Einstein's Gravitational Field Equations Exterior and Interior to an Oblate Spheroidal Body

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Abstract

In this paper we derive the exact Einstein's gravitational field equations exterior and interior to a homogeneous oblate spheroidal massive body to open the way for their solution and hence application to test particles in the universe.

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1.0 Introduction

Einstein's gravitational field equations[1] in the space-time exterior to a point particle and their solutions are very well known. Cylindrically symmetric solutions were first investigated by Weyl[2-3]. The field equations interior and exterior to a spherical massive body and their solutions were first constructed by K. Schwarzschild[4-8] in 1916 and have formed the basis of the general relativistic resolution of the phenomena of (i) anomalous orbital precession, (ii) gravitational spectral shift, (iii) gravitational time dilation, (iv) gravitational length contraction, (v) radar sounding and (vi) geodetic deviation in the Solar System. In the third place the field equations and their solutions due to a cylindrical body are well known[3]. And finally the field equations for the cosmos were first derived by H.P Robertson[9-14] and A.G. Walker[15] in Spherical coordinates and have constituted the basis of general relativistic cosmological theories as is well known. But beginning from 1950 it has become increasingly recognised that the Earth and the Sun as well as almost all major astronomical bodies are actually spheroidal and that that geometry has experimentally measurable and physically interesting effects on the motions of test particles in their gravitational fields.

Prior to 1950 theoretical gravitational study was restricted almost exclusively to the fields of massive bodies of perfect spherical geometry. For example in Newtonian theory of gravitation in the earth's atmosphere the motion of particles (such as projectiles, satellites and penduli) is treated under the assumption that the earth is a perfect sphere. Similarly in the Solar System the motion of bodies (such as planets, comets and asteroids) is treated under the assumption that the Sun is a perfect sphere. Also in Einstein's theory of gravitation called General Relativity the motion of bodies (such as planets) and particles (such as photons) is treated under the assumption that the Sun is a perfect sphere (the Schwarzschild's space-time). But it is well known that the only reason for these restrictions is mathematical convenience and simplicity. That the real fact of nature is that all rotating planets and stars and galaxies in the universe are spheroidal. And it is obvious that their spheroidal geometry will have corresponding consequences and effects in the motions of all particles in their gravitational fields. These effects will exist in Newton's Theory of Universal Gravitation as well as Einstein's Theory of General Relativity.

As an example it is now well know that satellite orbits around the earth are governed by not only the simple inverse distance squared gravitational fields due to perfect spherical goemetry. That they are also governed by second harmonics (pole of order 3) as well as fourth harmonics (pole of order 5) of gravitational scalar potential not due to perfect spherical geometry. Therefore towards the more precise explanation and hence prediction of satellite orbits around the earth Sterne[16] and Garfinkel[17] introduced the method of quadratures for approximating the second harmonics of the gravitational scalar potential of the earth due to its spheroidal shape. That method was improved by O' Keefe[18] in 1959. Then in 1960 Vinti[19] suggested a general mathematical form of the gravitational scalar potential of the spheroidal earth and how to estimate some of the parameters in it for use in the study of satellite orbits. Therefore in a recent paper[20] we derived the exact and complete Newton's universal gravitational fields interior and exterior to a homogeneous oblate spheroidal body to open the way for the extension of Newtonian mechanics from the field of perfectly spherical bodies to that of spheroidal bodies[21]. Similarly we now derive the exact Einstein's gravitational field equations exterior and interior to a homogeneous oblate spheroidal massive body to open the way for their solution and hence the extension of general relativistic mechanics from the space-time due to perfectly spherical bodies to that due to spheroidal bodies.

2.0 Mathematical Analysis

Consider a homogenous massive oblate spheriodal body of surface parameter ξ_0 and rest mass M_0 as shown in Figure 1.

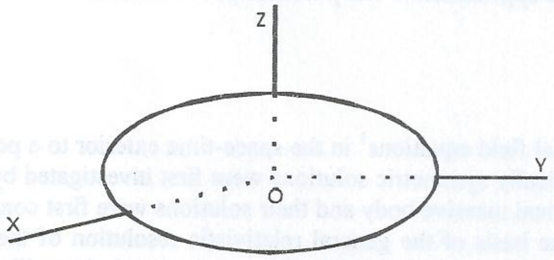

Figure 1: An oblate spheroidal massive body.

The oblate spheroidal coordinates of space (η, ξ, ϕ) are defined[22-23] in terms of the Cartesian coordinates (x, y, z) as

$$x = a\left(1-\eta^2\right)^{\frac{1}{2}}\left(1+\xi^2\right)^{\frac{1}{2}} \cos\phi \tag{2.1}$$

$$y = a\left(1-\eta^2\right)^{\frac{1}{2}}\left(1+\xi^2\right)^{\frac{1}{2}} \sin\phi \tag{2.2}$$

$$z = a\eta\xi \tag{2.3}$$

where a is constant parameter and

$$-1 \le \eta \le 1 \quad 0 \le \xi < \infty \; ; \; 0 \le \phi \le 2\pi \tag{2.4}$$

and the surface is given by the equation

$$\xi = \xi_0 \tag{2.5}$$

for some constant ξ_0. It therefore follows from definition and the invariance of the line element ds^2, as is well known, that the components of the covariant metric tensor in flat space-time are given in oblate-spheroidal coordinates by

$$g_{00} = 1 \tag{2.6}$$

$$g_{11} = \frac{a^2\left(\eta^2+\xi^2\right)}{\left(1-\eta^2\right)} \tag{2.7}$$

$$g_{22} = \frac{a^2\left(\eta^2+\xi^2\right)}{\left(1^2+\xi^2\right)} \tag{2.8}$$

$$g_{33} = -a^2\left(1-\eta^2\right)\left(1+\xi^2\right) \tag{2.9}$$

$$g_{\mu\nu} = 0 \; ; \; otherwise \tag{2.10}$$

Towards our goal in this paper it may be noted that (i) the oblate spheroid possesses azimuthal symmetry and (ii) the flat space-time metric tensor (2.6)-(2.10) is diagonal and is a function of only the coordinates η and ξ. Consequently we shall seek the covariant metric tensor exterior or interior to a massive oblate spheroidal body in the general form:

$$g_{00} = e^{-F} \quad (2.11)$$

$$g_{11} = -e^{-G} \quad (2.12)$$

$$g_{22} = -e^{-H} \quad (2.13)$$

$$g_{33} = -a^2(1-\eta^2)(1+\xi^2) \quad (2.14)$$

where F,G and H are functions at η and ξ only.

Now it follows from definition that the coefficients of affine connection

$$\Gamma_{\mu\ \lambda}^{\ \sigma} = \frac{1}{2}g^{\sigma\varepsilon}\left(g_{\mu\varepsilon,\lambda}+g_{\varepsilon\lambda,\mu}-g_{\mu\lambda,\varepsilon}\right) \quad (2.15)$$

are given explicitly by

$$\Gamma_{0\ 1}^{\ 0} = \Gamma_{0\ 1}^{\ 0} = \frac{1}{2}F_\eta \quad (2.16)$$

$$\Gamma_{0\ 2}^{\ 0} = \Gamma_{2\ 0}^{\ 0} = \frac{1}{2}F_\xi \quad (2.17)$$

$$\Gamma_{0\ 0}^{\ 1} = -\frac{1}{2}e^{G-F}F_\eta \quad (2.18)$$

$$\Gamma_{1\ 1}^{\ 1} = -\frac{1}{2}G_\eta \quad (2.19)$$

$$\Gamma_{1\ 2}^{\ 1} = \Gamma_{2\ 1}^{\ 1} = -\frac{1}{2}G_\xi \quad (2.20)$$

$$\Gamma_{2\ 2}^{\ 1} = \frac{1}{2}e^{G-H}H_\eta \quad (2.21)$$

$$\Gamma_{3\ 3}^{\ 1} = a^2\eta(1+\xi^2)e^G \quad (2.22)$$

$$\Gamma_{0\ 0}^{\ 2} = -\frac{1}{2}e^{H-F}F_\xi \quad (2.23)$$

$$\Gamma_{1\ 1}^{\ 2} = \frac{1}{2}e^{H-G}G_\xi \quad (2.24)$$

$$\Gamma_{1\ 2}^{\ 2} = \Gamma_{2\ 1}^{\ 2} = -\frac{1}{2}H_\eta \quad (2.25)$$

$$\Gamma_{2\ 2}^{\ 2} = -\frac{1}{2}H_\xi \quad (2.26)$$

$$\Gamma_{3\ 3}^{\ 2} = -a^2\xi(1-\eta^2)e^H \quad (2.27)$$

$$\Gamma_{1\ 3}^{\ 3} = \Gamma_{3\ 1}^{\ 3} = -\frac{\eta}{1-\eta^2} \quad (2.28)$$

$$\Gamma_{2\ 3}^{\ 3} = \Gamma_{3\ 2}^{\ 3} = \frac{\xi}{1+\xi^2} \quad (2.29)$$

$$\Gamma_{\mu\ \lambda}^{\ \sigma} = 0 \quad ; \text{ otherwise} \quad (2.30)$$

Now the Riemann curvature tensor $R^{\rho}_{\ \sigma\mu\nu}$ is defined by

$$R^{\rho}_{\ \sigma\mu\nu} = \Gamma^{\rho}_{\sigma\mu,\nu}-\Gamma^{\rho}_{\sigma\nu,\mu}-\Gamma^{\rho}_{\lambda\mu}\Gamma^{\lambda}_{\sigma\nu}+\Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\sigma\mu} \quad (2.31)$$

And the Ricci tensor $R_{\mu\nu}$ is defined by contraction as

$$R_{\mu\nu} = R^{\sigma}_{\ \mu\nu\sigma} \quad (2.32)$$

Consequently

$$R_{00} = R^{0}_{\ 000}+R^{1}_{\ 001}+R^{2}_{\ 002}+R^{3}_{\ 003} \quad (2.33)$$

and

$$R_{11} = R^0_{110}+R^1_{111}+R^2_{112}+R^3_{113} \tag{2.34}$$

and

$$R_{22} = R^0_{220}+R^1_{221}+R^2_{222}+R^3_{223} \tag{2.35}$$

and

$$R_{33} = R^0_{330}+R^1_{331}+R^2_{332}+R^3_{333} \tag{2.36}$$

But leaving vanishing terms it follows that

$$R^1_{001} = \Gamma^1_{00,1} - \Gamma^1_{00}\Gamma^0_{01} = -\frac{1}{2}e^{G-F}F_{\eta\eta} - \frac{1}{2}e^{G-F}F_\eta G_\eta + \frac{1}{4}e^{G-F}F_\eta{}^2 \tag{2.37}$$

where a subscript denotes one partial derivative with respect to that coordinate and

$$R^2_{002} = \Gamma^2_{00,2} - \Gamma^2_{00}\Gamma^0_{02} = -\frac{1}{2}e^{H-F}F_{\xi\xi} - \frac{1}{2}e^{H-F}F_\xi H_\xi + \frac{1}{4}e^{H-F}F_\xi{}^2 \tag{2.38}$$

$$R^3_{003} = \Gamma^3_{13}\Gamma^1_{00} + \Gamma^3_{23}\Gamma^2_{00} = \frac{\eta e^{G-F}}{2\left(1-\eta^2\right)}F_\eta - \frac{\xi e^{H-F}}{2\left(1+\xi^2\right)}F_\xi \tag{2.39}$$

$$R^0_{110} = -\Gamma^0_{10,1} + \Gamma^0_{20}\Gamma^2_{11} = -\frac{1}{2}F_{\eta\eta} - \frac{1}{4}e^{H-G}F_\xi G_\xi \tag{2.40}$$

$$R^2_{112} = \Gamma^2_{11,2} - \Gamma^2_{12,1} - \Gamma^2_{11}\Gamma^1_{12} - \Gamma^2_{21}\Gamma^2_{12} + \Gamma^2_{12}\Gamma^1_{11} + \Gamma^2_{22}\Gamma^2_{11} = \frac{1}{2}e^{H-G}G_{\xi\xi} - \frac{1}{4}e^{H-G}G_\xi{}^2 + \frac{1}{4}e^{H-G}H_\xi G_\xi + \frac{1}{2}G_{\eta\xi} - \frac{1}{4}H_\eta{}^2 + \frac{1}{4}H_\eta G_\eta \tag{2.41}$$

$$R^3_{113} = -\Gamma^3_{13,1} - \Gamma^3_{31}\Gamma^3_{13} + \Gamma^3_{13}\Gamma^1_{11} + \Gamma^3_{23}\Gamma^2_{11} = \frac{1}{\left(1-\eta^2\right)^2} + \frac{\eta}{2\left(1-\eta^2\right)}G_\eta + \frac{\xi e^{H-G}}{2\left(1+\xi^2\right)}G_\xi \tag{2.42}$$

$$R^0_{220} = -\Gamma^0_{20,2} - \Gamma^0_{02}\Gamma^0_{20} + \Gamma^0_{10}\Gamma^1_{22} + \Gamma^0_{20}\Gamma^2_{22} = \frac{1}{2}F_{\xi\xi} - \frac{1}{4}F_\xi{}^2 + \frac{1}{4}e^{G-H}F_\eta H_\eta + \frac{1}{4}F_\xi H_\xi \tag{2.43}$$

$$R^1_{221} = \Gamma^1_{22,1} - \Gamma^1_{21,2} - \Gamma^1_{12}\Gamma^1_{21} - \Gamma^1_{22}\Gamma^2_{21} + \Gamma^1_{11}\Gamma^1_{22} + \Gamma^1_{21}\Gamma^2_{22} = -\frac{1}{2}e^{G-H}H_{\eta\eta} + \frac{1}{2}e^{G-H}H_\eta{}^2 + \frac{1}{2}G_{\xi\xi} - \frac{1}{2}e^{G-H}H_\eta G_\eta + \frac{1}{4}G_\xi H_\xi - \frac{1}{4}G_\xi{}^2 \tag{2.44}$$

$$R^3_{223} = -\Gamma^3_{23,2} - \Gamma^3_{32}\Gamma^3_{23} + \Gamma^3_{13}\Gamma^1_{22} + \Gamma^3_{23}\Gamma^2_{22} = -\frac{1}{\left(1-\xi^2\right)^2} + \frac{\eta e^{G-H}}{2\left(1-\eta^2\right)}H_\eta - \frac{\xi}{2\left(1+\xi^2\right)}G_\xi \tag{2.45}$$

$$R^0_{330} = -\Gamma^0_{10}\Gamma^1_{33} + \Gamma^0_{23}\Gamma^2_{33} = -\frac{1}{2}a^2\eta\left(1+\xi^2\right)e^G F_\eta + \frac{1}{2}a^2\xi\left(1-\eta^2\right)e^H F_\xi \tag{2.46}$$

$$R^1_{331} = -\Gamma^1_{33,1} - \Gamma^1_{33}\Gamma^3_{31} + \Gamma^1_{11}\Gamma^1_{33} + \Gamma^1_{21}\Gamma^2_{33} = \frac{1}{2}a^2\eta\left(1+\xi^2\right)e^G G_\eta + \frac{a^2\left(1+\xi^2\right)e^G}{\left(1-\eta^2\right)} + \frac{1}{2}a^2\xi\left(1-\eta^2\right)e^H G_\xi \tag{2.47}$$

$$R_{332}^{2} = \Gamma_{33,2}^{2} - \Gamma_{33}^{2}\,\Gamma_{32}^{3} + \Gamma_{12}^{2}\Gamma_{33}^{1} + \Gamma_{22}^{2}\Gamma_{33}^{2} = \frac{a^2(1-\eta^2)(\xi^2-1)e^{H}}{(1-\xi^2)} - \frac{1}{2}a^2\xi(1-\eta^2)e^{H}H_{\xi} \quad (2.48)$$

It now follows from (2.33) and (2.37) – (2.39) that in explicit form

$$R_{00} = -\frac{1}{2}e^{G-F}F_{\eta\eta} - \frac{1}{2}e^{G-F}F_{\eta}G_{\eta} + \frac{1}{4}e^{G-F}F_{\eta}^{2} - \frac{1}{2}e^{H-F}F_{\xi\xi} - \frac{1}{2}e^{H-F}F_{\xi}H_{\xi} + \frac{1}{4}e^{H-F}F_{\xi}^{2} + \frac{\eta e^{G-F}}{2(1-\eta^2)}F_{\eta} - \frac{\xi e^{H-F}}{2(1+\xi^2)}F_{\xi} \quad (2.49)$$

Similarly it follows from (2.34) and (2.40) – (2.42) that in explicit form

$$R_{11} = \frac{1}{2}F_{\eta\eta} - \frac{1}{4}e^{H-G}F_{\xi}G_{\xi} + \frac{1}{2}e^{H-G}G_{\xi\xi} - \frac{1}{4}e^{H-G}G_{\xi}^{2} + \frac{1}{4}e^{H-G}H_{\xi}G_{\xi} + \frac{1}{2}H_{\eta\eta} - \frac{1}{4}H_{\eta}^{2} + \frac{1}{4}H_{\eta}G_{\eta} + \frac{1}{(1-\eta^2)^2} + \frac{\eta}{2(1-\eta^2)}G_{\eta} + \frac{\xi e^{H-G}}{2(1+\xi^2)}G_{\xi} \quad (2.50)$$

Similarly, it follows from (2.35) and (2.43) – (2.45) that in explicit form

$$R_{22} = \frac{1}{2}F_{\xi\xi} - \frac{1}{4}F_{\xi}^{2} + \frac{1}{4}F_{\xi}H_{\xi} - \frac{1}{4}e^{G-H}F_{\eta}H_{\eta} + \frac{1}{2}e^{G-H}H_{\eta\eta} - \frac{1}{4}e^{G-H}H_{\eta}^{2} + \frac{1}{4}e^{G-H}H_{\eta}G_{\eta} + \frac{1}{2}G_{\xi\xi} + \frac{1}{4}G_{\xi}H_{\xi} - \frac{1}{4}G_{\xi}^{2} - \frac{\eta e^{G-H}}{2(1-\eta^2)}H_{\eta} - \frac{\xi}{2(1+\xi^2)}H_{\xi} - \frac{1}{(1+\xi^2)^2} \quad (2.51)$$

Similarly, it follows from (2.36) and (2.46) – (2.49) that in explicit form

$$R_{33} = -\frac{1}{2}a^2\eta(1+\xi^2)e^{G}F_{\eta} + \frac{1}{2}a^2\xi(1-\eta^2)e^{H}F_{\xi} + \frac{1}{2}a^2\eta(1+\xi^2)e^{G}G_{\eta} + \frac{a^2(1+\xi^2)e^{G}}{(1-\eta^2)} + \frac{1}{2}a^2\xi(1-\eta^2)e^{H}G_{\xi} - \frac{1}{2}a^2\eta(1+\xi^2)e^{G}H_{\eta} - \frac{1}{2}a^2\xi(1-\eta^2)e^{H}H_{\xi} - \frac{a^2(1-\eta^2)e^{H}}{(1+\xi^2)} \quad (2.52)$$

Next, by contraction the Riemann curvature scalar R defined by

$$R = g^{\alpha\beta}R_{\alpha\beta} = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \quad (2.53)$$

is well defined.

In the first place the Einstein gravitational field equations exterior to the oblate spheroidal body are given by

$$G_{\mu\nu} = 0 \quad (2.54)$$

where $G_{\mu\nu}$ is the Einstein tensor. But by definition

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \quad (2.55)$$

and hence it follows by (2.49) - (2.52) and (2.53) that

$$G_{00} = \frac{1}{2}R_{00} + \frac{1}{2}e^{G-F}R_{11} + \frac{1}{2}e^{H-F}R_{22} + \frac{e^{-F}}{2a^2(1-\eta^2)(1+\xi^2)}R_{33} \quad (2.56)$$

$$G_{11}=\frac{1}{2}e^{F-G}R_{00}+\frac{1}{2}R_{11}-\frac{1}{2}e^{H-G}R_{22}-\frac{e^{-G}}{2a^2\left(1-\eta^2\right)\left(1+\xi^2\right)}R_{33} \tag{2.57}$$

$$G_{22}=\frac{1}{2}e^{F-H}R_{00}-\frac{1}{2}e^{G-H}R_{11}+\frac{1}{2}R_{22}-\frac{e^{-H}}{2a^2\left(1-\eta^2\right)\left(1+\xi^2\right)}R_{33} \tag{2.58}$$

$$G_{33}=\frac{1}{2}a^2\left(1-\eta^2\right)\left(1+\xi^2\right)\left[e^{F}R_{00}-e^{G}R_{11}-e^{H}R_{22}\right]+\frac{1}{2}R_{33} \tag{2.59}$$

It therefore follows from (2.56) - (2.59) and (2.54) that

$$0=e^{F}R_{00}+e^{G}R_{11}+e^{H}R_{22}+\frac{1}{a^2\left(1-\eta^2\right)\left(1+\xi^2\right)}R_{33} \tag{2.60}$$

and

$$0=e^{F}R_{00}+e^{G}R_{11}-e^{H}R_{22}-\frac{1}{a^2\left(1-\eta^2\right)\left(1+\xi^2\right)}R_{33} \tag{2.61}$$

$$0=e^{F}R_{00}-e^{G}R_{11}+e^{H}R_{22}-\frac{1}{a^2\left(1-\eta^2\right)\left(1+\xi^2\right)}R_{33} \tag{2.62}$$

and

$$0=e^{F}R_{00}-e^{G}R_{11}-e^{H}R_{22}+\frac{1}{a^2\left(1-\eta^2\right)\left(1+\xi^2\right)}R_{33} \tag{2.63}$$

Now the explicit forms of the field equations follow from (2.49) – (2.52) as

$$\left.\begin{aligned}0=e^{G}&\left\{H_{\eta\eta}+\frac{1}{4}F_{\eta}^{2}-\frac{1}{2}H_{\eta}^{2}-\frac{1}{2}F_{\eta}G_{\eta}-\frac{1}{4}F_{\eta}H_{\eta}+\frac{1}{2}H_{\eta}G_{\eta}-\frac{3\eta}{2\left(1-\eta^2\right)}H_{\eta}+\frac{\eta}{\left(1-\eta^2\right)^2}G_{\eta}+\frac{2}{\left(1-\eta^2\right)^2}\right\}\\ &+e^{H}\left\{G_{\xi\xi}-\frac{1}{2}G_{\xi}^{2}-\frac{1}{4}F_{\xi}G_{\xi}-\frac{1}{4}F_{\xi}H_{\xi}+\frac{1}{2}H_{\xi}G_{\xi}-\frac{\xi}{\left(1+\xi^2\right)}H_{\xi}+\frac{\xi}{\left(1+\xi^2\right)}G_{\xi}-\frac{2}{\left(1+\xi^2\right)^2}\right\}\end{aligned}\right\} \tag{2.64}$$

$$\left.\begin{aligned}0=e^{G}&\left\{\frac{1}{4}F_{\eta}^{2}+\frac{1}{4}F_{\eta}H_{\eta}-\frac{1}{2}F_{\eta}G_{\eta}+\frac{\eta}{\left(1-\eta^2\right)}F_{\eta}+\frac{3\eta}{2\left(1-\eta^2\right)}H_{\eta}\right\}\\ &+e^{H}\left\{-F_{\xi\xi}+\frac{1}{2}F_{\xi}^{2}-\frac{1}{4}F_{\xi}G_{\xi}-\frac{3}{4}F_{\xi}H_{\xi}-\frac{\xi}{\left(1+\xi^2\right)}F_{\xi}+\frac{\xi}{\left(1+\xi^2\right)}H_{\xi}+\frac{2}{\left(1+\xi^2\right)^2}\right\}\end{aligned}\right\} \tag{2.65}$$

$$\left.\begin{aligned}0=e^{G}&\left\{\frac{1}{4}F_{\xi}^{2}-\frac{1}{2}F_{\eta}G_{\eta}-\frac{1}{4}F_{\eta}H_{\eta}+\frac{\eta}{\left(1-\eta^2\right)}F_{\eta}-\frac{\eta}{\left(1-\eta^2\right)}G_{\eta}-\frac{2}{\left(1-\eta^2\right)^2}\right\}\\ &+e^{H}\left\{\frac{1}{4}F_{\xi}G_{\xi}-\frac{1}{4}F_{\xi}H_{\xi}-\frac{\xi}{\left(1+\xi^2\right)}F_{\xi}-\frac{\xi}{\left(1+\xi^2\right)}G_{\xi}\right\}\end{aligned}\right\} \tag{2.66}$$

$$\left.\begin{aligned} 0 &= e^{G}\left\{-H_{\eta\eta}+\frac{1}{4}F_{\eta}^{2}+\frac{1}{2}H_{\eta}^{2}-\frac{1}{2}F_{\eta}G_{\eta}+\frac{1}{4}F_{\eta}H_{\eta}-\frac{1}{2}H_{\eta}G_{\eta}\right\} \\ &+ e^{H}\left\{-F_{\xi\xi}-G_{\xi\xi}+\frac{1}{2}F_{\xi}^{2}+\frac{1}{2}G_{\xi}^{2}-\frac{3}{4}F_{\xi}H_{\xi}+\frac{1}{4}F_{\xi}G_{\xi}-\frac{1}{2}G_{\xi}H_{\xi}\right\}\end{aligned}\right\} \tag{2.67}$$

Finally it follows by the independence of the functions G and H that the equations (2.64) – (2.67) are equivalent to the following eight equations:

$$\left.\begin{aligned} 0 = H_{\eta\eta}+\frac{1}{4}F_{\eta}^{2}-\frac{1}{2}H_{\eta}^{2}-\frac{1}{2}F_{\eta}G_{\eta}-\frac{1}{4}F_{\eta}H_{\eta}+\frac{1}{2}H_{\eta}G_{\eta} \\ -\frac{3\eta}{2\left(1-\eta^{2}\right)}H_{\eta}+\frac{\eta}{\left(1-\eta^{2}\right)^{2}}G_{\eta}+\frac{2}{\left(1-\eta^{2}\right)^{2}}\end{aligned}\right\} \tag{2.68}$$

and

$$0 = G_{\xi\xi}-\frac{1}{2}G_{\xi}^{2}-\frac{1}{4}F_{\xi}G_{\xi}-\frac{1}{4}F_{\xi}H_{\xi}+\frac{1}{2}H_{\xi}G_{\xi}-\frac{\xi}{\left(1+\xi^{2}\right)}H_{\xi}+\frac{\xi}{\left(1+\xi^{2}\right)}G_{\xi}-\frac{2}{\left(1+\xi^{2}\right)^{2}} \tag{2.69}$$

and

$$0 = \frac{1}{4}F_{\eta}^{2}+\frac{1}{4}F_{\eta}H_{\eta}-\frac{1}{2}F_{\eta}G_{\eta}+\frac{\eta}{\left(1-\eta^{2}\right)}F_{\eta}+\frac{3\eta}{2\left(1-\eta^{2}\right)}H_{\eta} \tag{2.70}$$

and

$$0 = -F_{\xi\xi}+F_{\xi}^{2}-\frac{1}{4}F_{\xi}G_{\xi}-\frac{3}{4}F_{\xi}H_{\xi}-\frac{\xi}{\left(1+\xi^{2}\right)}F_{\xi}+\frac{\xi}{\left(1+\xi^{2}\right)}H_{\xi}+\frac{2}{\left(1+\xi^{2}\right)^{2}} \tag{2.71}$$

and

$$0 = \frac{1}{4}F_{\eta}^{2}-\frac{1}{2}F_{\eta}G_{\eta}-\frac{1}{4}F_{\eta}H_{\eta}+\frac{\eta}{\left(1-\eta^{2}\right)}F_{\eta}-\frac{\eta}{\left(1-\eta^{2}\right)}G_{\eta}-\frac{2}{\left(1-\eta^{2}\right)^{2}} \tag{2.72}$$

and

$$0 = \frac{1}{4}F_{\xi}G_{\xi}-\frac{1}{4}F_{\xi}H_{\xi}-\frac{\xi}{\left(1+\xi^{2}\right)}F_{\xi}-\frac{\xi}{\left(1+\xi^{2}\right)}G_{\xi} \tag{2.73}$$

and

$$0 = -H_{\eta\eta}+\frac{1}{4}F_{\eta}^{2}+\frac{1}{2}H_{\eta}^{2}-\frac{1}{2}F_{\eta}G_{\eta}+\frac{1}{4}F_{\eta}H_{\eta}-\frac{1}{2}H_{\eta}G_{\eta} \tag{2.74}$$

and

$$0 = -F_{\xi\xi}-G_{\xi\xi}+\frac{1}{2}F_{\xi}^{2}+\frac{1}{2}G_{\xi}^{2}-\frac{3}{4}F_{\xi}H_{\xi}+\frac{1}{4}F_{\xi}G_{\xi}-\frac{1}{2}G_{\xi}H_{\xi} \tag{2.75}$$

The results (2.68) – (2.75) are the eight Einstein gravitational field equations exterior to a homogeneous oblate spheroidal body in the natural oblate spheroidal coordinates. They are henceforth opened up for mathematical analysis and integration and solution. Their solutions will pave the way for the extension of general relativistic mechanics from the space-times exterior to perfect spherical bodies to those exterior to spheroidal bodies.

It may be noted that one of the conditions on the metric tensor in the space-time exterior to the oblate spheroidal body is that it should reduce to the flat oblate spheroidal metric tensor in the limit of vanishing rest mass. In other words if $(M_0 \to 0)$ then:

$$g_{00} = e^{-F} \to 1 \tag{2.76}$$

$$g_{11} = -e^{-G} \to -a^{2}\left(\frac{\eta^{2}+\xi^{2}}{1-\eta^{2}}\right) \tag{2.77}$$

$$g_{22} = -e^{-H} \rightarrow -a^2\left(\frac{\eta^2+\xi^2}{1+\eta^2}\right) \tag{2.78}$$

In the second place the Einstein gravitational field equation interior to the oblate spheroidal body is given by

$$G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.79}$$

where G is the universal gravitational constant, c is the speed of light in vacuo, $G_{\mu\nu}$ is the Einstein tensor (2.56) - (2.59) and $T_{\mu\nu}$ is stress-energy tensor. And assuming that the matter is an incompressible fluid the stress-energy tensor can be taken[12] as

$$T_{\mu\nu} = (P_0 + P_0) u_\mu u_\nu - P_0 g_{\mu\nu} \tag{2.80}$$

where ρ_0 is the proper mass density and P_0 is the proper pressure. Furthermore since the body is at rest the spatial components of the four velocity u_μ must vanish so that only u_0 is nonzero. Then since

$$u_\mu u_\nu = 1 \tag{2.81}$$

it follows that

$$u_0 = (g_{00})^{-\frac{1}{2}} \tag{2.82}$$

Consequently,

$$T_{00} = \rho_0 g_{00} = \rho_0 e^{-F} \tag{2.83}$$

$$T_{11} = -P_0 g_{11} = P_0 e^{-G} \tag{2.84}$$

$$T_{22} = -P_0 g_{22} = P_0 e^{-H} \tag{2.85}$$

$$T_{33} = -P_0 g_{33} = P_0 a^2 (1-\eta^2)(1+\xi^2) \tag{2.86}$$

Consequently, substituting (2.83) - (2.86) and (2.56) - (2.59) into (2.79) we obtain the equations

$$-\frac{16\pi G}{c^4}\rho_0 = e^F R_{00} + e^G R_{11} + e^H R_{22} + \frac{1}{a^2(1-\eta^2)(1+\xi^2)} R_{33} \tag{2.87}$$

and

$$-\frac{16\pi G}{c^4} P_0 = e^F R_{00} + e^G R_{11} - e^H R_{22} - \frac{1}{a^2(1-\eta^2)(1+\xi^2)} R_{33} \tag{2.88}$$

$$-\frac{16\pi G}{c^4} P_0 = e^F R_{00} - e^G R_{11} + e^H R_{22} - \frac{1}{a^2(1-\eta^2)(1+\xi^2)} R_{33} \tag{2.89}$$

and

$$\frac{16\pi G}{c^4} P_0 = e^F R_{00} - e^G R_{11} - e^H R_{22} + \frac{1}{a^2(1-\eta^2)(1+\xi^2)} R_{33} \tag{2.90}$$

Now the explicit forms of the field equations follow from (2.49) - (2.52) as

$$-\frac{16\pi G}{c^4}\rho_0 = e^G\left\{H_{\eta\eta} + \frac{1}{4}F_\eta^2 - \frac{1}{2}H_\eta^2 - \frac{1}{2}F_\eta G_\eta - \frac{1}{4}F_\eta H_\eta + \frac{1}{2}H_\eta G_\eta - \frac{3\eta}{2(1-\eta^2)}H_\eta + \frac{\eta}{(1-\eta^2)^2}G_\eta + \frac{2}{(1-\eta^2)^2}\right\}$$
$$+ e^H\left\{G_{\xi\xi} - \frac{1}{2}G_\xi^2 - \frac{1}{4}F_\xi G_\xi - \frac{1}{4}F_\xi H_\xi + \frac{1}{2}H_\xi G_\xi - \frac{\xi}{(1+\xi^2)}H_\xi + \frac{\xi}{(1+\xi^2)}G_\xi - \frac{2}{(1+\xi^2)^2}\right\} \tag{2.91}$$

$$-\frac{16\pi G}{c^4}P_0 = e^G\left\{\frac{1}{4}F_\eta^2 + \frac{1}{4}F_\eta H_\eta - \frac{1}{2}F_\eta G_\eta + \frac{\eta}{\left(1-\eta^2\right)}F_\eta + \frac{3\eta}{2\left(1-\eta^2\right)}H_\eta\right\}$$
$$+ e^H\left\{-F_{\xi\xi} + \frac{1}{2}F_\xi^2 - \frac{1}{4}F_\xi G_\xi - \frac{3}{4}F_\xi H_\xi - \frac{\xi}{\left(1+\xi^2\right)}F_\xi + \frac{\xi}{\left(1+\xi^2\right)}H_\xi + \frac{2}{\left(1+\xi^2\right)^2}\right\} \quad (2.92)$$

$$-\frac{16\pi G}{c^4}P_0 = e^G\left\{\frac{1}{4}F_\xi^2 - \frac{1}{2}F_\eta G_\eta - \frac{1}{4}F_\eta H_\eta + \frac{\eta}{\left(1-\eta^2\right)}F_\eta - \frac{\eta}{\left(1-\eta^2\right)}G_\eta - \frac{2}{\left(1-\eta^2\right)^2}\right\}$$
$$+ e^H\left\{\frac{1}{4}F_\xi G_\xi - \frac{1}{4}F_\xi H_\xi - \frac{\xi}{\left(1+\xi^2\right)}F_\xi - \frac{\xi}{\left(1+\xi^2\right)}G_\xi\right\} \quad (2.93)$$

$$-\frac{16\pi G}{c^4}P_0 = e^G\left\{-H_{\eta\eta} + \frac{1}{4}F_\eta^2 + \frac{1}{2}H_\eta^2 - \frac{1}{2}F_\eta G_\eta + \frac{1}{4}F_\eta H_\eta - \frac{1}{2}H_\eta G_\eta\right\}$$
$$+ e^H\left\{-F_{\xi\xi} - G_{\xi\xi} + \frac{1}{2}F_\xi^2 + \frac{1}{2}G_\xi^2 - \frac{3}{4}F_\xi H_\xi + \frac{1}{4}F_\xi G_\xi - \frac{1}{2}G_\xi H_\xi\right\} \quad (2.94)$$

The results (2.91) – (2.94) are the four Einstein gravitational field equations interior to homogeneous oblate spheroidal body in the oblate spheroidal coordinates. They are henceforth opened up for mathematical analysis and solution. Their solutions will pave the way for the extension of general relativistic mechanics from the space-times interior to perfect spherical bodies to those interior to spheroidal bodies.

3.0 Summary and Conclusion

In this paper we derived the exact and complete Einstein's gravitational field equations in the space-time exterior and interior to a stationary homogeneous oblate spheroidal body (2.68) - (2.75) and (2.91) - (2.94). Consequently, the door is henceforth opened for their mathematical analysis and solution for application to extend general relativistic mechanics from the well known space-time of bodies of spherical geometry to those of spheroidal geometry. The results of the will include the effects of the spheroidal geometry of the Sun on the orbital periods, eccentricities and amplitudes of the planets as well as (i) anomalous orbital precession (ii) gravitational spectral shift, (iii) gravitational time dilation, (iv) gravitational length contraction, (v) radar sounding and (vi) geodetic deviation in the Solar System.

Now it may be most interesting and instructive to compare Einstein's gravitational field equations exterior and interior to a homogeneous oblate spheroidal body in this paper with the corresponding Newton's gravitational field equations derived by us in a paper[20-21] accepted for publication in 2005. It may be remarked that in those two papers we did not only derive the Newton's field equations but we also solved then exactly and completely and went on to apply them to test particles with very interesting results.

Finally the work in this paper may be extended to a homogeneous prolate spheroidal body.

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