k^2 a_k z^{k-p}}{p + \sum_{k=p+i}^{\infty} k a_k z^{k-p}}\right\} > \beta \qquad (2.10)$$

for $z \in E$, choose the value of z on half line $z = re^{i0}$ $(0 \le r < 1)$, then

$$\operatorname{Re}\left\{\frac{p^2 + \sum_{k=p+i}^{\infty} k^2 a_k z^{k-p}}{p + \sum_{k=p+i}^{\infty} k a_k z^{k-p}}\right\} = \frac{p^2 + \sum_{k=p+i}^{\infty} k^2 a_k r^{k-p}}{p + \sum_{k=p+i}^{\infty} k a_k r^{k-p}} > \beta \qquad (2.11)$$

Since $p^2 + \sum_{k=p+i}^{\infty} k^2 a_k r^{k-p} > 0$ we have

$$p^2 + \sum_{k=p+i}^{\infty} k^2 a_k r^{k-p} > \beta\left(p + \sum_{k=p+i}^{\infty} k a_k r^{k-p}\right) \qquad (2.12)$$

By letting $r \to 1$ through half line $z = re^{i0}$ $(0 \le r < 1)$ in (2.12) we have

$$p^2 + \sum_{k=p+i}^{\infty} k^2 a_k \ge p\beta + \beta \sum_{k=p+i}^{\infty} k a_k$$

which finally yields

$$\sum_{k=p+i}^{\infty} k(\beta - k) a_k \le p(p - \beta)$$

$(a_k \ge 0, \quad p, i \in N = \{1, 2, \ldots..\}), \ 0 \le \beta < 1$ which concludes the proof of Theorem 2.2.

3.0 Convolution Properties

***Theorem* 3.1**

If $f_j(z) \in T_n^{\alpha}(p, \beta_j)$, $(j = 1, \cdots, m)$, *then* $(f_1 * \ldots * f_m) \in T_n^{\alpha}(p, \beta_j)$. *where*

$$\rho = p + \frac{i \prod_{k=1}^{m} (p - \beta_j)}{\prod_{k=1}^{m} (p - \beta_j) + \prod_{k=1}^{m} (\beta_j - k)} \qquad (3.1)$$

The result is sharp for functions

$$f_j(z) = z^p + \left(\frac{p - \beta_j}{\beta_j - p - i}\right) z^{\rho + i} \qquad (3.2)$$

Proof

Our method of proof shall follow the work of Owa [2], Owa and Srivastava [4], and we shall use the principle of Mathematical induction in our proof of Theorem 3.1

For $m=1$, we see that $p=\beta_1$. For $m=1$ Theorem 2.1 gives

$$\sum_{k=p+i}^{\infty}\sqrt{\frac{\beta_j-k}{p-\beta_j}}\,a_{k,j}\quad\le 1\quad (j=1,2) \tag{3.3}$$

Thus, by applying the Cauchy–Schwarz inequality we have

$$\left|\sum_{k=p+i}^{m}\sqrt{\frac{(\beta_1-k)(\beta_2-k)}{(p-\beta_1)(p-\beta_2)}(a_{k,1})(a_{k,2})}\right|^2 \le \left(\sum_{k=p+i}^{m}\left(\frac{\beta_1-k}{p-\beta_1}\right)(a_{k,1})\right)\left(\sum_{k=p+i}^{m}\left(\frac{\beta_2-k}{p-\beta_2}\right)(a_{k,2})\right) \tag{3.4}$$

Therefore, if

$$\sum_{k=p+i}^{\infty}\left(\frac{\delta-k}{p-\delta}\right)(a_{k,1})(a_{k,2})\le\sum_{k=p+i}^{\infty}\sqrt{\frac{(\beta_1-k)(\beta_2-k)}{(p-\beta_1)(p-\beta_2)}(a_{k,1})(a_{k,2})}$$

That is, if

$$\sqrt{(a_{k,1})(a_{k,2})}\le\frac{p-\delta}{\delta-k}\sqrt{\frac{(\beta_1-k)(\beta_2-k)}{(p-\beta_1)(p-\beta_2)}} \tag{3.5}$$

then $(f_1*f_2)(z)\in T_n^{\alpha}(p,\delta)$. We also note that the inequality (3.3) yields

$$\sqrt{a_{k,j}}\ \le\ \sqrt{\frac{p-\beta_j}{\beta_j-k}}\quad (j=1,2,\quad k=p+i,\quad p,i\in N) \tag{3.6}$$

Consequently, if

$$\frac{(p-\beta_1)(p-\beta_2)}{(\beta_1-k)(\beta_2-k)}\ \le\ \frac{p-\delta}{\delta-k}\sqrt{\frac{(\beta_1-k)(\beta_2-k)}{(p-\beta_1)(p-\beta_2)}}$$

that is, if

$$\frac{p-\delta}{\delta-k}\ \le\ \frac{(\beta_1-k)(\beta_2-k)}{(p-\beta_1)(p-\beta_2)} \tag{3.7}$$

then we have $(f_1*f_2)(z)\in T_n^{\alpha}(p,\delta)$. It follows from

$$\delta\le p+\frac{(k-p)(p-\beta_1)(p-\beta_2)}{(p-\beta_1)(p-\beta_2)+(\beta_1-k)(\beta_2-k)}\ =h(k)$$

since $h(k)$ is increasing, for $k\ge p+i$ we have

$$\delta\le p+\frac{(k-p)(p-\beta_1)(p-\beta_2)}{(p-\beta_1)(p-\beta_2)+(\beta_1-k)(\beta_2-k)}$$

which shows that $(f_1*f_2)(z)\in T_n^{\alpha}(p,\delta)$ where

$$\delta = p + \frac{i(p-\beta_1)(p-\beta_2)}{(p-\beta_1)(p-\beta_2)+(\beta_1-k)(\beta_2-k)} \tag{3.8}$$

Therefore, the result is true for $m=2$. Suppose that the result is true for any positive integer m. That is $(f_1 * f_2)(z) \in T_n^{\alpha}(p,\gamma)$ where

$$\gamma = p + \frac{(k-p)\prod_{k=1}^{m}(p-\beta_j)}{\prod_{k=1}^{m}(p-\beta_j)+\prod_{k=1}^{m}(\beta_j-k)} \quad (j=1,\dots m) \tag{3.9}$$

Then by means of the above technique, we can show that $(f_1 * f_2)(z) \in T_n^{\alpha}(p,\rho)$ where

$$\rho = p + \frac{i\prod_{k=1}^{m+1}(p-\beta_j)}{\prod_{k=1}^{m+1}(p-\beta_j)+\prod_{k=1}^{m+1}(\beta_j-k)(\beta_j-p-i)} \tag{3.10}$$

This shows that the result is true for $m+1$ Therefore, by mathematical induction, the result is true for any positive integer m.

Further, taking the functions $f_j(z)$ defined by (3.2) we have

$$(f_1 * f_2 * \dots * f_m)(z) = z^p + \left\{ \pi_{j=1}^{m} \left(\frac{p-\beta_j}{B_j-p-i} \right) \right\} z^{p+i} = z^p + A_{p+i} z^{p+i} \tag{3.11}$$

which shows that

$$\sum_{k=p+i}^{\infty} \frac{\rho-k}{p-\rho} A_k = \left(\frac{\rho-p-i}{p-\rho} \right) \left\{ \pi_{j=1}^{m} \left(\frac{p-\beta_j}{B_j-p-l} \right) \right\} = 1 \tag{3.12}$$

Consequently, the result is sharp for functions $f_j(z)$ given by (3.2)

Letting $\beta_j = \beta\ (j=1,2,\dots,m)$ in Theorem 3.1, we have

Corollary **A**

If, $f_j(z) \in T_n^{\alpha}(p,\beta_j),\ (j=1,\cdots,m)$ *then* $(f_1 * f_2 * \dots * f_m)(z) \in T_n^{\alpha}(p,\rho),$ *where*

$$\rho = p + \frac{(p-\beta)^m}{(p-\beta)^m + (\beta-p-i)^m} \tag{3.13}$$

The result is sharp for functions

$$f_j(z) = z^p + \left(\frac{(p-\beta)}{(\beta-p-i)^m} \right) z^{p+i} \quad (j=1,2,.,m) \tag{3.14}$$

Setting $p=1,\ i=1\cdots$, and in Theorem 3.1 we have

***Corollary* B**

If $f_j(z) \in T_n^{\alpha}(p,\beta_j),\ (j=1,2,\cdots,m)$, *then* $(f_1 * f_2 * \dots * f_m)(z) \in T_n^{\alpha}(1,\rho)$

where

$$\rho = 1 + \frac{\prod_{j=1}^{m}\left(1-\beta_j\right)}{\prod_{j=1}^{m}\left(1-\beta_j\right) + \prod_{j=1}^{m}\left(\beta_j - 2\right)} \tag{3.15}$$

The result is sharp for functions

$$f_j(z) = z + \left(\frac{1-\beta_j}{\beta_j - 2}\right) z^2 \quad (j = 1, 2, ., m) \tag{3.16}$$

***Theorem* 3.2**

If $f_j(z) \in T_n^{\alpha}(p, \beta_j)$ $(j = 1, 2, . .m)$ *then* $(f_1 * f_2 * \ldots * f_m)(z) \in T_1^{\alpha}(p, \rho)$ *where*

$$\rho = p + \frac{ip^{m-1}\prod_{j=1}^{m}\left(p-\beta_j\right)}{(p-i)^{m-1}\prod_{j=1}^{m}\left(\beta_j - p - i\right) + p^{m-1}\prod_{j=1}^{m}\left(p-\beta_j\right)} \tag{3.17}$$

The result is sharp for functions

$$f_j(z) = z^p + \frac{p-\beta_j}{(p+i)\left(\beta_j - p - i\right)} z^{p+i} \quad (j = 1, 2, \ldots m) \tag{3.18}$$

Proof

It is clear that the result is true for m = 1. For m = 2, theorem 2.2gives

$$\sum_{k=p+i}^{\infty}\left\{\frac{k\left(\beta_j - k\right)}{p\left(p-\beta_j\right)}\right\} a_{k,j} \le 1, \qquad (j = 1,2) \tag{3.19a}$$

which implies

$$\sum_{k=p+i}^{\infty}\frac{\sqrt{k\left(\beta_1 - k\right)\left(\beta_2 - k\right)}}{p\sqrt{\left(p-\beta_1\right)\left(p-\beta_2\right)}}\sqrt{a_{k,1}\, a_{k,2}} \le 1 \tag{3.19b}$$

Since we have to get the largest ρ such that

$$\frac{\rho - k}{p-\rho} \le \frac{k\left(\beta_1 - k\right)\left(\beta_2 - k\right)}{\left(p-\beta_1\right)\left(p-\beta_2\right)} \tag{3.20}$$

From the above, we need to find the largest ρ such that

$$\rho \ge p + \frac{p(k-p)\left(p-\beta_1\right)\left(p-\beta_2\right)}{k\left(\beta_1 - k\right)\left(\beta_2 - k\right) + p\left(p-\beta_1\right)\left(p-\beta_2\right)} \quad (k \ge p+i) \tag{3.21}$$

Further, noting that the function

$$h(k) = p + \frac{p(k-p)\left(p-\beta_1\right)\left(p-\beta_2\right)}{k\left(\beta_1 - k\right)\left(\beta_2 - k\right) + p\left(p-\beta_1\right)\left(p-\beta_2\right)}$$

is increasing for k, we have

$$\rho \ge h(p+i) = p + \frac{ip\left(p-\beta_1\right)\left(p-\beta_2\right)}{(p+i)\left(\beta_1 - p - i\right)\left(\beta_2 - p - i\right) + p\left(p-\beta_1\right)\left(p-\beta_2\right)} \tag{3.22}$$

Thus the result is true for $m = 2$.

Next, by using mathematical induction, we conclude that $(f_1 * f_2 * \ldots * f_m)(z) \in T_n^{\alpha}(p, \rho)$.
Also, it is easy to show that the result is sharp for functions $f_j(z)$ given by 3.18

***Corollary* C**

If $f_j(z) \in T_n^{\alpha}(p, \beta_j)$ $(j = 1, 2, . . m)$ *then* $(f_1 * f_2 * \ldots * f_m)(z) \in T_1^{\alpha}(1, \rho)$ *where*

$$\rho = 1 + \frac{\prod_{j=1}^{m}(1-\beta_j)}{2^{m-1}\prod_{j=1}^{m}(1-\beta_j) + \prod_{j=1}^{m}(1-\beta_j)} \tag{3.22}$$

The result is sharp for functions

$$f_j(z) = z + \frac{1-\beta_j}{2(\beta_j - 2)} z^2 \quad (j = 1, 2, . ., m). \tag{3.23}$$

4.0 Conclusion

The author has been able to establish the coefficient inequalities for the functions in the class $T_n^{\alpha}(p, \beta)$ a subset of class $T_n^{\alpha}(\beta)$ and its convolution behaviour.

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