Antiplane Shear Analysis for a Non-homogeneous Semi-infinite Layer

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Abstract.

Closed form stress field, $\sigma_{iyz}(x,0)$, $i = 1,2$, near the free surface-interface junction of a nonhomogeneous layer of height, h are obtained and shown to depend on material constants if loads are applied on straight line segments of unequal lengths on the free surface. The special case of self-equilibrating loads applied on segments of equal lengths lead to independence of stresses on material constants. Stresses are not singular at the junction. In a graph, we study the variation of $\frac{\sigma_{yz}(x,0)}{T_1}$ with $\frac{b_1}{h}$ as the length, $a_1 - b_1$, of loaded segment increases, when a_1 is proportional to b_1.

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1.0 Introduction

Layers made of different materials are used to construct a solid semi-infinite layer of height *h*, perfectly bonded along their common surface so that stress and displacement fields are continuous across the interface. Anti-plane shear loads of magnitudes T_i, $i = 1,2$, are applied on straight line segments of the free surface so that the first material, of elastic constant μ_1, has displacement $w_1(x,y)$ in the *z*-direction while the second material, of elastic constant μ_2, experiences the only non-vanishing displacement $w_2(x,y)$ also in the *z*-direction. The applied loads are not necessarily symmetric with respect to the origin. The subscript *i* = 1 will be attached to items concerning materials 1 while *i* = 2 will be associated with items of materials 2 (see Figure 1).

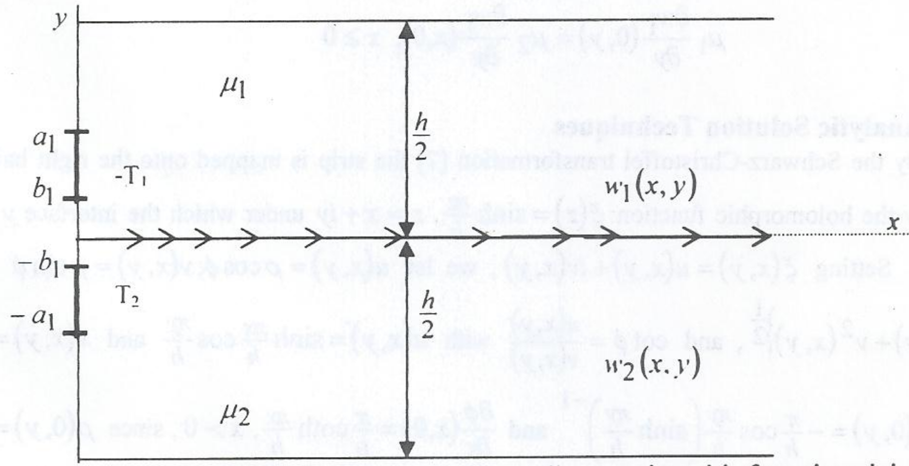

Figure 1: Bonded Layer of height *h*, and loads sites at distances b_1 and b_2 from the origin

Fractures parameters arising from bonded dissimilar materials with interfacial cracks often require the knowledge of stress raising character of the materials at the interface, in the absence of the crack. Rigorous analysis different from the one used in investigating crack tip stresses, is often called into play when stresses are studied in a non-homogeneous uncracked material of the same geometry as the cracked one (see example [1] cited in [2] and [3] cited in [4]). Our technique of analysis differs from those in [1-4]. A related problem for a cracked homogeneous semi-infinite layer known to us was solved in [5].

2.0 Governing Equations

The differential equation governing the problem is

$$\nabla^2 w_i(x,y)=0,\ x\geq 0,\ -\tfrac{h}{2}\leq y\leq \tfrac{h}{2},\ i=1,2 \tag{2.1a}$$

$$\left(\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\ \text{is the Cartesian coordinate Laplacian}\right)$$

The stress are given in terms of displacement gradients as [6]

$$\sigma_{ixz}(x,y)=\mu_i\frac{\partial w_i}{\partial x}(x,y),\quad \sigma_{iyz}(x,y)=\mu_i\frac{\partial w_i}{\partial x}(x,y) \tag{2.1b}$$

The prescribed stresses are

$$\sigma_{1xz}(0,y)=-\mathrm{T}_1,\ b_1\leq y\leq a_1;\ \mu_i\frac{\partial w_i}{\partial x}(x,y),\ \sigma_{2yz}(0,y)=\mathrm{T}_2,\ -a_2\leq y\leq -b_2$$

$$b_i\geq 0,\ a_i\leq \tfrac{h}{2},\ b_i\prec a_i,\ i=1,2.$$

There (2.1a) has to be solved subject to the boundary conditions :

$$\frac{\partial w_1}{\partial x}(0,y)=\frac{-\mathrm{T}_1}{\mu_1},\qquad b_1\leq y\leq a_1 \tag{2.2a}$$

$$\frac{\partial w_2}{\partial x}(0,y)=\frac{\mathrm{T}_2}{\mu_2}\qquad -a_2\leq y\leq -b_2 \tag{2.2b}$$

$$\frac{\partial w_i}{\partial x}(0,y)=0,\qquad i=1,\ 2,\ otherwise \tag{2.2c}$$

and the continuity conditions

$$w_1(x,0)=w_2(x,0); \tag{2.3a}$$

$$\mu_1\frac{\partial w_1}{\partial y}(0,y)=\mu_2\frac{\partial w_1}{\partial y}(x,0),\ x\geq 0 \tag{2.3b}$$

3. Analytic Solution Techniques

By the Schwarz-Christoffel transformation [7] the strip is mapped onto the right half of a ξ − plane defined by the holomorphic function: $\xi(z)=\sinh\frac{\pi z}{h}$, $z=x+iy$ under which the interface $y=0$, is invariant, Figure 2. Setting $\xi(x,y)=u(x,y)+iv(x,y)$, we let $u(x,y)=\rho\cos\phi; v(x,y)=\rho\sin\phi$ so that $\rho(x,y)=\left\{u^2(x,y)+v^2(x,y)\right\}^{\frac{1}{2}}$, and $\cot\phi=\frac{u(x,y)}{v(x,y)}$ with $u(x,y)=\sinh\frac{\pi x}{h}\cos\frac{\pi y}{h}$ and $v(x,y)=\cosh\frac{\pi x}{h}\sin\frac{\pi y}{h}$.

Then $\frac{\partial\phi}{\partial x}(0,y)=-\frac{\pi}{h}\cos\frac{\pi y}{h}\left(\sinh\frac{\pi y}{h}\right)^{-1}$ and $\frac{\partial\phi}{\partial x}(x,0)=\frac{\pi}{h}\coth\frac{\pi x}{h}$, $x\succ 0$, since $\rho(0,y)=\left\{v^2(0,y)\right\}^{1/2}=\left|\sin\frac{\pi y}{h}\right|$, $\rho(x,0)=\sinh\frac{\pi x}{h}$ and $\cos\frac{\pi x}{h}=\left(1-\sin^2\frac{\pi y}{h}\right)^{\frac{1}{2}}$ it is easily seen that for $\rho\prec 1$

$$\frac{\partial\phi}{\partial x}(0,y)=-\frac{\pi}{h}\rho^{-1}\left(1-\rho^2\right)^{\frac{1}{2}},\quad 0\prec y\leq\frac{h}{2},\quad \phi=\frac{\pi}{2} \tag{3.1a}$$

$$= \frac{\pi}{h}\rho^{-1}\left(1-\rho^2\right)^{\frac{1}{2}}, -\frac{h}{2} \le y \prec 0,\ \phi = -\frac{\pi}{2} \tag{3.1b}$$

$$\frac{\partial \phi}{\partial x}(0,y) = 0,\ -\frac{h}{2} \le y \le \frac{h}{2},\ \phi \pm \frac{\pi}{2} \tag{3.2a}$$

$$\frac{\partial \phi}{\partial x}(x,0) = 0,\ \ x \succ 0 \tag{3.2b}$$

As a consequence of the conformal mapping $w_i(x,y) \equiv w_i(\rho,\phi),\ i = 1,2.$ Hence for $\rho \prec 1,\ -\frac{h}{2} \le y \le \frac{h}{2}$

$$\frac{\partial w_i}{\partial x}(0,y) = \frac{\partial w_i}{\partial \phi}\left(\rho, \pm\frac{\pi}{2}\right)\frac{\partial \phi}{\partial x}(0,y),\ i = 1.2 \tag{3.3}$$

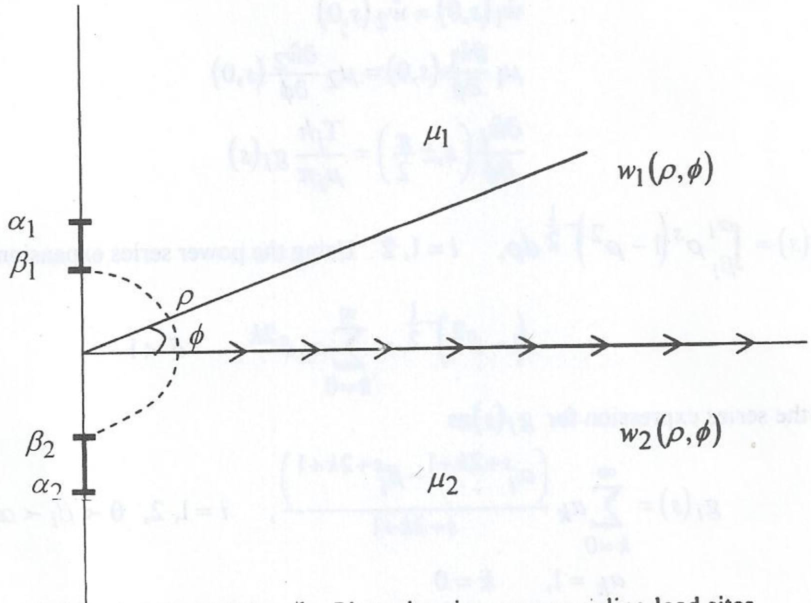

Figure 2: The right half of the ξ – Plane showing corresponding load sites

Therefore, instead of solving (2.1) to (2.3), we solve for $w_i(\rho,\phi),\ i = 1,2$ in the problem

$$\nabla^2 w_i(\rho,\phi) = 0,\ -\frac{\pi}{2} \le \phi \le \frac{\pi}{2},\ \rho \succ 0,\ i = 1.2 \tag{3.4}$$

$$\left(\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right)$$

$$w_1(\rho,\phi) = w_2(\rho,\phi), \tag{3.5a}$$

$$\mu_1\frac{\partial w_1}{\partial \phi}(\rho,0) = \mu_2\frac{\partial w_2}{\partial \phi}(\rho,0),\ \rho \succ 0 \tag{3.5b}$$

$$\frac{\partial w_1}{\partial \phi}\left(\rho, \pm\frac{\pi}{2}\right) = \frac{T_i h}{\mu_i \pi}\rho\left(1-\rho^2\right)^{-\frac{1}{2}},\ \beta_i \prec \rho \prec \alpha_i,\ i = 1,2 \tag{3.6a}$$

$$= 0 \ \text{otherwise} \tag{3.6b}$$

$$\alpha_i = \sin\frac{\pi a_i}{h};\ \beta_i = \sin\frac{\pi b_i}{h},\ i = 1,2 \tag{3.6c}$$

The boundary conditions (3.5a, b, c) and (3.6a,b,c) were obtained using (2.2a,b,c) to (3.3). Considering the equation in (3.6a,b,c), the asymptotic behaviours of the displacements $w_i(\rho,\phi) = 0(\rho)$ as $\rho \to 0$, and $w_i(\rho,\phi) = 0(1)$ as $\rho \to \infty$. $i = 1, 2$ To analyze (3.4) to (3..6,a,b,c) by method of integral transform, we apply to them, the Mellin integral transform of $w_i(\rho,\phi)$ defined by

$$\hat{w}_i(s,\phi)\int_0^\infty w_j(\rho,\phi)\rho^{s-1}d\rho,\quad -1 \prec s \prec 1,\ i = 1, 2$$

to get the ordinary differential equation

$$\frac{d^2}{d\phi^2}\hat{w}_i + s^2\hat{w}_i = 0,\ -1 \prec s \prec 1,\quad i = 1.2 \tag{3.7}$$

$$\hat{w}_1(s,0) = \hat{w}_2(s,0) \tag{3.8a}$$

$$\mu_1\frac{\partial\hat{w}_1}{\partial\phi}(s,0) = \mu_2\frac{\partial\hat{w}_2}{\partial\phi}(s,0) \tag{3.8b}$$

$$\frac{\partial\hat{w}_i}{\partial\phi}\left(s,\pm\frac{\pi}{2}\right) = \frac{\mathrm{T}_i h}{\mu_i\pi}g_i(s) \tag{3.9a}$$

where $g_i(s) = \int_{\beta_i}^{\alpha_i}\rho^s\left(1-\rho^2\right)^{-\frac{1}{2}}d\rho,\quad i = 1, 2$. Using the power series expansion [8]

$$\left(1-\delta^2\right)^{-\frac{1}{2}} = \sum_{k=0}^{\infty}a_k\delta^{2k},\quad \delta \prec 1 \tag{3.9b}$$

we obtain the series expression for $g_i(s)$ as

$$g_i(s) = \sum_{k=0}^{\infty}a_k\frac{\left(\alpha_i^{s+2k+1} - \beta_i^{s+2k+1}\right)}{s+2k+1},\quad i = 1, 2,\ 0 \prec \beta_i \prec \alpha_i \tag{3.10}$$

$$a_k = 1,\quad k = 0$$

$$= \frac{1}{2^k k!}\prod_{m=1}^{k}(2m-1)\quad k \succ 0$$

Analysis of the solution of (3.7) given as

$$\hat{w}_i(s,\phi) = \mathrm{A}_i(s)\cos\phi + \mathrm{B}_i(s)\sin s\phi,\ i = 1,2 \tag{3.11}$$

subject to (3.8a,b) and (3.9a,b) lead to

$$\mathrm{A}_1(s) = \mathrm{A}_2(s), \tag{3.12a}$$

$$\mu_1\mathrm{B}_1(s) = \mu_2\mathrm{B}_2(s), \tag{3.12b}$$

$\mathrm{A}_1(s)\sin\frac{\pi}{2}s + \mathrm{B}_1(s)\cos\frac{\pi}{2}s = \frac{\mathrm{T}_1 h g_1(s)}{\mu_1\pi s}$, $\mathrm{A}_2(s)\sin\frac{\pi}{2}s + \mathrm{B}_2(s)\cos\frac{\pi}{2}s = \frac{\mathrm{T}_2 h g_2(s)}{\mu_2\pi s}$. Thus

$$\left[\mathrm{A}_2(s) - \mathrm{A}_1(s)\right]\sin\frac{\pi}{2}s + \left[\mathrm{B}_1(s) + \mathrm{B}_2(s)\right]\cos\frac{\pi}{2}s = \frac{h}{\pi s}\left[\frac{\mathrm{T}_1}{\mu_1}g_1(s) + \frac{\mathrm{T}_2}{\mu_2}g_2(s)\right]$$

In view of (3.12a,b) $\mathrm{B}_1(s) = \frac{\mu_2}{(\mu_1+\mu_1)\pi}\left[\frac{\mathrm{T}_1}{\mu_1}g_1(s) + \frac{\mathrm{T}_2}{\mu_2}g_2(s)\right]\frac{1}{s\cos\frac{\pi}{2}s}$ Also (3.12a,b) together with

$[A_2(s)-A_1(s)]\sin\frac{\pi}{2}s+[B_2(s)+B_1(s)]\cos\frac{\pi}{2}s=\frac{h}{\pi s}\left[\frac{T_2}{\mu_2}g_2(s)+\frac{T_1}{\mu_1}g_1(s)\right]$ lead to

$$A_1(s)=\frac{h}{2\pi}\left[(1+\gamma)\frac{T_2}{\mu_2}g_2(s)-(1+\gamma)\frac{T_1}{\mu_1}g_1(s)\right]\frac{1}{s\sin\frac{\pi}{2}s}$$

where $\lambda=\frac{\mu_2-\mu_1}{\mu_2+\mu_1}$. The expression for $B_2(s)$ is obtained from that of $B_1(s)$ by applying (3.12b). Hence (3.11) can be written as

$$\hat{w}_i(\rho,\phi)=\frac{h}{2\pi}\left\{\left[(1+\gamma)\frac{T_2}{\mu_2}g_2(s)-(1+\gamma)\frac{T_1}{\mu_1}g_1(s)\right]\frac{\cos s\phi}{s\sin\frac{\pi}{2}s}\right.$$
$$\left.+\frac{1}{\mu_i}[(1+\gamma)T_1g_1(s)+(1+\gamma)T_2g_2(s)]\frac{\sin s\phi}{s\cos\frac{\pi}{2}s}\right\} \tag{3.13}$$

The displacement fields $w_i(\rho,\phi)$, $i=1,2$ are derived from the inverse Mellin transform denoted by

$$w_j(\rho,\phi)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\hat{w}_j(s,\phi)\rho^{s-1}ds,\quad -1\prec c\prec 1,\quad j=1,2 \tag{3.14}$$

The residue theorem [7] which will be applied in evaluating (3.14) require considerations obtained by setting $G\left(s,\frac{\rho}{t_i}\right)=\sum_{k=0}^{\infty}a_k\frac{\left(a_k t^{2k+1}\right)}{s+2k+1}\left(\frac{\rho}{t_i}\right)^{-s}$, $i=1,2$. Using (3.10), we get

$$g_i(s)\rho^{-s}=G\left(s,\frac{\rho}{\alpha_i}\right)-G\left(s,\frac{\rho}{\beta_i}\right), i=1,2,\quad \beta_i\prec\alpha_i\prec 1$$

Note that for $b_i=\succ 0, i=1,2$ and $n=1,2,3,\cdots$. $g_i(s)$ has removable singularities at $s=1-2n$ and that

$$g_i(1-2n)=a_{n-1}\mathrm{In}\left(\frac{\alpha_i}{\beta_i}\right)+\sum_{\substack{k=0\\k\neq n-1}}^{\infty}a_k\frac{\left(\alpha_i^{2(k+1-n)}-\beta_i^{2(k+1-n)}\right)}{2(k+1-n)} \tag{3.15}$$

Each $G\left(s,\frac{\rho}{t_1}\right)$, $i=1,2$ enter the analysis by consideration of the validity of $w_i(\rho,\phi)$ for $0\prec\rho\prec\beta_i$, $\beta_i\prec\rho\prec\alpha_i$ and $\rho\prec\alpha_i$. The form of $w_i(\rho,\phi)$ is obtained for $\beta_i\prec\rho\prec\alpha_i$ by writing (3.14) as

$$\hat{w}_k(\rho,\phi)=\left(\frac{h}{2\pi}\right)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left\{\left\{(1+\gamma)\frac{T_2}{\mu_2}\left[G\left(s,\frac{\rho}{\alpha_2}\right)-G\left(s,\frac{\rho}{\beta_2}\right)\right]-(1-\gamma)\frac{T_1}{\mu_1}\left[G\left(s,\frac{\rho}{\alpha_1}\right)\right.\right.\right.$$
$$\left.\left.-G\left(s,\frac{\rho}{\beta_1}\right)\right]\right\}\frac{\cos s\phi}{s\sin\frac{\pi}{2}s}+\frac{1}{\mu_k}\left\{(1+\gamma)T_i\left[G\left(s,\frac{\rho}{\alpha_2}\right)-G\left(s,\frac{\rho}{\beta_2}\right)\right]+(1+\gamma)T_2\left[G\left(s,\frac{\rho}{\alpha_2}\right)-G\left(s,\frac{\rho}{\beta_2}\right)\right]\right. \tag{3.16}$$
$$\left.\left.-\left[G\left(s,\frac{\rho}{\alpha_2}\right)-G\left(s,\frac{\rho}{\beta_2}\right)\right]\right\}\frac{\sin s\phi}{s\cos\frac{\pi}{2}s}\right\}ds,\ k=1,2$$

For this case, we observe that $\frac{\rho}{\alpha_i} \prec 1$ *and* $\frac{\rho}{\beta_i} \succ 1$ simultaneously. Thus, the contours involving $G\left(s, \frac{\rho}{\alpha_i}\right)$ are closed the left half plane Res $\prec 0$, while contours involving $G\left(s, \frac{\rho}{\beta_i}\right)$ are closed in the right half-plane Res $\succ 0$ as prescribed by Jordan's lemma. Next, the integrand is separated into terms involving $G\left(s, \frac{\rho}{\alpha_i}\right)$ and those involving $G\left(s, \frac{\rho}{\beta_i}\right)$ and we consider the simple poles of $G\left(s, \frac{\rho}{\alpha_i}\right)\frac{\cos\phi}{s\sin\frac{\pi}{2}s}$ at $s=-2n$ and $s=-(2n-1)$, $n=1,2,3,\cdots$ as well as the double poles of $G\left(s, \frac{\rho}{\alpha_i}\right)\frac{\sin s\phi}{s\cos\frac{\pi}{2}s}$, $i=1,2$ at $s=-(2n-1), n=1,2,3,\cdots$. All the poles of $G\left(s, \frac{\rho}{\beta_i}\right)\frac{\cos\phi}{s\sin\frac{\pi}{2}s}$ and $G\left(s, \frac{\rho}{\beta_i}\right)\frac{\sin s\phi}{s\cos\frac{\pi}{2}s}$, $i=1,2$, at $s=2n$ and $s=2n-1$, $n=1,2,3,\cdots$ respectively, are simple. Consequently, for $\beta_i \prec \rho \prec \alpha_i$, $i=1,2$

$$w_i(\rho,\phi)=\frac{h}{2\pi}\left\{\left\{(1+\gamma)\frac{T_2}{\mu_2}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_{n-1}\rho^{2n-1}\cos(2n-1)\phi-\frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}a_k\frac{\alpha_2^{2k-2n+1}}{2k-2n+1}\rho^{2n}\cos 2n\phi\right]\right.\right.$$

$$-(1-\gamma)\frac{T_1}{\mu_1}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_{n-1}\rho^{2n-1}\cos(2n-1)\phi-\frac{2}{\pi}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}a_k\frac{\alpha_1^{2k-2n+1}}{2k-2n+1}\rho^{2n}\cos 2n\phi\right]\Bigg\}+\frac{1}{\mu_1}\Big[(1+$$

$$\gamma)T_1\overline{G}\left(1-2n,\frac{\rho}{\alpha_1}\right)+(1-\gamma)T_2\overline{G}\left(1-2n,\frac{\rho}{\alpha_2}\right)\Big]\Bigg\}-\frac{h}{2\pi}\Bigg\{\frac{2}{\pi}(1-\gamma)\frac{T_1}{\mu_1}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}a_k\frac{\beta_1^{2k-2n+1}}{2k-2n+1}\rho^{-2n}\cos 2n\phi$$

$$-\frac{2}{\pi}(1+\gamma)\frac{T_2}{\mu_2}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}a_k\frac{\beta_2^{2k-2n+1}}{2k-2n+1}\rho^{-2n}\cos 2n\phi+\frac{1}{\mu_i}\Bigg\{\frac{2}{\pi}\Big[(1-\gamma)T_1\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_k\frac{\beta_1^{2(k-n)}}{2(k+n)}$$

$$+(1-\gamma)T_2\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_k\frac{\beta_2^{2(k+n)}}{2(k+n)}\Big]\rho^{1-2n}\sin(2n-1)\phi\Bigg\}\Bigg\} \tag{3.17}$$

where

$$\overline{G}\left(1-2n,\frac{\rho}{\alpha_i}\right)=\lim_{s\to 1-2n}\frac{d}{ds}\left\{(s+2n-1)^2G\left(s,\frac{\rho}{\alpha_i}\right)\frac{\sin s\phi}{s\cos\frac{\pi}{2}s}\right\}=\frac{2}{\pi}\sum_{\substack{k=0\\ k\neq n-1}}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)}\frac{\alpha_i^{2(k+1-n)}}{2(k+2n-1)}\rho^{2n-1}\times$$

$$\left.\sin(2n-1)\phi-\ln\frac{\rho}{\alpha_i}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_{n-1}\rho^{2n-1}\sin(2n-1)\phi-\phi\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_{n-1}\rho^{2n-1}\cos(2n-1)\phi\right\}$$

The expression in (3.9b) may be useful in ascertaining that (3.17) fulfils the boundary conditions (3.6a). On the interval $0 \prec \rho \prec \beta_i$, $i=1,2$, the conditions $\frac{\rho}{\alpha_i} \prec 1$ and $\frac{\rho}{\beta_i} \prec 1$ area simultaneously satisfied. Therefore, the integrand in (3.14) is precisely the equation given in (3.13). The contour is closed in the left half-plane, Res $\prec 0$ where $g_i(s)$, $i=1,2$ have no singularities. There are poles at $s=-2n$ and $s=-(2n-1)$, $n=1,2,3,\cdots$. Hence $0 \prec \rho \prec \beta_i$, $i=1,2$

$$w_i(\rho,\phi)=\frac{h}{\pi^2}\left\{-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\left[(1+\gamma)\frac{T_2}{\mu_2}g_2(-2n)-(1-\gamma)\frac{T_1}{\mu_2}g_1(-2n)\right]\rho^{2n}\cos 2n\phi\right.$$
$$\left.+\frac{1}{\mu}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left[(1+\gamma)T_1g_1(1-2n)+(1+\gamma)T_2g_2(1-2n)\right]\rho^{2n-1}\sin(2n-1)\phi\right\},\ i=1,2 \quad (3.18)$$

4.0 The Fields Near the free Surface-interface Junction

The displacement fields near the free surface-interface junction are obtained from (3.18) as $\rho\to 0$ are given by

$$w_i(\rho,\phi)=\frac{h}{\mu_i\pi}\left[(1-\gamma)T_1g_1(-1)+(1-\gamma)T_2g_2(-1)\right]\rho\sin\phi,\quad \text{as}\ \rho\to 0,\ i=1,2 \quad (4.1)$$

In terms of the original Cartesian coordinates, (4.1) becomes

$$w_i(x,y)=\frac{h}{\mu_i\pi}\left[(1-\gamma)T_1g_1(-1)+(1-\gamma)T_2g_2(-1)\right]y(x,y)\quad x\to 0,\ y\to 0,\ 0,\ i=1,2 \quad (4.2)$$

Now from (3.15), $g_i(-1)=\ln\frac{\alpha_i}{\beta_i}+\sum_{k=1}^{\infty}a_k\frac{\alpha_i^{2k}}{2k}-\sum_{k=1}^{\infty}a_k\frac{\beta_i^{2k}}{2k}$, $i=1,2$. But $\sum_{k=1}^{\infty}a_k\frac{t^k}{k}=-2\ln\left(1+\sqrt{1-t}\right)+$ $2\ln$, $|t|\prec 1$ implies that $\sum_{k=1}^{\infty}a_k\frac{t^{2k}}{2k}=2\ln\left(1+\sqrt{1-t^2}\right)$. Therefore, for $b_i\succ 0(\beta_i\succ 0)$, $i=1,2$.

$$g_i(-1)=\ln\left(\frac{\sin\frac{\pi a_i}{h}}{\sin\frac{\pi b_i}{h}}\right)+\ln\left(\frac{1+\cos\frac{\pi b_i}{h}}{1+\cos\frac{\pi a_i}{h}}\right),\ i=1,2 \quad (4.3)$$

Thus, with (4.3) the junction neighbourhood fields can be expressed in close form. The stress fields are obtained using (2.1b) and (4.2) as

$$\sigma_{ixz}(x,y)=\frac{1}{\pi}\sinh\frac{\pi}{h}\sin\frac{\pi y}{h}\left[(1-\gamma)T_1g_1(-1)+(1-\gamma)T_2g_2(-1)\right],\ i=1,2 \quad (4.4a)$$

$$\sigma_{iyz}(x,y)=\frac{1}{\pi}\cosh\frac{\pi}{h}\cos\frac{\pi y}{h}\left[(1-\gamma)T_1g_1(-1)+(1-\gamma)T_2g_2(-1)\right],\ i=1,2 \quad (4.4b)$$

Equations (4.2) and (4.3) agree with (2.3a,b) near the origin. Hence, under this mode of loading the interface near the origin is not displaced but experiences stress given by

$$\sigma_{iyz}(x,0)=\frac{1}{\pi}\left[(1+\gamma)T_1g_1(-1)+(1-\gamma)T_2g_2(-1)\right]\ \text{as}\ x\to 0=\sigma_{yz}(0,0) \quad (4.5)$$

(the stress at the free surface-interface junction)

On the other hand, if $b_i=0$ (*i.e.* $\beta_i=0$), $i=1,2$, the loading begins from the origin and so the analysis changes slightly because $g_i(s)=\sum_{k=1}^{\infty}a_k\frac{\alpha_i^{s+2k+1}}{s+2k+1}$, implies that $g_i(s)$ now has simple poles $s=-(2n-1)$, $n=1,2,3,\cdots$. Substituting this form of $g_i(s)$ into (3.13) and inserting the result in (3.14) and noting that the interval $0\prec\rho\prec\beta_i$, $i=1,2$ vanishes while $0\prec\rho\prec\alpha_i$ and $\rho\succ\alpha_i$ remain, the solution for $0\prec\rho\prec\alpha_i$ is

$$w_i(\rho,\phi)=\frac{h}{2\pi^2\mu_i}\left\{\frac{\pi}{2}\mu_i\left[(1+\gamma)\frac{T_2}{\mu_2}-(1-\gamma)\frac{T_1}{\mu_1}\right]\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}a_{n-1}\rho^{2n-1}\cos(2n-1)-\sum_{n=1}^{\infty}\frac{(-1)^n}{2n}\left[(1+\gamma)\frac{T_2}{\mu_2}g_2(-2n)\right.\right.$$

$$-(1-\gamma)\frac{\mathrm{T}_1}{\mu_1}\hat{g}_1(-2n)\Big]\rho^{2n}\cos 2n\phi+\frac{\pi}{2}\left[(1+\gamma)\overline{G}\left(1-2n,\frac{\rho}{\alpha_1}\right)+(1-\gamma)\overline{G}\left(1-2n,\frac{\rho}{\alpha_2}\right)\right]\Bigg\}\quad i=1,2$$

Close to the free surface-interface junction, the displacement fields are obtained as $\rho \to 0$. The result is

$$w_i(\rho,\phi)=\frac{h}{2\pi^2\mu_i}\left\{\left(\frac{\pi\mu_i}{2}\left[(1+\gamma)\frac{\mathrm{T}_2}{\mu_2}-(1-\gamma)\frac{\mathrm{T}_1}{\mu_1}\right]-\phi\left[(1+\gamma)\mathrm{T}_1+(1-\gamma)\mathrm{T}_2\right]\right)\rho\cos\phi+\left[(1+\gamma)\mathrm{T}_1\sum_{k=1}^{\infty}a_k\frac{\alpha_1^{2k}}{2k}\right.\right.$$

$$\left.\left.+\operatorname{In}\alpha_1\right)+(1-\gamma)\mathrm{T}_2\left(\sum_{k=1}^{\infty}a_k\frac{\alpha_2^{2k}}{2k}+\operatorname{In}\alpha_2\right)\right]\rho\sin\phi+2\left[(1+\gamma)\mathrm{T}_1+(1-\gamma)\mathrm{T}_2\right]\rho\sin\phi\Big\},\ -\frac{\pi}{2}\le\phi\le\frac{\pi}{2},\ i=1,2,\text{ as}$$

$\rho \to 0$. In terms of Cartesian coordinates, the displacements are, for $i = 1, 2$,

$$w_i(\rho,\phi)=\frac{h}{2\pi^2\mu_i}\left\{\left(\frac{\pi\mu_i}{2}\left[(1+\gamma)\frac{\mathrm{T}_2}{\mu_2}-(1-\gamma)\frac{\mathrm{T}_1}{\mu_1}\right]-\tan^{-1}\frac{v(x,y)}{u(x,y)}\left[(1+\gamma)\mathrm{T}_1+(1-\gamma)\mathrm{T}_2\right]\right)u(x,y)\right.$$

$$+\left[(1+\gamma)\mathrm{T}_1\left(\operatorname{In}2\alpha_i-\operatorname{In}\left(1+\sqrt{1-\alpha_1^2}\right)\right)+(1+\gamma)\mathrm{T}_2\left(\operatorname{In}2\alpha_2-\operatorname{In}\left(1+\sqrt{1-\alpha_2^2}\right)\right)\right]v(x,y)$$

$$\left.+2\left[(1+\gamma)\mathrm{T}_1+(1-\gamma)\mathrm{T}_2\right]v(x,y)\right\},\qquad -\frac{\pi}{2}\le\phi\le\frac{\pi}{2},\ \text{as } x\to 0,\ y\to 0,\ i=1,2,$$

In this case, the free surface-interface junction is displaced. The displacement along the interface is

$$w(x,0)=\frac{1}{4}\left[(1+\gamma)\frac{\mathrm{T}_2}{\mu_2}-(1-\gamma)\frac{\mathrm{T}_1}{\mu_1}\right]x,\ \text{as } x\to 0,$$

while the stresses there are

$$\sigma_{ixz}(x,0)=\frac{\mu_i}{4}\left[(1+\gamma)\frac{\mathrm{T}_2}{\mu_2}-(1-\gamma)\frac{\mathrm{T}_1}{\mu_1}\right]x,\ \text{as } x\to 0,\ i=1,2.$$

$$\sigma_{ixz}(x,0)=\frac{1}{2\pi}\left\{(1+\gamma)\mathrm{T}_1\operatorname{In}\left(\frac{2\sin\frac{\pi a_1}{h}}{1+\cos\frac{\pi a_1}{h}}\right)+(1-\gamma)\mathrm{T}_2\operatorname{In}\left(\frac{2\sin\frac{\pi a_2}{h}}{1+\cos\frac{\pi a_2}{h}}\right)+2\left[(1+\gamma)\mathrm{T}_1+(1-\gamma)\mathrm{T}_2\right]\right\},\tag{4.6}$$

as $x \to 0$, $i = 1, 2$

5.0 Conclusion

The fields for $\rho \succ \alpha_i$ can be deduced by the same technique used for the other cases. The stress fields for $\beta_i \prec \rho \prec \alpha_i$, $i=1,2$ can be obtained from (3.17). From (4.5) and (4.6) various load situations can be derived near the free surface-interface junction.

For instance, when the segments are loaded so that loadings do not start from the origin, $b_i \succ 0$, $i=1,2$, stress corresponding to equal and opposite loads is derived from (4.5) as:

$$\sigma_{yz}(x,0)=\frac{\mathrm{T}}{\pi}\left[(1+\gamma)\left\{\operatorname{In}\left[\frac{\left(1+\cos\frac{\pi b_1}{h}\right)\sin\frac{\pi a_1}{h}}{\left(1+\cos\frac{\pi a_1}{h}\right)\sin\frac{\pi b_1}{h}}\right]\right\}+\left\{\frac{\left(1+\cos\frac{\pi b_2}{h}\right)\sin\frac{\pi a_2}{h}}{\left(1+\cos\frac{\pi a_2}{h}\right)\sin\frac{\pi b_2}{h}}\right\}\right]\tag{5.1}$$

when length of loaded segments are unequal but not symmetric about the origin; and

$$\sigma_{yz}(x,0)=\frac{2\mathrm{T}}{\pi}\ln\left[\frac{\left(1+\cos\frac{\pi b}{h}\right)\sin\frac{\pi a}{h}}{\left(1+\cos\frac{\pi a}{h}\right)\sin\frac{\pi b}{h}}\right] \tag{5.2}$$

when lengths of loaded segments, $a-b$, are equal (and possibly symmetric about the origin). Observe that (5.2) indicates independence on material constants. We may set $\mathrm{T}_i=0$, $i=1$ or 2 (but not both) in (4.5) to get stress states when only the upper ($i=1$) or lower ($i=2$) layer is loaded. Let $\mathrm{T}_2=0$ in (4.5) we use

$$\sigma_{yz}(x,0)=\frac{(1+\gamma)\mathrm{T}_1}{\pi}\ln\left[\frac{\left(1+\cos\frac{\pi b_1}{h}\right)\sin\frac{\pi a_1 b_1}{b_1 h}}{\left(1+\cos\frac{\pi a_1 b_1}{b_1 h}\right)\sin\frac{\pi b_1}{h}}\right],\ a_1\succ b_1 \tag{5.3}$$

to study, in a graph, the variation of $\frac{\sigma_{yz}(x,0)}{\mathrm{T}_1}$ with $\frac{b_1}{h}$, $b_1\succ 0$. The form (5.3) permits the study of the stress states when a_1 is proportional to b_1, so that if l the denotes the length of a loaded line segment, then $l=(q-1)b_1$, where $a_1=qb_1$, $q\succ 1$ and $2l\prec h$, that is, $\left(b_1\prec qb_1\le\frac{h}{2}\right)$. For the homogeneous case, the versions of (5.1) to (5.3) are deduced from (4.5) and (4.6) by setting $\mu_1=\mu_2$, in which case $\gamma=0$. The graph in Figure IV shows the variation of $\frac{\sigma_{yz}(x,0)}{\mathrm{T}_2}$ with $\frac{a_2}{h}$, when $T_1=0$ in (4.6) $a_2\succ 0$. The stresses are not singular near the origin.

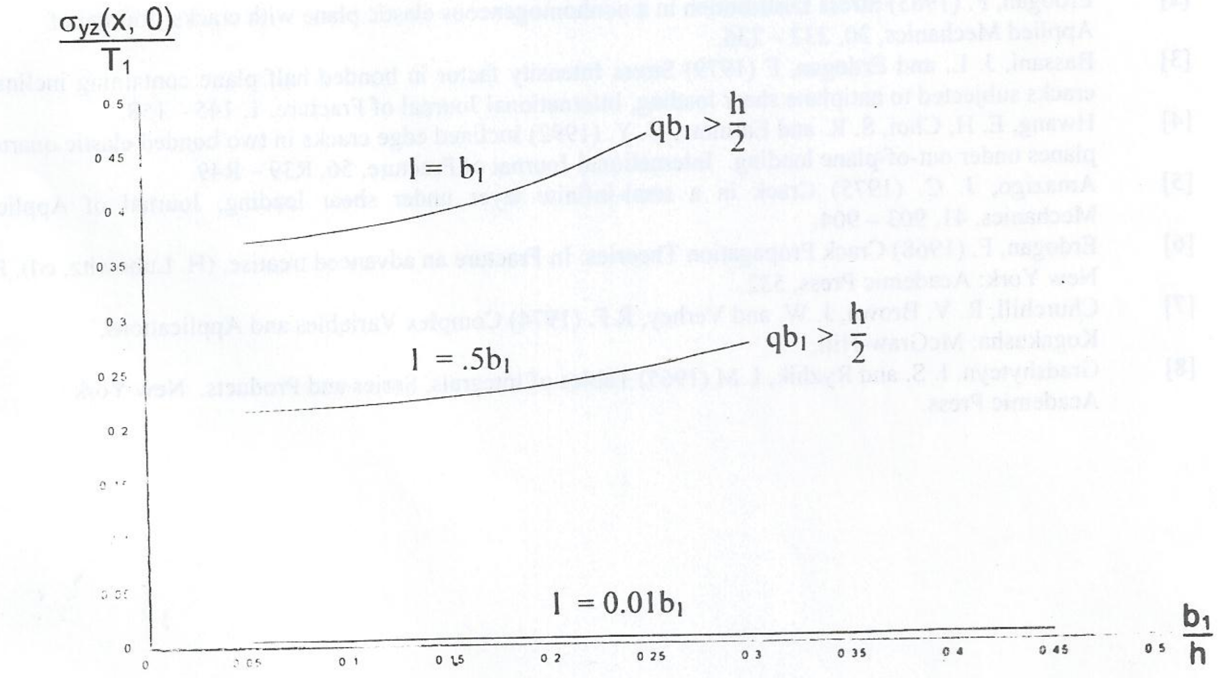

Figure 3: $\frac{\sigma_{yz}(x,0)}{\mathrm{T}_1}$ versus $\frac{b_1}{h}$. For increase in $l=(q-1)b_1$, $\left(\mu_2=5\mu,\ \gamma=\frac{2}{3}\right)$

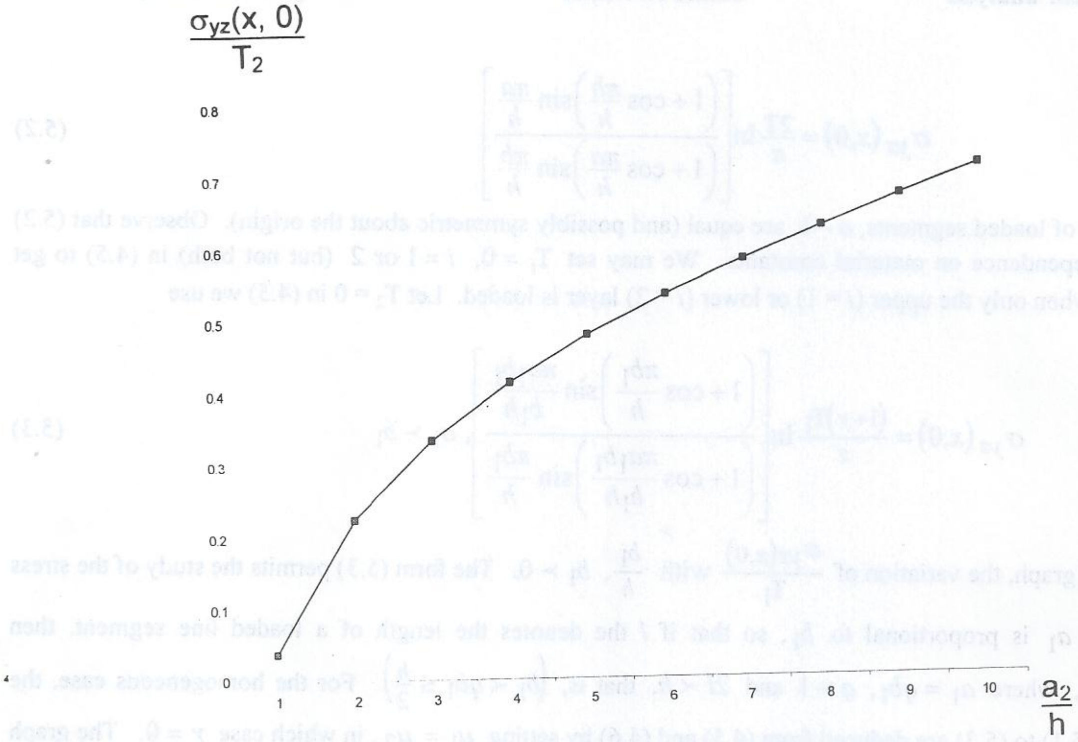

Figure 4: $\frac{\sigma_{yz}(x,0)}{T_2}$ versus $\frac{a_2}{h}$ where loading starts from the origin; $\mu_1 = 5\mu_2$

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