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Non-well posed evolution equations and Fredholm operators

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Abstract

We shall consider the operator

$$(F\xi)(t) = \dot{\xi}(t) + A(t)\xi(t) \qquad (*)$$

where $A(t) \in L(W, H)$ ***is continuously differentiable in the uniform operator topology with*** $W \to H$***, a continuous dense injection. Both W and H are Hilbert spaces each with its own norm. At the same time we assume that W is a dense linear subspace of H, with A(t) selfadjoint when regarded as an unbounded operator on H and domain D(A(t)) = W. We consider F as an operator***

$$F : W^{1,2}(R;H) \cap L^2(R;W) \to L^2(R;W) \qquad (**)$$

and show that this is Fredholm provided $W \subset H$ ***is a compact embedding and the limit operator*** $A^{\pm} = \lim_{t\to\pm\infty} A(t)$ ***is bijective.***

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1.0 Introduction

The operator F as defined in equation (*) has been extensively studied in both the finite dimensional case [13] and in the infinite dimensional situation [1, 5, 6, 7, 11], some of which are proved in [2, 3] as special cases of equation (**). Since $W \subset H$ is compact, *A(t)* has compact resolvent and thus has a discrete spectrum consisting of real eigenvalues and the Fredholm index of the operator F can be characterized in terms of spectral flow [1].This will however not be proved in this paper.

We have made no assumptions on the eigenvalues of *A(t)* but we see from [4], that its spectrum can be unbounded below and above which would therefore show that the differential equation

$$\dot{X}(t) = A(t)X(t)\ ,\ X(0) = X_0 \qquad (1.1)$$

which corresponds to the kernel of the operator F, is not well-posed, that is will not in general have a unique solution $x \in C[0,T;W] \cap C^1[0,T;W]$ for a given $x_0 \in W$.

2.0 Some Preliminary Results.

Fredholm operators form a very interesting class and arise very frequently in applications. The reader is referred to [8, 9, 10, 12,14] for more details of the theory of Fredholm and linear operators.

To show that A is bijective, we require the following lemma which will also be necessary to prove the main theorem.

3.0 Lemma

Let X, Y, Z be Banach spaces. Suppose $F \in L(X,Y)$ is a bounded linear operator and $K \in L(X, Z)$ a compact bounded linear operator. If

$$||x||_X \le c[||Fx + ||_Y \; ||xK||_Z] \;, \forall \; x \in X \;, \qquad (3.1.)$$

c a constant, then F has closed range and finite dimensional kernel.

Proof

It suffices to show that the unit ball in ker$F$ is compact to show that $dimKerF$ is finite. Let $B = \{x \in X{:}Fx = 0, ||x|| \le 1\}$. Consider $x_n \in B$, and then there exists a subsequence such that Kx_{nk} converges since K is compact. Therefore

$$||x_{nk} - x_{nl}||_X \le c||Kx_{nk} - Kx_{nl}||_Z \to 0 \text{ as } k, l \to \infty \qquad 3.2)$$

Thus x_{nk} is Cauchy and because X is complete, $x_{nk} \to x \in X$. Therefore B is compact.

Let $y_n = Fx_n \in$ *Range F* such that $y_n \to y$, it remains to prove that $y \in$ Range F to show that Range F is closed. Suppose there exists a sequence $\xi_n \in$ Ker F such that $x_n + \xi_n$ *is* bounded. Hence there exists a subsequence $\tilde{y}_{nk} = x_{nk} + \xi_{nk}$ such that $K\tilde{y}_{nk} \to z$. Therefore $F\tilde{y}_{nk} \to y$ by our assumption. Hence $\tilde{y}_{nk}$ is Cauchy and $\tilde{y}_{nk} \to x \in X$ and $y = \lim_{n=\infty} F\tilde{y}_{nk} = Fx$ showing that $y \in$ Range F.

That there exists a sequence $\xi_n \in$ Ker F such that $x_n + \xi_n$ is bounded will now be proved. Suppose not, then $\inf ||x_n + \xi_n|| = c_n$ has unbounded sequence. Without loss of generality $c_n \to \infty$ .We choose ξ_n such that $c_n \le || x_n + \xi_n || \le 2c_n$ then $K(x_n + \xi_n)/c_n$ has a converging subsequence and $F(x_n + \xi_n)/c_n = F(x_n)/c_n \to 0$ as $n \to \infty$. Therefore $(x_n + \xi_n)/c_n$ is Cauchy and converges to some $x \in X$ and $Fx = \lim_{n\to\infty} \frac{F(x_n + \xi_n)}{c_n} = 0$. Hence for $\xi \in$ *Ker F* we see that

$$\|x + \xi\| = \lim_{n\to\infty} \left\| \frac{x_n + \xi_n}{c_n} + \xi \right\| \ge 1,$$

contradicting the fact that $\xi \in$ Ker F.

In addition to the above lemma we require the following.

3.1 Lemma

Let $W \subset H$ be a compact dense injection. Suppose H is separable then

$\hat{W} = W^{1,2}([0,T]; H) \cap L^2([0,T];W) \subset \hat{H} = L^2([0,T]; H)$ is compact.

Proof

Since H is separable, let $e_1, e_2, e_3,\ldots$ be orthonormal basis for H. Let $P_n \in L(H)$ be the orthogonal projection $p_n x = \sum_{j=1}^{n} \langle e_j, x \rangle e_j$ and define $Q_n = P_n \;|_w{:}W \to H$.

Applying the general fact that if $T_n{:}X \to Y$ converges strongly to $T{:}X \to Y$ and $K{:}W \to X$ is compact $\lim_{n\to\infty} \|T_n oK - ToK\|_{L(W,Y)} = 0$,we obtain

$$\|Q_n - i\|_{L(W,Y)} = \sup_{x\in W, \|x\|=1} \|Q_n x - x\|_H \to 0 \text{ as } n \to \infty.$$

Let j be the embedding; $j{:}\hat{W} \to \hat{H}$ and $E_n =$ span $(e_1, e_2, e_3, \ldots e_n)$. Consider $P_n \text{ o } j := \hat{J} \text{ o } P_n{:}\hat{W} \to W^{1,2}[0,T;E_n] \to L^2[0,T;E_n] \subset L^2[0,T;H]$. Since $\hat{J}$ is compact

$$\|P_n x - x\|_H^2 = \int_0^T \|P_n x(t) - x(t)\|_H^2 \, dt \qquad (3.3)$$

$$\le \|P_n oi - i\|_{L(W,H)}^2 \|x\|_{L(0,T;W)}^2 \le \|P_n oi - i\|_{L(W,H)}^2 \|x\|_W^2 \to 0$$

as $n \to \infty$. Therefore $\lim_{n\to\infty} \|P_n o j - j\|_{L(W,H)} = 0$ and by the general fact above, j is compact. Hence P_n o j is compact i.e. $\hat{W} \subset H$ is compact.

4.0 Main Theorem

4.1 Basic Assumptions and Notation

$A(1)$ $\quad c_0 = sup(||A(t)||_{L(W,H)} + ||\dot{A}(t)||_{L(W,H)}) < \infty$

$A(2)$ $\quad ||\xi||^2_W \le c_1(||A(t)\xi||^2_H + ||\xi||^2_H)$

$\wp = L^2(\Re;H)$, $\varpi = L^2(\Re;W) \cap W^{1,2}(\Re;H)$

where c_0 and c_1 are constants.

4.2 Theorem

Assuming A(1), A(2), let $A^{\pm} = \lim_{n\to\infty} A(t)$ *in the uniform operator topology where* $A(t) \in L(W,H)$ *and if* $A^{\pm}$ *are bijective then* $F : \varpi \to \wp$ *defined by* $(F\xi)(t) = \dot{\xi}(t) + A(t)\xi(t)$ *is a Fredholm operator.*

Proof

It is sufficient to prove that for some constant $c > 0$ and large enough $T > 0$,

$$\|\xi\|_{\varpi} \le c[\|F\xi\|_{\wp} + \|\xi\|_{L^2([-T,T],H)}] \tag{4.1}$$

Then by Lemma 3.0 and 3.1, range F is closed and dimker $F < 0$ if we observe that here $\varpi = X, \wp = Y$ and Z = $L^2([-T,T],H)$ where K:$\varpi \to Z$ is compact. A similar inequality holds for $F'\eta = \dot{\eta} - A\eta$ [3], which is the adjoint of F. Thus co-kernel of F is also finite dimensional, therefore F is Fredholm for $\xi(t)$ with compact support. Now consider

$$\|F\xi\|^2_{\wp} = \int_{-\infty}^{\infty} \|\dot{\xi}(t) + A(t)\xi(t)\|^2_H dt$$

$$= \int_{-\infty}^{\infty} (\|\dot{\xi}(t)\|^2_H + 2\langle \dot{\xi}(t), A(t)\xi(t)\rangle_H + \|A(t)\xi(t)\|^2_H)dt \tag{4.2}$$

$$= \int_{-\infty}^{\infty} (\|\dot{\xi}(t)\|^2_H + \|A(t)\xi(t)\|^2_H - 2\langle \xi(t), \dot{A}(t)\xi(t)\rangle_H)dt$$

By assumption $A(1)$ we obtain

$$\|(F\xi)(t)\|^2_{\wp} \ge \int_{-\infty}^{\infty} (\|\dot{\xi}(t)\|^2_H + \|A(t)\xi(t)\|^2_H - 2c_0\|\xi(t)\|_W \|\xi(t)\|_H)dt$$

$$\ge \int_{-\infty}^{\infty} (\|\dot{\xi}(t)\|^2_H + \|A(t)\xi(t)\|^2_H - \frac{c_0\varepsilon}{2}\|\xi(t)\|^2_W - \frac{2c_0}{\varepsilon}\|\xi(t)\|^2_H)dt \tag{4.3}$$

$$\ge \int_{-\infty}^{\infty} (\|\dot{\xi}(t)\|^2_H + (\frac{1}{c_1} - \frac{c_0\varepsilon}{2})\|\xi(t)\|^2_W - (1 + \frac{2c_0}{\varepsilon})\|\xi(t)\|^2_H)dt$$

by $A(2)$ $\quad \ge k_1\|\xi(t)\|^2_{\varpi} - k_2\|\xi(t)\|^2_{\wp}$

where $k_1 = \frac{1}{c_1} - \frac{c_0\varepsilon}{2}$ *with* ε *chosen so that* $c_0 < \frac{2}{c_1\varepsilon}$ *and* $k_2 = 1 + \frac{2c_0}{\varepsilon}$.

Therefore

$$k_1\|\xi(t)\|_{\varpi}^2 \le \|F(\xi(t))\|_{\wp}^2 + k_2\|\xi(t)\|_{\wp}^2$$

$$\le k_2\left(\|F(\xi(t))\|_{\wp}^2 + \|\xi(t)\|_{\wp}^2\right) \le k_2\left(\|F(\xi(t))\|_{\wp} + \|\xi(t)\|_{\wp}\right)^2 \tag{4.4}$$

and $\|\xi(t)\|_{\varpi} \le c\left(\|F(\xi(t))\|_{\wp} + \|\xi(t)\|_{\wp}\right)^{\frac{1}{2}}$

where $c = \left(\frac{k_2}{k_1}\right)$.

Now consider the special case where $A(t) = A$, independent of t to show that if $A : W \to H$ is non-singular then F is bijective and since W is a Hilbert space.

$$\begin{aligned} |w|\|\xi\|_W^2 &= |\langle \xi, iw\xi\rangle| \\ &\le |\langle \xi, iw\xi\rangle + \langle \xi, A\xi\rangle| = |\langle \xi, iw\xi + A\xi\rangle| \\ &\le |\xi||iw\xi + A\xi| \end{aligned} \tag{4.5}$$

$$\therefore\ |w||\xi| \le |iw\xi + A\xi| \Rightarrow |w|\|\xi\|_W \le \|iw\xi + A\xi\|_H$$

for all real constant w. By the inverse mapping theorem, there exists a constant k_0 such that $\|\xi(t)\|_W \le k_0\|A\xi(t)\|_H$. Therefore since A is selfadjoint, for all real constant w,

$$\begin{aligned} (1+|w|)\|\xi(t)\|_W &\le k_0\|A\xi(t)\|_H + \|iw\xi(t) + A\xi(t)\|_H \\ &= k_0\|iw\xi(t) + A\xi(t) - iw\xi(t)\|_H + \|iw\xi(t) + A\xi(t)\|_H \\ &\le k_0\|iw\xi(t) + A\xi(t)\|_H + k_0\|iw\xi(t)\|_W + \|iw\xi(t) + A\xi(t)\|_H \\ &\le (2k_0 + 1)\|iw\xi(t) + A\xi(t)\|_H \end{aligned} \tag{4.6}$$

Hence $\|\xi(t)\|_W \le 2k_0\|iw\xi(t) + A\xi(t)\|_H$. Using Fourier transform for $\xi(t)$; $\hat{\xi}(iw) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iwt}\xi(t)dt$ and related properties together with Plancherel's theorem; $\int_{-\infty}^{\infty}|\hat{\xi}(iw)|^2 dw = \int_{-\infty}^{\infty}|\xi(t)|^2 dt$, $\hat{\dot{\xi}}(t) = iw\hat{\xi}(iw)$ if $\xi(t)$ has compact support. Therefore

$$\|\xi(t)\|_{\varpi}^2 = \int_{-\infty}^{\infty}\left(\|\dot{\xi}(t)\|_H^2 + \|\xi(t)\|_W^2\right)dt$$

$$= \int_{-\infty}^{\infty}\left(\left|\hat{\dot{\xi}}(iw)\right|_H^2 + \left|\hat{\xi}(iw)\right|_W^2\right)dw = \int_{-\infty}^{\infty}(1+|w|)\left|\hat{\xi}(iw)\right|_W^2 dw$$

$$\le \int_{-\infty}^{\infty}\left((1+|w|)\left|\hat{\xi}(iw)\right|_W\right)^2 dw \le \int_{-\infty}^{\infty}\left((2k_0+1)\left|iw\hat{\xi}(iw) + A\hat{\xi}(iw)\right|_H\right)^2 dw \tag{4.7}$$

$$\leq (2k_0+1)^2 \int_{-\infty}^{\infty} \left|\dot{\xi}(t) + A\xi(t)\right|_H^2 dt = (2k_0+1)^2 \int_{-\infty}^{\infty} \left|F\xi(t)\right|_H^2 dt$$

Therefore $\|\xi(t)\|_{\varpi}^2 \leq k_1 \|F\xi\|_{\wp}^2$, $k_1 = (1+2k_0)^2$. Hence F and similarly $F^{\pm}$ are bijective.

This implies that for large enough T > 0 and $\xi(t) = 0$ for $|t| \leq T$

$$\|\xi\|_{\varpi} \leq k_2 \|F\xi\|_{\wp} \tag{4.8}$$

where $k_2 = (k_1)^{\frac{1}{2}}$.

Let T be fixed and define $\|A(t) - A^+\| \leq \varepsilon, t \geq T; \|A(t) - A^-\| \leq \varepsilon, t \leq -T$.

Suppose $\xi(t) = 0$ for $t \leq T$, then

$$\begin{aligned} \|F\xi\|_{\wp}^2 &= \int_T^{\infty} \left\|\dot{\xi}(t) + A(t)\xi(t)\right\|_H^2 dt \\ &\geq \int_T^{\infty} \left\|\dot{\xi}(t) + A^+\xi(t)\right\|_H^2 dt - \int_T^{\infty} \left\|A(t)\xi(t) - A^+\xi(t)\right\|_H^2 dt \qquad (4.9) \\ &\geq \frac{1}{k_1^2} \int_T^{\infty} (\|\dot{\xi}(t)\|_H^2 + \|\xi(t)\|_W^2) dt - \varepsilon \int_T^{\infty} \|\xi(t)\|_H^2 dt \geq \left(\frac{1}{k_1^2} - \varepsilon\right) \|\xi\|_{\varpi}^2 \end{aligned}$$

Hence (4.8) is proved.

Now making use of cut off function $\beta : \Re \to [0,1]$ such that β (t) = 1 for $|t| \leq T-1$ and β (t) = 0 for $|t| \geq T$, $\xi(t) = \beta\xi(t) + (1-\beta)\xi(t)$ implies that,

$$\begin{aligned} \|\xi(t)\|_{\varpi} &\leq \|\beta\xi(t)\|_{\varpi} + \|(1-\beta)\xi(t)\|_{\varpi} \\ &\leq c\left(\|F(\beta\xi)\|_{\wp} + \|\beta\xi\|_{\wp}\right) + k_2 \|F((1-\beta)\xi)\|_{\wp} \end{aligned} \tag{4.10}$$

by (4.4) and (4.8).But $F(\beta\xi)(t) = (\beta\xi)'(t) + A\beta\xi(t) = \beta(\xi'(t) + A\xi(t)) + \beta'\xi(t)$.

Similarly $F((1-\beta)\xi(t)) = (1-\beta)(\xi'(t) + A\xi(t)) - \beta'\xi(t)$.

Hence

$$\|F(\beta\xi\|_{\wp} \leq \|F\xi\|_{\wp} + c_3\|\xi\|_{\wp} \; and \; \|F(1-\beta)\xi\|_{\wp} \leq \|F\xi\|_{\wp} + c_4\|\xi\|_{\wp} \tag{4.11}$$

for some constants c_3 and c_4 . If ξ has compact support in ([-T,T],H) then Theorem 4.2 is proved.

The fact that Fredholm index is invariant under continuous deformation of the operator family [13] will now be used to prove an easy consequence of the main theorem.

4.3 **Theorem**

Suppose the eigenvalues of A(t) are non-zero for all $t \in \Re$ then the Fredholm index of F is Zero.

Proof

Consider a sequence of Fredholm Operators $F_\lambda : \varpi \to \wp$ defined by

$$(F_\lambda\xi)(t) = \dot{\xi}(t) + A_\lambda(t)\xi(t) \tag{4.12}$$

which are continuously deformed to each other and using the fact above we are done. Let a cutoff function as described above be chosen such that

$$A_\lambda(t) = \begin{cases} [(1-\beta(t)]A(\lambda t), t \geq T \\ [(1-\beta(t)]A(-\lambda t), t \leq -T \end{cases} \quad (4.13)$$

Suppose that we set λ = 0, then A(t) = A(0), for all $t \in \Re$. Since A(0) is bijective by our assumption, F_o is bijective and thus index F_o = dim Ker F_o - dim co-Ker F_o = 0.

Now consider

$$\|A(t) - A_1(t)\| = \begin{cases} \|1-\beta(t)\|\|A(t) - A(T)\|, t \geq T \\ \|1-\beta(t)\|\|A(t) - A(-T), t \leq -T\| \end{cases} \quad (4.14)$$

For sufficiently large T > 0

$$\|F\xi - F_1\xi\|_{\wp}^2 = \int_{-\infty}^{\infty} \|A(t)\xi(t) - A_1(t)\xi(t)\|_H^2 dt \leq \lim\sup \|A(t) - A_1\|_{L(W,H)}^2 \int_{-\infty}^{\infty} \|\xi(t)\|_H^2 dt$$

$$\leq \varepsilon^2 \|\xi(t)\|_{\varpi}^2 \quad (4.15)$$

Therefore $F_1 = F + (F_1 - F)$ is Fredholm and index F = index F_1. Also

$$\lim_{t \to \pm\infty} A_\lambda(t) = A(\pm\lambda T)$$

is bijective, by assumption for each λ. From the main theorem therefore F_λ is Fredholm and from the general fact index F = index $F_\lambda = 0, \forall\ \lambda = 0, 1, 2 \ldots$ since index F_o = 0.

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