Transverse Vibration of Euler-Bernoulli Beams on Elastic Foundation under Mobile Distributed Masses

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Abstract

Uniform distributed moving masses vibration analysis is presented for Euler Bernoulli beams on elastic foundation. The partial differential equation governing the beam's motion is reduced to ordinary differential equation and then expressed as a system of linear equations by finite difference scheme. The analysis is valid for Euler Bernoulli beams with various boundary conditions. However, simply supported boundary conditions are used as an illustrated example. The numerical results are presented in graphical forms and the limiting cases compared well with known existing results. The numerical analysis shows that the foundation stiffness and loads' distribution have significant effects on the dynamic deflection of the beam.

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1.0 Introduction

The dynamic effects of a mobile load play the most significant role in highway and railway bridges in which stresses under moving loads are very much greater the stresses produced by the same static loads. This important consideration has led many researchers to carry out the analysis of beams under the móving masses [1-6]. Many of the publications on the dynamic response of beams to mobile loads have been on concentrated (point) load. However, distributed moving load problems have been the subject of recent investigations [4-6] being more practically realistic than the concentrated problems. Of particular interest in this paper is the work of Esmailzadeh and Ghorashi [4] who studied the problem of vibration of beams traversed by uniform partially distributed moving masses. As mentioned by Lin [8], the result of this investigation is not satisfactory for higher velocities. Another motivation for studying this problem is the fact that the structures of roadway or runway, concrete or reinforced concrete rest on various foundation and of the various foundation models, Pasternak type of foundation is a possible mechanical model for the generalized foundation [7].

Consequently the present study examines the transverse vibration of Euler-Bernoulli beams on elastic foundation under mobile uniform partially distributed masses. The governing equation of motion utilized satisfies both the lower and higher velocities. The cross section of the beam remains constant and the load, moving with constant velocity, maintains a continuous contact with the beam throughout the motion on the beam.

2.0 The Method of Solution

The transverse displacement v(x, t) of a uniform Euler-Bernoulli beam satisfies the partial differential equation [1] where E is the Young's modulus of the beam, I is the constant moment of inertia of the beam cross-section, m is the constant mass per unit length of the beam, q(x, t) is the restoring force due to the foundation reaction, p(x, t) is the impressed force distribution defined by

$$EI\frac{\partial^4 v}{\partial x^4} + m\frac{\partial^2 v}{\partial t^2} = p(x,t) - q(x,t) \tag{2.1}$$

$$p(x,t) = \left[-\overline{M}g - \overline{M}\frac{d^2 v}{dt^2} \right] \frac{1}{(2r)} \{U(x-a) - U(x-b)\} \tag{2.2}$$

$$U(x-a) = \begin{cases} 0, & x \angle a \\ 1, & x \leq a \end{cases}$$

a = δ-r, b = δ +r and r is the half–length of the load as showed in Figure 1

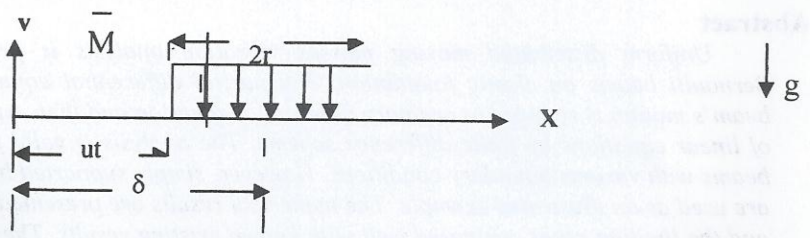


Figure 1

The simplest representations of continous elastic foundation is the Winkler foundation model given by $q = K_1 v$ where K_1 is the modulus of sub-grade reaction. Since a large class of foundation materials in practice caannot be adequately represented by Winkler types of foundation, Pasternak type of foundation which is the most natural extensions of Winkler mode for homogeneous foundations [7] has be used in this analysis.and its restoring force q(x, t) is defined as The total derivative in equation (2.2) is expressed *as* where C is a constant and u is the velocity of the load. The substitution of equation (2.3) into equation (2.1) yields

$$q(x,t) = k_1 v - k_2 v \frac{\partial^2 v}{\partial x^2} + mf \frac{\partial^2 v}{\partial t^2} \tag{2.3}$$

$$\frac{d^2 v}{dt^2} = \frac{\partial^2 v}{at^2} + C\left(2u \frac{\partial^2 v}{\partial x \partial t} + u^2 \frac{\partial^2 v}{\partial x^2} \right) \tag{2.4}$$

$$EI \frac{\partial^4 v}{\partial x^4} + (m + m_f)\frac{\partial^2 v}{\partial t^2} + K_1 v - K_2 \frac{\partial^2 v}{\partial x^2} = p\,(x,t) \tag{2.5}$$

Both the expressions for the displacement and the load may be expanded in series (see reference [6,8]) where $q_i(t)$ and $f_i(t)$ are functions of time to be determined and the $V_i(x)$ are the normalized deflection curves for the i th mode of vibration. Substituting equations (2.4), (2.6) and (2.7) into (2.2) and multiply both sides of the

$$v(x,t) = \sum_{i=1}^{\infty} q_i(t) V_i(x) \tag{2.6}$$

$$p(x,t) = \sum_{i=1}^{\infty} f_i(t)\, V_i(x). \tag{2.7}$$

resultant equation by $V_j(x)$, then taking the integral over the length of the beam gives

$$-\frac{\overline{M}g}{2r}\int_0^L V_j(x)\{U(x-a)-U(x-b)\}dx-\frac{\overline{M}}{2r}\sum_{i-1}^{\infty}\left[\ddot{q}_i(t)\int_0^L V_j(x)V_i(x)\right.$$

$$\left.+2Cu\,\dot{q}_i(t)\int_0^L V_i'(x)V_j(x)+Cu^2q_i(t)\int_0^L V_i''(x)\right][U(x-a)-U(x-b)]dx$$

$$=\sum_{i-1}^{\infty} f_i(t)\int_0^L V_i(x)V_j(x)dx \qquad (2.8)$$

The integrals in equation (2.8) are evaluated with the help of Taylor series expansion and truncating it at the 5th term as in [6]

$$\frac{1}{2r}\int_0^L V_j(x)\{U(x-a)-U(x-b)\}dx\approx V_j(\delta)+\frac{r^2}{3!}V_j^{\;II}(\delta) \qquad (2.9)$$

The properties of orthogonal functions $V_i(x)$ and $V_j(x)$ and the substitution of equations (2.9), (2.10) and (2.11) into equation (2.8) reduce the equation to

$$\frac{1}{2r}\int_0^L V_j(x)V_i'(x)\{U(x-a)-U(x-b)\}dx\approx V_j(\delta)V_i'(\delta)+\frac{r^2}{3!}\left[V_j(\delta)V_i'(\delta)\right]'' \qquad (2.10)$$

and

$$\frac{1}{2r}\int_0^L V_j(x)V_i''(x)\{U(x-a)=U(x-b)\}dx=V_j(\delta)V_i''(\delta)+\frac{r^2}{3!}\left[V_j(\delta)V_i''\,\delta\right]'' \qquad (2.11)$$

$$f_i(t)=-\overline{M}g\left(V_j(\delta)+\frac{r^2}{3!}V_j''(\delta)\right)-\overline{M}\sum_{i=1}^{\infty}\left[\ddot{q}_i(t)\left\{V_j(\delta)V_i(\delta)+\frac{r^2}{3!}\left(V_j(\delta)V_j(\delta)\right)''\right.\right.$$

$$+2Cu\;\dot{q}_i(t)\left\{V_i'(\delta)V_j(\delta)+\frac{r^2}{3!}\left(V_i'(\delta)V_j(\delta)\right)''\right\}+Cu^2q_i(t)\left\{-V_i''(\delta)V_j(\delta)\right.$$

$$\left.\left.+\frac{r^2}{3!}\left(V_i''(\delta)V_j(\delta)\right)''\right\}\right] \qquad (2.12)$$

By virtue of equations (2.6) and (2.7), equation (2.5) becomes

$$\sum_{i=0}^{\infty}\left[\left(EI\,V_i^{iv}(x)+K_1V_i(x)-K_2V_i'''(x)\right)q_i(t)+(m+m_f)\ddot{q}(t)V_i(x)-f_i(t)V_i(x)\right]=0 \qquad (2.13)$$

The equation of free vibration of Euler-Bernoulli beam on Pasternak foundation is

$$EI\,V_i^{iv}(x) + K_1 V_i(x) - K_2 V_i''(x) = p_i^2 (m + m_f) V_i(x) \tag{2.14}$$

where *pi* is the angular frequencies. Applying this free vibration equation, equation (2.13) may now be written as

$$(m + m_f)\ddot{q}_i(t) + p_i^2 (m + m_f) q(t) = -\overline{M}g\left(V_j(\delta) + \frac{r^2}{3!}V_j''(\delta) - \overline{M}\sum_{i=1}^{\infty}\left[\ddot{q}_i(t)\left\{V_j(\delta)V_i(\delta) + \frac{r^2}{3!}\left(V_j(\delta)V_i(\delta)\right)''\right\} + 2Cu\,\dot{q}_i(t)\left\{V_i'(\delta)V_j(\delta) + \frac{r^2}{3!}\left(V_i'(\delta)V_j(\delta)\right)''\right\} + Cu^2 q_i(t)\left\{V_i''(\delta)V_j(\delta) + \frac{r^2}{3!}\left(V_i''(\delta)V_j(\delta)\right)''\right\}\right]\right) \tag{2.15}$$

Equation (2.15) represents the general equation to be solved subject to the boundary conditions of the beam

3.0 Simply Supported Boundary Conditions

Many highway and railway bridges consist of simply supported girders [9]. Therefore, useful insight into the dynamics behaviours of bridges may be obtained by studying the response of simply supported beams under the moving loads.

The boundary conditions of a simply supported beam may be written in the forms

$$v(0,t) = 0 \quad and \quad EI\frac{d^2 v}{dx^2}(0,t) = 0 \tag{3.1}$$

$$v(L,t) = 0 \quad and \quad EI\frac{d^2 v}{dx^2}(L,t) = 0 \tag{3.2}$$

The shapes of the normalized deflection curves for the various modes of vibration satisfying the simply supported boundary conditions are given as follows

$$V_i(x) = \sqrt{(2/L)}\,Sin\frac{i\pi x}{L} \tag{3.3}$$

with corresponding angular frequencies

$$p_i^2 = \frac{EI\left(\frac{i\pi}{L}\right)^4 + K_2\left(\frac{i\pi}{L}\right)^2 + K_1}{m + m_f} \tag{3.4}$$

The expression for $f_i(t)$ is obtained from equation (2.8) by carrying out the direct integration with respect to x having substituted equation (3.3) into it. The truncation error resulted from Taylor series expansion has be completely eliminated by the possibility of direct integration. Hence

$$f_i = -\frac{\sqrt{(2L)}\,\overline{M}g}{rj\pi}Sin\left(\frac{j\pi\delta}{L}\right)Sin\left(\frac{j\pi r}{L}\right) - \frac{\overline{M}}{Lr}\sum_{i=1}^{\infty}\left[\ddot{q}_i(t)Aij + \frac{2Cu(i\pi)}{L}\dot{q}_i(t)Bij - Cu^2 q_i(t)\left(\frac{i\pi}{L}\right)^2 Aij\right] \tag{3.5}$$

where

$$Aij = \begin{cases} \dfrac{L}{\pi(i-j)} Cos\left(\dfrac{(i-j)\pi\delta}{L}\right) Sin\left(\dfrac{(i-j)\pi r}{L}\right) - \dfrac{L}{\pi(i+j)} Cos\left(\dfrac{(i+j)\pi\delta}{L}\right) Sin\left(\dfrac{(i+j)\pi r}{L}\right), & i \neq j \\ r - \dfrac{L}{\pi(i+j)} Cos\left(\dfrac{(i+j)\pi\delta}{L}\right) Sin\left(\dfrac{(i+j)\pi r}{L}\right), & i = j \end{cases}$$

$$Bij = \begin{cases} \dfrac{L}{(i+j)\pi} Sin\left(\dfrac{(i+j)\pi\delta}{L}\right) Sin\left(\dfrac{(i+j)\pi r}{L}\right) - \dfrac{L}{(i-j)\pi} Sin\left(\dfrac{(i-j)\pi\delta}{L}\right) Sin\left(\dfrac{(i-j)\pi r}{L}\right), & i \neq j \\ \dfrac{L}{(i+j)\pi} Sin\left(\dfrac{(i+j)\pi\delta}{L}\right) Sin\left(\dfrac{(i+j)\pi r}{L}\right), & i = j \end{cases}$$

Similar expression for fi(t) corresponding to the point load can be obtained by taking the limit as $r \to 0$ of the right hand side of the equation (3.5)

$$f_i(t)\Big|_{Po\,int\,load} = -\frac{\left(\sqrt{2L}\right)}{L}\overline{Mg}\ Sin\left(\frac{j\pi ut}{L}\right) - \frac{M}{L}\sum_{i=1}^{\infty}\left[\ddot{q}_i(t)\overline{Aij} + \frac{2Cui\pi}{L}\dot{q}_i(t)\overline{Bij} - Cu^2 q_i(t)\left(\frac{i\pi}{L}\right)^2 \overline{Aij}\right] \quad (3.6)$$

where $\overline{Aij} = Cos\left(\frac{(i-j)\pi ut}{L}\right) - Cos\left(\frac{(i+j)\pi ut}{L}\right)$, $\overline{Bij} = Sin\left(\frac{(i+j)\pi ut}{L}\right) - Sin\left(\frac{(i-j)uit}{L}\right)$. Here, the following limit is used, $\lim_{r \to 0} \frac{Sin\ r}{r} = 1$.

The ordinary differential equation to be solved subject to the different boundary conditions of the beam may be written as

$$\left(m + m_f\right)\ddot{q}_i(t) + p_i^2\left(m + m_f\right)q_i(t) = -\frac{\left(\sqrt{2L}\right)Mg}{rj\pi}\sin\left(\frac{j\pi c}{L}\right)\sin\left(\frac{j\pi r}{L}\right) - \frac{\overline{M}}{Lr}\sum_{i=0}^{\infty}\left[\ddot{q}(t)Aij + \frac{2Cui\pi}{L}q_i(t)\left(\frac{i\pi}{L}\right)^2 Aij\right] \quad (3.7)$$

If there is no elastic foundation and C=0, reduced form of equation (3.7) is the same with as the one given by Esmailzadel and Ghorashi in [4]. That is setting $C = K_1 = K_2 = m_f = 0$ in equation (3.7), the equation is consistent with the known result

4.0 Numerical Analysis and Discussion of Result

The above equation (3.7) is solved using central difference method. The method transformed the equation into a system of linear algebraic equations. Computer package is utilized to ensure a fine mesh that produces a higher degree accurate solution.

For the purpose of comparison with the work in [4] the computation is carried out with the data in the analysis defined as follows: $E = 2.07 \times 10^{11}$ N m^2, $I = 1.04 \times 10^{-6}$ m^4, u = 3.333m s, M = 70kg, C = 1, g = 9.81 m/s^2, m = 7.04 kg/m and L= 10m. The beam is supported with a Pasternak type of elastic foundation stiffness (k_1, k_2).

Figure 2 shows the defection profile of the beam at time 0.8 seconds for $c = k_1 = k_2 = 0$ and r = 0.5 which gives the same pattern and result with reference [4]. Moreover, setting c = k_1 = k_2 = 0 and r = 0.5, equation [20] is the same with similar equation in reference [4]

Figure 3 and 4 give the effect of the foundation stiffness on the dynamic behaviours of mid-span of the beam against time for r = 0.5. It shows that the foundation stiffness has a significant effect on the dynamic deflection of the beam. Increased foundation stiffness reduces the deflection of the beam.

Figure 5 shows the dynamic deflection of the beam on elastic foundation stiffness ((k_1,k_2) = (200, 40)) for partially uniform distributed load (r = 0.5) and point load.

The variation of the ratio of the load's length to the beam's length for $(k_1,k_2) = (200, 40)$ is given in Figure 6. An increase in ratio of the length of the load to the beam results in corresponding increase in the maximum deflection for relatively small value while for $\frac{2r}{L} \geq 0 \cdot 37$, $\frac{2r}{L} < 0 \cdot 37$ a relatively sharp decrease in the maximum deflection is recorded.

5.0 Conclusions

This paper presents a vibration analysis for Euler-Bernoulli beams resting on non-Winkler foundation under the action of partially uniform distributed masses. The governing characteristic differential equation is reduced to ordinary coupled differential equation by employing separation of variables which are expressed in series forms. Recourse was made to finite difference method in order to solve the ordinary coupled differential equation obtained. To illustrate the validity and accuracy of the analysis, some limiting cases and their numerical results are presented. The presentation is valid for different boundary conditions. However, simply supported boundary conditions are used in the numerical analysis. The result shows that the foundation stiffness and the load's length has a significant effect on the dynamic deflection of the beam.

Figure 2

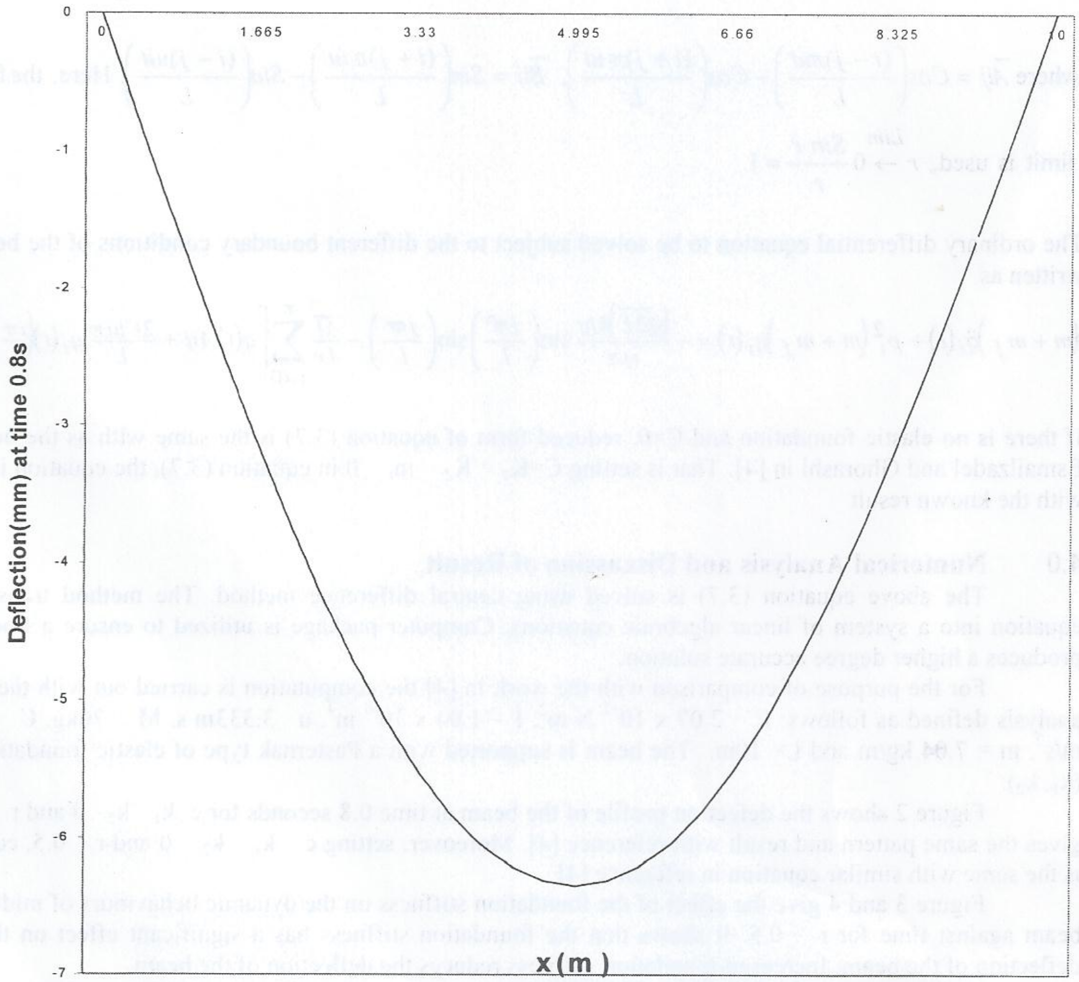
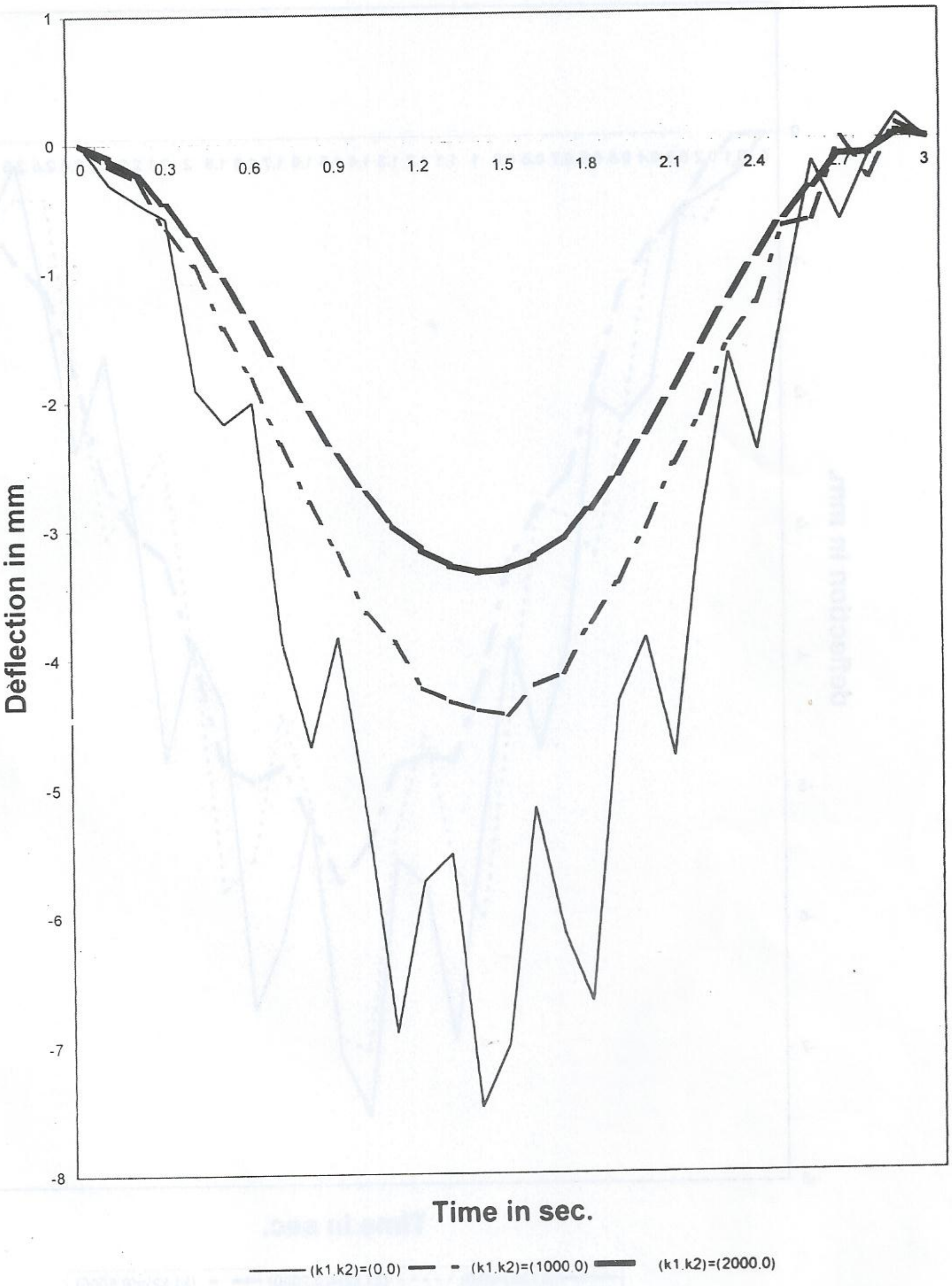


Figure 3.



1
0
-1
-2
-3
-4
-5
-6
-7
-8
0
0.3
0.6
0.9
1.2
1.5
1.8
2.1
2.4
3
Deflection in mm
Time in sec.
(k1,k2)=(0,0)
(k1,k2)=(1000,0)
(k1,k2)=(2000,0)

Figure 4.

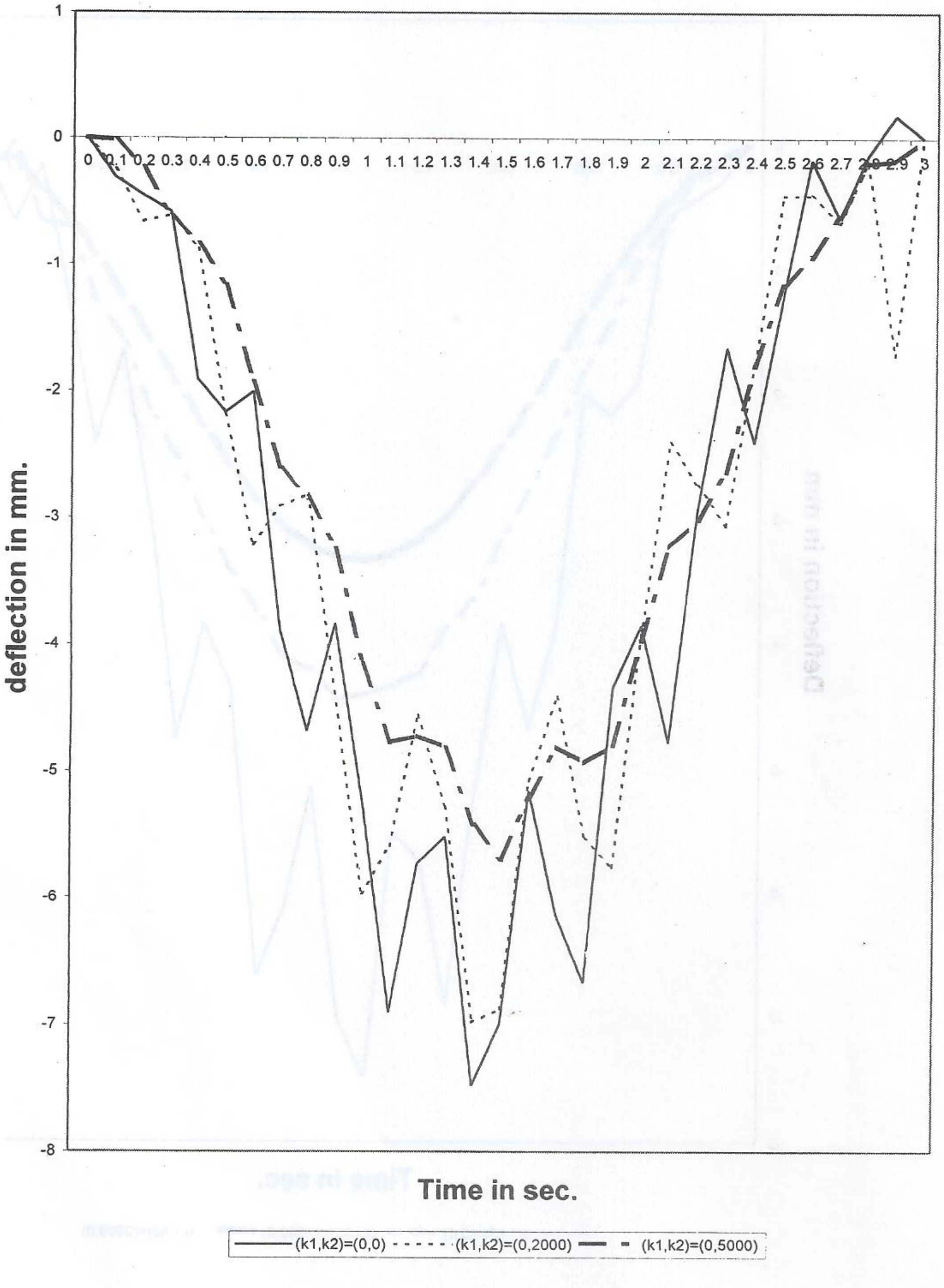

1
0
-1
-2
-3
-4
-5
-6
-7
-8
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 2.9 3
deflection in mm.
Time in sec.
(k1,k2)=(0,0)
(k1,k2)=(0,2000)
(k1,k2)=(0,5000)

Figure 5.

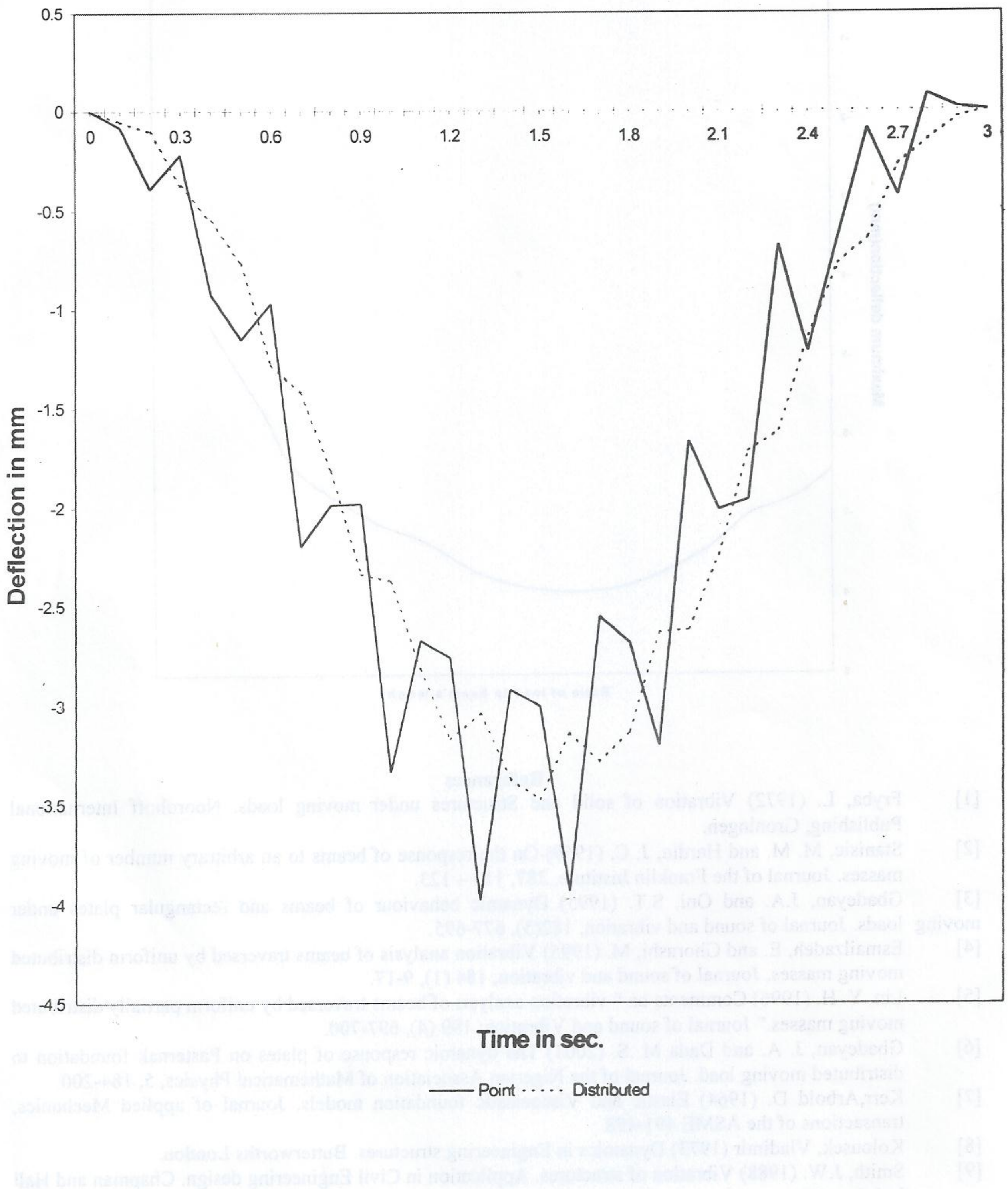
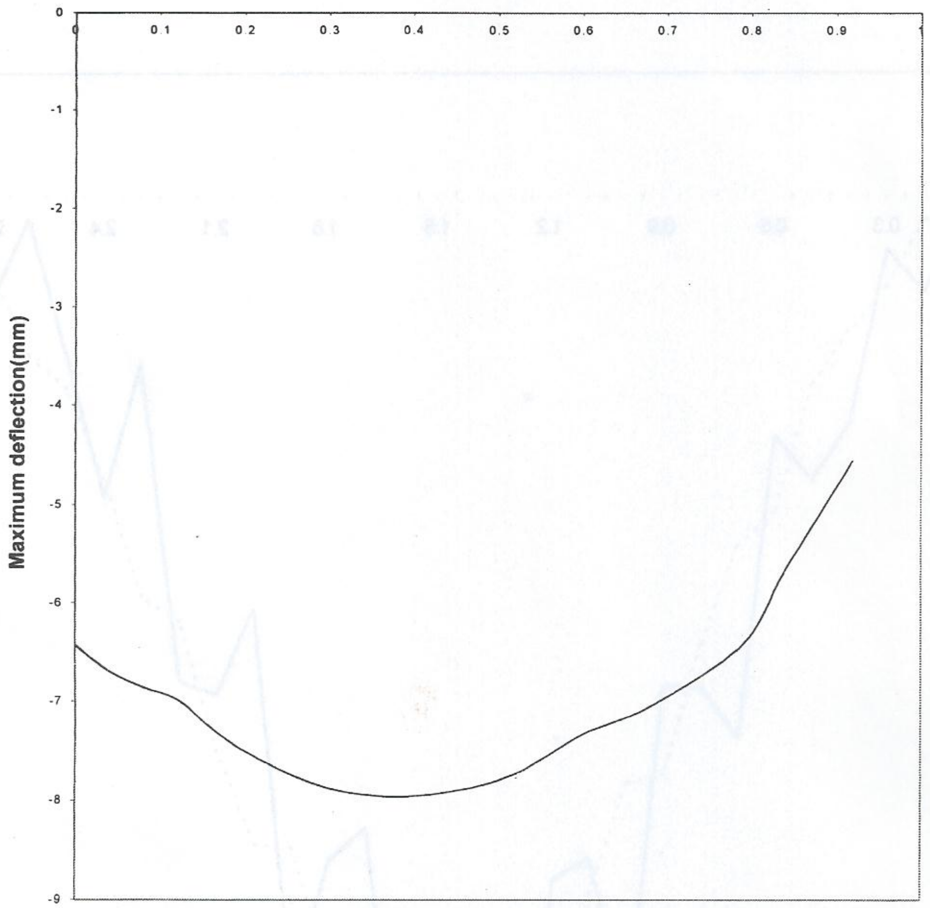

0.5
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
-4
-4.5
0
0.3
0.6
0.9
1.2
1.5
1.8
2.1
2.4
2.7
3
Deflection in mm
Time in sec.
Point
Distributed

Figure 6.



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