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A stochastic Ergodic Theorem in Von-Neumann algebras

A. A. Tijjani
Department of Mathematics, Ahmadu Bello University, Zaria
Federal Republic of Nigeria

Abstract

In this paper we introduce the notion of stochastic convergence of $\tau-$ measurable operators and prove a noncommutative extension of pointwise ergodic theorem of G. D. Birkhoff by means of it by using the techniques developed by Petz in [12].

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1.0 Introduction and Preliminaries

The noncommutative extension of the classical pointwise ergodic theorem of G. D. Birkhoff was first obtained by E. C. Lance [9] for the state invariant automorphism of σ - finite von Neumann algebra using the notion of almost uniform convergence. Since then many pointwise ergodic theorems have been studied by many authors [3], [5], [6], [7], [8], [12], [13], [14], [15], [20] in the context of von Neumann algebras. In [12], quasi-uniform convergence was introduced by Petz who showed that quasi-uniform convergence is stronger than almost uniform convergence and that the ergodic averages converge in this sense. In this paper we introduce the notion of stochastic convergence of $\tau-$measurable operators and show that the ergodic averages converge in this sense. The motivation for this work is due to Batty [1].

Let A be a σ-finite von Neuman algebra with a faithful normal state μ. We assume that A acts standardly on a Hilbert space H such that μ is implemented by a cyclic and separating vector Ω in H. We denote by $\sigma_t^{\mu}, t\in IR,$ the Modula automorphisim group associated with A and μ. Let $\overline{A}$ denote the crossed product $R\left(A,\sigma_t^{\mu}\right)$ of A by σ_t^{μ}. Then $\overline{A}$ is a semifinite von Neumann algebra acting on $\overline{H}=L^2\left(IR,H\right)$ [18] and is generated by the operators $\pi(x), x\in A$ and $\lambda(s), s\in IR,$ $(\pi(x)\xi)(t)=\sigma_{-t}^{\mu}(x)\,\xi(t), \xi\in L^2(IR,H), t\in IR,$ where $(\lambda(s)\xi)(t)=\xi(t-s),\ \xi\in L^2(IR,H), s,\ t\in IR.$ A is identified with its image $\pi(A)\ in\,\overline{A}$. If τ is the canonical faithful normal semifinite trace on $\overline{A}$ then $\tau\, o\,\theta=e^{-s}\,\tau, s\in IR$ [18], where $\{\theta s: s\in IR\}$ is the dual action of IR on $\overline{A}$ determined by

$$\theta s(\pi(x))=\pi(x),\ x\in A,\ s\in IR$$

$$\theta s(\lambda(t))=e^{-ist}\,\lambda(t),\ s,t\in IR$$

A closed densely defined operator h, with domain D(h), is said to be affiliated with $\overline{A}$, denoted by $h\eta\overline{A}$, if $\overline{A}'D(h)\subseteq D(h)$ and $h'h\subseteq hh'$ for all h' in the commutant $\overline{A}'$ of $\overline{A}$, i.e, if $\xi\in D(h)$ and $h'\in\overline{A}'$ then $h'\xi\in D(h)$ and $hh'\xi=h'h\xi$. An affiliated operator h is called a $\tau-$ measurable if its domain is $\tau-$dense, i.e, if $\forall\delta>0\exists p\in\overline{A}_{proj}: pH\subseteq D(h)$ and $\tau(I-p)\leq\delta,$ where $\overline{A}_{proj}$ is the

lattice of projections in $\overline{A}$. The set of all $\tau -$ measurable operators affiliated with $\overline{A}$, denoted by $\tilde{\overline{A}}$, is a *-algebra of operators on $\overline{H}$, where the sum and the product operators are in the strong sense [19].
The sets

$$N(\varepsilon,\delta)=\left\{h\in \approx{A}:\exists p \in \overline{A}_{proj} : pH \subseteq D(h), \|hp\|\leq \varepsilon, \tau(I-p)\leq \delta\right\},$$

where $\varepsilon > 0, \delta > 0,$ form a basis for the neighbourhoods of 0 for the topology on $\tilde{\overline{A}}$ that makes $\tilde{\overline{A}}$ a topological vector space. We call this topology the topology of convergence in measure. $\tilde{\overline{A}}$ endowed with this topology is a complete Hausdorff topological *-algebra in which $\overline{A}$ is a dense subset. For details see [19].

Let $\overline{A}^{+}$ and $\tilde{\overline{A}}^{+}$ denote the positive parts of $\overline{A}$ and $\tilde{\overline{A}}$ respectively. We note that if h, k are $\tau -$ measurable operators with $\tau -$ dense domains D(*h*), D(*k*) respectively and $h|E = k|E$ then $h = k$ [19], where *E* is a $\tau -$ dense subspace contained in $D(h)\cap D(k)$ and $h|E$ means h restricted to *E*. And that h is positive if and only if $\langle hx, x\rangle \geq 0$ for x in some $\tau -$ dense subspace of D(*h*). So $h \geq k$ if and only if $\langle hx, x\rangle \geq \langle kx, x\rangle$ for x in some $\tau -$ dense subspace of $D(h)\cap D(k)$ [19].

2.0 Convergence of $\tau -$ measurable Operators

2.1 *Definition*

A sequence (h_n) of elements of *A* is said to converge almost uniformly to *h* in *A,* if $\forall \varepsilon > 0 \exists p \in A_{proj}$ such that $\mu(I-p) < \varepsilon \, and \, \|(h_n - h)p\| \to 0$ as $n \to \infty$.

2.2 *Definition*

A sequence (h_n) of elements of A is said to converge quasi-uniformly to h in A, if for every non-zero projection $q \in A_{proj} \, and \in > 0$ there is a non-zero sub-projection $p \leq q$ in *A* such that $\mu(q-p) < \varepsilon \, and \, \|(h_n - h)p\| \to 0 \, as \, n \to \infty.$

2.3 *Remark*

Note that quasi-uniform limit is additive while almost uniform limit is non additive and the quasi-uniform convergence is stronger than almost uniform convergence. Many classical convergence theorems have been generalised to A by means of definitions 2.1 and 2.2 see [4], [8], [10], [12], [13] and [20].

2.4 *Definition*

A sequence (h_n) of elements of $\tilde{\overline{A}}$ is said to converge in measure to *h* in $\tilde{\overline{A}}$ (in the sense of Nelson [11]), if for $\varepsilon, \delta > 0$ there is an $n_\varepsilon \in IN$ such that $h_n - h \in N(\varepsilon, \delta)$ for all $n \geq n_\varepsilon$. A set $D \subset \tilde{\overline{A}}$ is said to be bounded in measure if and only if $\forall \delta > 0 \exists c < \infty$ such that $D \subset N(c, \delta)$.

2.5 ***Definition***

A sequence (h_n) of elements of $\tilde{\overline{A}}$ is said to converge stochastically to h in $\tilde{\overline{A}}$, if there exist a sequence (p_n) of projections in $\overline{A}$ such that $\tau(I-p_n)\underset{n\to\infty}{\to}0$ and $\|(h_n-h)p_n\|\underset{n\to\infty}{\to}0.$

2.6 ***Remark***

In this set up, the almost uniform convergence will take the following form which is called Segal convergence [17]. $(h_n)\subset\tilde{\overline{A}}$ converges almost uniformly to h in $\tilde{\overline{A}}$, if $\forall\varepsilon>0\exists\, p\in A_{proj}$ such that $\tau(I-p)<\in$ and $\|(h_n-h)p\|\underset{n\to\infty}{\to}0.$ In this set up it is easy to see that almost uniform limit is additive due to subadditivity of the trace and that Segal convergence implies Nelson convergence. It is also easy to see that stochastic convergence implies Nelson convergence if we put

$$P=\overset{\infty}{\underset{n=1}{\wedge}}\ p_n,$$

where $\overset{\infty}{\underset{n=1}{\wedge}}\ p_n$ is the projection onto $\bigcap_{n=1}^{\infty}p_n\overline{H}.$

Note that

$$\left(\bigwedge_{n=1}^{\infty}p_n\right)^{\perp}=\bigvee_{n=1}^{\infty}p_n{}^{\perp} \text{and } \tau\left(\bigvee_{n=1}^{\infty}p_n\right)\le\sum_{n=1}^{\infty}\tau(p_n).$$

2.7 ***Lemma***

Let (h_n) be a sequence of $\tau-$measurable operators converging stochastically to $\tau-$measurable operators h and k. Then h = k. That is, the limit of stochastic convergence is unique.

Proof

$h_n\underset{n\to\infty}{\to}h\Rightarrow\exists(p_n)\subset\overline{A}_{proj}$ such that $\tau(I-p_n)\underset{n\to\infty}{\to}0$ and $\|(h_n-h)p_n\|\underset{n\to\infty}{\to}0$ $h_n\underset{n\to\infty}{\to}k\Rightarrow\exists(q_n)\subset\overline{A}_{proj}$ such that $\tau(I-q_n)\underset{n\to\infty}{\to}0$ and $\|(h_n-k)q_n\|\underset{n\to\infty}{\to}0.$ Let $r_n=p_n\wedge q_n$ then $r_n\le p_n,\ r_n\le q_n$ and $\|(h-k)r_n\|\le\|(h-h_n)r_n\|+\|(h_n-k)r_n\|\le\|(h-h_n)p_n\|+\|(h_n-k)q_n\|\underset{n\to\infty}{\to}o,$ which shows that $h = k$. □

2.8 ***Lemma***

Let (h_n) and (k_n) be sequences of $\tau-$measurable operators converging stochastically to $\tau-$measurable operators h and k respectively. Then (h_n+k_n) converges stochastically to h + k. That is the limit of stochastic convergence is additive.

Proof

Let (p_n) and (q_n) be sequences of projections in $\overline{A}$ such that $\tau(I-p_n)\underset{n\to\infty}{\to}0$ and $\|(h_n-h)p_n\|\underset{n\to\infty}{\to}0,\tau(I-q_n)\underset{n\to\infty}{\to}0$ and $\|(k_n-k)q_n\|\underset{n\to\infty}{\to}0$ then putting $r_n=p_n\wedge q_n$ we see that

$$\tau(I-r_n)\le\tau(I-p_n)+\tau(I-q_n)\underset{n\to\infty}{\to}0$$

and $$\|[(h_n + k_n) - (h+k)]r_n\| \le \|(h_n - h)p_n\| + \|(k_n - k)q_n\| \to 0,$$
which shows that $h_n + k_n \underset{n\to\infty}{\to} h+k$. □

2.9 ***Theorem***

Let T be a normal positive linear contraction of $\tilde{\bar{A}}$ to itself such that

$$\tau(T(h)) \le \tau(h),\ \forall h \in \tilde{\bar{A}}^{+}.$$

Then for each $h \in \tilde{\bar{A}}$ the ergodic averages

$$s_n(h) := \frac{1}{n}\sum_{i=0}^{n-1} T^i(h) \underset{n\to\infty}{\to} \tilde{h} \in \tilde{\bar{A}}$$

stochastically.

2.10 ***Remark***

Since the stochastic limit is additive by lemma 2.8, it is enough to prove the theorem for $h \in \tilde{\bar{A}}^{s}$, where $\tilde{\bar{A}}^{s}$ is the set of self- adjoint elements of $\tilde{\bar{A}}$. The next three lemmas are due to Saito [16], Lance [9] and Petz [12] respectively. We do not want repeat their proofs here.

2.11 ***Lemma*** [16]

Let $(h_k) \subset \tilde{\bar{A}}^{s}$ be bounded in measure. Suppose (h_k) converges in measure to zero then for any $\varepsilon > 0$ there exist a subsequence (h_{kn}) in $\tilde{\bar{A}}$ and a sequence (p_n) of projections in $\bar{A}$ such that $\tau(I - p_n) \underset{n\to\infty}{\to} 0$ and $\lim_n \|h_{kn}\, p_n\| = 0$.

2.12 ***Maximal Lemma*** [9]

For every $h \in \tilde{\bar{A}}_1^{+}$ there exists $c \in \bar{A}^{+}$ with $\|c\| \le 2, \tau(c) \le 4\varepsilon^{\frac{1}{2}}$ such that $s_n(h) \le c, \forall n,$ where $\tilde{\bar{A}}_1$ is the unit ball of $\tilde{\bar{A}}$.

2.13 ***Lemma*** [12]

Let $b_k = h - s_k(h) + \tilde{h}$ with $h \in \tilde{\bar{A}}^{s}$, $k \in IN$ then $\left\| s_n(b_k - \tilde{h}) \right\| \le 2kn^{-1}\|h\|$, for n > k.

2.14 ***Lemma***

Let $(c_k) \subset \tilde{\bar{A}}_1^{s}$ be a bounded sequence converging in measure to zero. Then there is a sequence (p_n) of projections and a subsequence (c_{kn}) such that $\tau(I - p_n) \underset{n\to\infty}{\to} 0$ and $\|s_n(c_{kn})p_n\| \underset{n\to\infty}{\to} 0$

Proof

We may assume without loss of generality that $(c_k) \in \tilde{\overline{A}}_1^+$. By maximal lemma there are elements $h_k \in \overline{A}^+$ such that $s_n(c_k) \le h_k$ and $\tau(h_k) \le 4\varepsilon^{\frac{1}{2}}$. By Saito's lemma there is a subsequence (h_{kn}) and a sequence (p_n) of projections such that $\tau(I - p_n) \underset{n\to\infty}{\to} 0$ and $\|h_{kn}\, p_n\| \underset{n\to\infty}{\to} 0$. Hence

$\|s_n(c_{kn}) p_n\| \underset{n\to\infty}{\to} 0$ □

Proof of theorem 2.9

Let b_k be as in lemma 2.13 then $s_n(h) - \tilde{h} = s_n\left(b_k - \tilde{h}\right) + s_n(h - b_k)$. Using lemma 2.14 for $c_k = h - b_k$ we see that $\|s_n(c_{kn}) p_n\| \underset{n\to\infty}{\to} 0$ and $\tau(I - p_n) \underset{n\to\infty}{\to} 0$ and so

$$\left\|\left(s_n(h) - \tilde{h}\right) p_n\right\| \le \left\|s_n(b_k - \tilde{h})\right\| + \|s_n(c_{kn}) p_n\| \underset{n\to\infty}{\to} 0. \qquad \square$$

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