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rho-supermanifolds and the Clifford bundle

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Abstract

Let $T(E_p)$ be the body of the supertangent bundles $T(E)$ and $\overline{T}(E)$, where E is a $\rho-$supermanifold. Let T_xM be the tangent space to the Lorentzian manifold M at $x \in M$. In the case m = n = 4, where m and n are the integers labelling the associated superspace, we identify T_xE_p with T_xM and proceed to construct a Clifford bundle of algebras on the tangent space of the body of the supertangent bundle.

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1.0 Introduction

In 1975, T. T. Wu and C. N. Yang [1] asserted that a principal fiber bundle $P(X,G)$ where *X* is the base space and *G* is the structure group (the guage group) describes a pure guage theory. It was consequently found imperative in order to meet the demands of supersymmetry to go beyond the concepts of manifold and space-time into those of supermanifold and superspace. The most fruitful approach to supermanifolds and superspaces has been that initiated by Rogers [2] in which G^{∞}-supermanifolds are topological spaces locally G-diffeomorphic to superspaces. Jadcyzk and Pilch [3] pointed out that a certain theorem in Rogers, approach is valid only for a particular class of open sets. However, the deficiency that a body was not guaranteed for ever G^{∞}-supermanifold still remained. A generalized theory of supermanifolds and superspaces was formulated by Hoyos et al [4], [5] and [6] in a series of three fundamental papers. In their approach they defined $\rho-$supermanifolds and G^{∞}-functions defined on them; $\rho-$supermanifolds being supermanifolds with well-behaved, that is, operative, bodies. After expatiating on G^{∞}-vector fields and G^{∞}-derivations on supermanifolds, they introduced supervector bundles, that is, L_B-bundles whose G^{∞}-sections are the G^{∞}-vector fields. Finally, they discussed the two supertangent bundles and the cotangent bundle on $\rho-$supermanifolds.

Mosna and Rodrigues, Jr [7] discussed extensively the bundles of algebraic and Dirac-Hestenes spinor fields. The purpose of the present paper is to identify in the case $m = n = 4$ the body of the two supertangent bundles $T(E)$ and $\overline{T}(E)$ with the tangent bundle *TM* of Ref. [7] where *M* is a Lorentzian manifold; and consequently be in a position to define the Clifford bundle of algebras on T_xE_ρ. Let us note that *m* and *n* refer respectively to the integers labelling the even and the odd coordinates of the superspace $S^{m,n}$ which will be discussed in detail later. We may note that the sections of the Clifford bundle, the Clifford fields ψ_Ξ, satisfy the Dirac-Hestenes equation.

In their discussion of Clifford and other bundles [7], they considered the four-dimensional, real, connected, paracompact, and noncompact manifold *M*. (As an example of a paracompact space one may note that R^m is paracompact [8]. Let us note that for *m* = *4*, $R^m \equiv R^{1,3}$ is the Minkowshi spacetime. Since we wish to construct the Clifford bundle of algebras on the body of superbundles on generalized supermanifolds, which are manifolds endowed with both commuting and anticommuting coordinates, let us start with a few preliminary definitions and concepts [5], [6]. We shall limit ourselves to *N* = 1 supersymmetry. Hence, it will be necessary to extend Minkowski spacetime $R^{1,3}$ to superspace (x,θ) where, in addition to the spacetime coordinates $x = (x_0, x)$, we also have the Grassmann-valued coordinates θ with *n* = 4 in the superspace $S^{m,n}$.

Suppose B is a Z_2-graded Banach algebra, then $a \in B$ is said to be homogeneous if $a \in B^r$ $(r = 1,2)$. Suppose *J* is a fixed (finite or countably infinite) set of indices while *F(J)* is the set of finite parts of *J*, then a Grassmann-Banach algebra is a Z_2-graded commutative Banach algebra satisfying the following two properties.

(i) There exists a linearly independent subset $\{\beta_i\}_{i\in J} \subset B^1$ such that the set $\{\beta_M\}_{M\in F}$ is a Banach basis for *B* where $\beta_M = \beta_{i_1}, \beta_{i_2} \cdots \beta_{i_r} \neq 0$, $\beta_\Theta = 1$ and $M = \{i_1, i_2, \cdots, i_r\} \in F$ with $i_1 \prec \cdots \prec i_r$.

(ii) For each $H \subset F$ and $a = \sum_{M\in F} \alpha_M \beta_M \in B$, the body of a is by definition $p_H(a) = \sum_{M\in H} \alpha_M \beta_M$. The linear function $r \equiv p_H : B \to B$ is called the body map.

Suppose *B* is a Grassmann-Banach algebra, then Banach Z_2 − graded left *B*-module is a Banach space *V* which is also a left *B*-module and which can be written as the direct sum $V = V^0 \oplus V^1$ with $B^r V^s \subset V^{r+s}$, $r,s =$ integers mod 2. Also, $\|av\| \le \|a\|\|v\|, a \in B, v \in V$. In the sequel, we shall concentrate on free superspaces. Hence, one should note that a Z_2 − graded B − module *V* is said to be free if it has a basis whose elements are homogeneous. We may note that by the basis of a B − module we mean the sequence $\{e_1, \cdots, e_m, e_{m+1}, \cdots, e_{m+n}\}$, where e_1 is even for $1 \le i \le m$ and odd for $m+1 \le i \le m+n$. Superspaces are a subclass of objects called B − spaces, in a category [9] whose morphisms are L_B − operators (B − linear operators between B − spaces). Hence, it will be helpful to recall the following definition of B − spaces [4].

Suppose we are given an *m* + *n* sequence $\pi = \{p_1, p_2, \cdots, p_{m+n}\}$ of β − projections, that is, p_H projections, of B and $V(\pi)$ is the following Banach subspace of $V : V(\pi) = \left\{\sum_{i=1}^{m+n} a^i e_i \in V \mid a_i \in \operatorname{Im} p_i\right\}$. A B − space of dimension (π, σ) is the quotient Banach space $V(\pi,\sigma) = V(\pi)/V(\sigma)$. We now consider a B − space with a body. Let $V(\pi,\sigma)$ be a B − space, F_c a real vector space, *t* a linear map from $\mathbf{R}^{m+n}$ onto F_c and *p* the canonical projection of $V(\pi)$ onto $V(\pi,\sigma)$. Then *c* is called a *k*-body map of $V(\pi,\sigma)$ and F_c the body of $V(\pi,\sigma)$ if *c* is onto and $k = \dim F_c$. Moreover, with *p* being the quotient projection, $c \cdot p = t \cdot R|_{v(\pi)}$ where $t \cdot R|_{v(\pi)} : V(\pi) \to F_c$ and $c : V(\pi,\sigma) \to F_c$.

Here *R* is the body map associated with the fixed basis $\{e_i\}_{i=1}^{m+n}$ of ***V***:

$$R(\sum_{i=1}^{m+n} a^i e_i) = \left(r\left(a^1\right), \cdots, r\left(a^{m+n}\right)\right).$$

We are now in a position to define a superspace. It is a bodied $B-$space (S, r) where $S = V(\pi, \sigma)^0$, the elements of the $m + n$ sequence being $p_i \equiv p_{F(K_i)}, \rho \neq K_i \subset J$ for $m+1 \leq i \leq m+n$, and r is defined by $t: R^{m+n} \to R^m, t(x_1, \cdots x_{m+n}) = (x_1, \cdots, x_m)$. Superspaces are a subclass of $B-$spaces, the latter being objects in a category whose morphisms are L_B operators defined as follows:

Let V and V' be $B-$modules, then a $B-$linear operator from $V \to V'$ is a continuous linear operator $T \in L(V, V')$ such that $T(av) = aT(v)$, $a \in B$, $v \in V$.

The set of $B-$linear operators from $V \to V'$ is deonoted by $L_B(V, V')$. Let us now recall the definition of a $G^\infty-$supermanifold [6]. Suppose S is a superspace and E is a $C^\infty-$manifold modelled on S. (Heuristically, this means that E is obtained by taking the union of patches of S subject to the operation of an equivalence relation between the patches). (a) A $G^\infty-$atlas on E is a $C^\infty-$atlas $\{(U_\alpha, \psi_\alpha)\}\alpha \in \Lambda$ on E such that $\psi_{\beta\alpha} = \psi_\beta \psi_\alpha^{-1}$ is G^∞ from $\psi_\alpha(U_\alpha \cap U_\beta)$ to $\psi_\beta(U_\alpha \cap U_\beta)$, (b) A $G^\infty-$structure on E is a maximal $G^\infty-$atlas on E. (c) A $G^\infty-$supermanifold E is a Banach $C^\infty-$manifold endowed with a $G^\infty-$structure. A free $G^\infty-$supermanifold is such that S is free.

Let us also note what is meant by an open subset $U \subset S$ of a superspace being G-connected [5]. We first note that the r-saturation of U is $\overline{U} = r^{-1}(r(U))$. Then U is said to be G-connected (G-convex) if, for all $(x, \theta) \in U, \{(\tilde{x}, \theta)\} \cap U$ is connected (convex), note the tilde on (x, θ). The last important definition which we wish to recall is that of a $\rho-$supermanifold [6]. Suppose E is a $C^\infty-$manifold modelled on a superspace $S^{m,n}, E_\rho$ a $C^\infty-$manifold modelled on R^m and ρ a $G^\infty-$map from $E \to E_\rho$. A $\rho-$atlas on the triple (E, E_ρ, ρ) is a pair comprising a $C^\infty-$atlas $\{(U_\alpha, \psi_\alpha)\}$ on E and a $C^\infty-$atlas $\{(U_{\rho\alpha}, \psi_{\rho\alpha})\}$ on E_ρ such that (i) $\psi_\alpha(U_\alpha)$ is cG-connected (that is, both G-connected and connected), (ii) $U_\alpha = \rho^{-1}(U_{\rho\alpha})$ and (iii) $\psi_{\alpha\rho} \cdot \rho = r \cdot \psi_\alpha$ (where r is the body map on $S^{m,n}$). A $G^\infty - \rho-$structure on the triple (E, E_ρ, ρ) is a maximal $\rho-$atlas, while a $\rho-$supermanifold is a triple (E, E_ρ, ρ) with a $G^\infty - \rho-$structure on the triple.

2.0 Supervector bundles and the Clifford bundle

The purpose of this article is to identify he body of the supervector bundles *T(E)* and $\overline{T}(E)$ with the tangent bundle *TM* in the case $m = n = 4$, and proceed to construct a Clifford bundle of algebras on the said body. In their exposition of the algebraic and Dirac-Hestenes spinor fields, Mosna and Rogrigues Jr. [7] defined the Clifford bundles *Cl(M,g)* in two ways. In one of these, with *(M,g)* being a Lorentzian manifold, the corresponding Clifford bundle is defined as a disjoint union for all $x \in M$ of the Clifford algebras $Cl(T_x M, g_x)$, that is,

$$Cl(M,g) = \bigcup_{x \in M} Cl(T_x M, g_x)$$

where $g \in \sec T^{2,0}M$ is the Lorentzian metric of signature (1,3), (with *sec* denoting section), g_x is the restriction of *g* to *x* and $T^{2,0}M$ is the 2-covariant tensor bundle. It was also shown that to each Dirac-Hestenes spinor field $\psi \in \sec Cl^l{}_{Spin^e_{1,3}}(M)$ and to each spin frame $\Xi \in \sec P_{Spin^e_{1,3}}(M)$ there exists a well-defined $\psi\Xi \in \sec Cl(M,g)$. We may note that the vector bundle $Cl^l{}_{Spin^e_{1,3}}(M)$ is the left real spin Clifford bundle of *M* and ψ_Ξ a well-defined sum of even multivector fields.

Let *T* be a functor from the category of $\rho-$supermanifolds into the category of supervector bundles. If *E* is a $\rho-$supermanifolds, then the $G^\infty-$sections of the supervector bundles *T*(*E*) and $\overline{T}(E)$ are applicable to super-symmetry. In the approach to superfiber bundles expatiated upon by Hoyos et al [4], [5] and [6], Clifford algebras do not feature. Rather, a supervector bundle, for example, is constructed on a base which is a $\rho-$supermanifold. In fact, a supervector bundle or an L_B-bundle, having *E* a $\rho-$supermanifold as a base, and fiber a *B*-space *F*, is denoted by $M(E,F,\Pi)$ and is such that there exists a trivializing covering $\{(U_i,\tau_i)\}$ for Π such that the mapping $(x,\theta)\rightarrow$ the transition functions, a vector bundle morphism,

$$\psi_{ij}(x\theta) = \tau_i(x,\theta)\tau_j^{-1}(x,\theta) : U_i \cap U_j \rightarrow c - L_B(F)^0$$

are $G^\infty \forall_{i,j}$. Here now Π denotes the canonical projection: $\Pi : \mathrm{M} \rightarrow E$. Also, for a given body *c* of *F*, the subspace $c - L_B(F) = \{T \in L_B(F) \mid c(Tu) = 0 \forall c(u) = 0\}$. We also note that since τ_i is a trivializing map, $\tau_i : \Pi^{-1}(U_i) \rightarrow U_i \times F$.

The first two superbundles of interest are the two supertangent bundles *T(E)* with fiber being the superspace $S^{m,n}$), $\overline{T}(E)$ which is modelled after the fiber $\overline{S} = \left\{\sum_{i=1}^{m+n} a_i e_i \in V \mid a_i \beta_{Ki} = 0 m+1 \le i \le m+n\right\}$ Let us note that if $\mathrm{M} \rightarrow E$ is a supervector bundle and the functor $\mathrm{M} \rightarrow L(\mathrm{M})$ comprises continuous linear forms, then *L*(M) is called the dual bundle. In particular, if M = *T(E)* is the tangent bundle, then the dual bundle *T*(E)* is the cotangent bundle. It is the third of the three superbundles considered in [6] and its fiber is $S^* = L_B(S,B)$. (In connection with the terminology supervector bundles for these superbundles having the quotient Banach space $F = V(\pi,\sigma) = V(\pi)/V(\sigma)$ as a fiber, let us note that a Banach space is a topological vector space [9]).

Suppose $f : E_1 \rightarrow E_2$ is a vector bundle morphism where $E_i(i=1,2)$ are $\rho-$supermanifolds, then we can define the following:

$Tf : TE_1 \rightarrow TE_2$. It is usual to denote *Tf* by f_*. With the transition function ψ_{ij} being an example of a vector bundle morphism, it is clear that, with $\{(U_i,\psi_j)\}$ being the $\rho-$atlas of *E*, the transition function for the tangent bundles *T(E)* and $\overline{T}(E)$ are ψ_{ij_*}. Let us now note that a section *X* of a supervector bundle is the mapping:

$$X : E \to M.$$

Let us also recall that if *E* is a $\rho -$supermanifold, then a differential form of degree *r* or simply an *r*-form is a section of $L_{\alpha}^{r}(T(E))$ where the functor L_{α}^{r} comprises *r*-multilinear continuous alternating forms. These *r*-forms are denoted by $\Omega^{r}(E)$. The spaces of $G^{\infty} -$sections of *T(E)*, $\overline{T}(E)$, and *T*(E)* constitute respectively the spaces of supervector fields $\chi(E), \overline{\chi}(E)$, and 1-superforms $\Omega(E)$.

As noted in Ref [6], if M is a supervector bundle while F_c is the body of its fiber *F*, then there exists an associated real vector bundle M_c with base E_{ρ} and fiber F_c such that the transition functions are $(\psi_c)_{ij}$. Here $t \cdot R|_{V(\pi)}: V(\pi) \to F_c$ or alternatively $c : F \to F_c$ where *c* is a *k*-body map of $F = V(\pi, \sigma)$ with *k* being *dim* F_c. The body $r = t \cdot R$ for *T(E)* and $\overline{T}(E)$ where *t* is the projection on the first *m*(= 4) coordinates is the usual tangent bundle $\overline{T}(E_{\rho})$. Hence, if the spacetime part of the superspace coordinate (x, θ), where $\{\theta_{\alpha}\}, a = 1, \cdots, 4$ are the Grassmann valued coordinates, is chosen to be a Lorentzian manifold, we can identify this $T(E_{\rho})$ with the tangent bundle *TM*.

Let *M* be a Lorentzian manifold and (U, φ) a chart at $x \in M$. Let *v* be a vector in the vector space in which $\varphi(U)$ lies. We say that the triples (U, φ, v) and (V, ψ, w) are equivalent if the derivative of $\psi\varphi^{-1}$ at $\varphi(x)$ maps *v* into *w*. An equivalent class of such triples is called a tangent vector of *M* at *x*. The set of such tangent vectors is called the tangent space of *M* at *x*, and is dented by T_xM. The tangent bundle *TM* is then the disjoint union

$$TM = \bigcup_{x \in M} T_x M .$$

On the other hand, let $Cl(T_x M, g_x)$ be the Clifford algebra associated with $(T_x M, g_x)$. It is to be emphasized that $T(E_{\rho})$ has now been identified with *TM* for *m* = *n* = 4. Once this identification has been made we can now proceed to construct the Clifford bundle of algebras on the tangent space $T_x(E_{\rho})$ where $x \in E_{\rho}$. The first step, therefore, in defining $Cl(T_x E_{\rho}, g_x)$ is first to give the definition of the exterior algebra of $T_x E_{\rho}$, that is,

$$\Lambda T_x E_{\rho} = \sum_{i=0}^{n} \Lambda^{i} T_x E_{\rho}, \text{ where}$$

$\Lambda^{i} T_x E_{\rho}$ is the space of $i -$parts of the multivectors $X \in \Lambda T_x E_{\rho}$ [11]. After defining both the grade evolution and the reversion operator on this exterior algebra, the algebra of multivectors is defined followed by the definitions of the exterior product, the contracted products, and the Clifford product. The desired Clifford algebra is the unital associative algebra which is the vector space of multivectors over $T_x E_{\rho}$ and which is endowed with the Clifford product.

3.0 Acknowledgement

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