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Asymptotic Analysis of imperfection sensitivity of the toroidal shell segment with random imperfection

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Abstract.

The bifurcation of a toroidal shell segment with initial imperfection which are subjected to lateral or hydrostatic pressure is studied under the assumption that the initial imperfection are Gaussian random stress-free displacement whose mean and autocorrelation function are given. We use perturbation scheme developed by Amazigo [1]. A simple approximate asymptotic expression is obtained for the bifurcation load for small magnitudes of the imperfection. The result is compared with results obtained earlier under secondary bifurcation analysis for the imperfections in the shape of the buckling mode.

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1.0 Introduction

The imperfection sensitivity of structures has occupied many analyses on buckling since the classic work of Koiter [2]. Budiansky and Amazigo [3] applied a reworked version [4] of Koiter's theory and derived an asymptotic formula for the buckling of load of externally pressurized cylinders. Oyesanya [5] used the same procedure to derive an asymptotic formula for cylindrical shell of finite length for simultaneous buckling mode under a multimode analysis. In Oyesanya [6] a measure of imperfection sensitivity of the toroidal shell segment was derived while in an analysis [7] similar to the analysis in [5] an asymptotic formula was derived for the toroidal shell segment.

In general the imperfections in structures are stochastic rather than deterministic. The stochastic approach has been used for analysis of many imperfection sensitive structures like the column on nonlinear foundations [1, 12] and cylindrical shells under various loading [8, 11, and 13]. The toroidal shell segment is an imperfection sensitive structures [6, 14] which has been shown to exhibit secondary bifurcation. In this paper therefore, we consider the toroidal shell segment subjected to lateral or hydrostatic pressure under simple support. We assume that the imperfections are Gaussian. We use a perturbation scheme similar to that developed by Amazigo [1]. We obtain an asymptotic formula for the bifurcation load. We found that the loss in the bifurcation load parallels that obtained in [7]. We also found that our result reduce to their results for the cylindrical shell as derived in [8].

2.0 Formulation and analysis

For the imperfect structure the governing equations for toroidal shell segment are for the Karman-Donnell shell theory [3]

$$D\nabla^4 w + \frac{1}{a} f_{,xx} + \frac{1}{b} f_{,yy} + p_e a\left[\frac{1}{2}(w+\overline{w})_{,xx} + \left(1 - \frac{a}{2b}\right)(w+\overline{w})_{,yy}\right] = S(w+\overline{w}, f) \qquad (2.1)$$

$$\frac{1}{Eh}\nabla^4 f - \frac{1}{a} w_{,xx} - \frac{1}{b} w_{,yy} = -\frac{1}{2} S(w+\overline{w}, w) \qquad (2.2)$$

$$w = w_{,xx} = f = f_{,xx} = 0 \quad \text{on} \quad X = 0, L \qquad (2.3)$$

where *a, b* are the inner and outer radius respectively, p_e the external pressure and

$$S(p,q) = p_{,xx}q_{,yy} + p_{,yy}q_{,xx} - 2p_{,xy}q_{,xy} \qquad (2.4)$$

In non-dimensional form the boundary value problem that we consider for the Karman-Donnel shell theory is

$$\overline{\nabla}^4 w - K(\xi)\left(f_{xx} + \xi r f_{yy}\right) + \lambda\left[\frac{1}{2}\alpha\left(w+\overline{w}\right)_{xx} + \left(1-\frac{1}{2}\alpha\right)\xi\left(w+\overline{w}\right)_{yy}\right]$$
$$= -K(\xi) H S\left(f, w+\overline{w}\right)\overline{\nabla}^4 f - \left(1+\xi n^2\right)^2\left(w_{xx} + \xi r w_{yy}\right) = -\frac{1}{2}H\left(1+\xi n^2\right)S\left(w+\overline{w}, w\right) \tag{2.5}$$

$$w = w_{xx} = f = f_{xx} = 0, \text{ on } x = 0,\ \pi$$

where

$$\overline{\nabla}^4 \frac{\partial^2}{\partial x^2} + \xi\frac{\partial^2}{\partial y^2} \text{ and } S(p,q) = p_{xx}q_{yy} + p_{yy}q_{xx} - 2p_{xy}q_{xy} \tag{2.6}$$

w, f, are the non-dimensional displacement and stress functions respectively and the non-dimensional quantities are defined by

$$x = \frac{\pi X}{L},\ y = \frac{Y}{L},\ r = \frac{a}{b},\ \xi\left(\frac{L}{\pi r}\right)^2,\ \lambda = \frac{p_e \alpha L^2}{D\pi^2},\ H = \frac{1}{r},\ -K(\xi) = \frac{L^2}{\pi^2 \alpha D}\left(1+\xi n^2\right)^2 = \frac{EhL^2}{\pi^2\alpha} \tag{2.7}$$

where X is the axial coordinate, Y the circumferential coordinate, r the ratio of the inner and outer radii, L the length, h the thickness, p_e the external lateral pressure, D the flexural rigidity and E the Young's modulus.

We consider the toroidal shell as having an initial stress-free displacement of the form

$$\overline{w}(x,y) = \varepsilon W_0(y)\sin x \tag{2.8}$$

ε a small parameter characterizing the amplitude of displacement. Here $W_0(y)$ is assumed to be a sample function from an ensemble of twice continuously differentiable zero mean, stationary Gaussian random function with known autocorrelation function $R_{00}(z)$. Thus,

$$\langle W_0(y)\rangle = 0,\ \langle W_0(y+z)W_0(y)\rangle = R_{00}(z) \tag{2.9}$$

where $\langle\cdot\rangle$ denotes ensemble average and $-\infty \prec y \prec \infty$ so that the periodicity condition on y is no longer enforced. The power spectral density $S_{00}(w)$ of W_0 is defined by the Fourier transform of the autocorrelation function

$$S_{00}(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{00}(z)e^{-iwz}dz \tag{2.10}$$

We consider λ to be given and such that $0 \prec \lambda \prec \lambda_c$ where λ_c is the classical buckling load given by

$$\lambda_c = \left(1+\xi n^{-2}\right)^{-2}\left[\frac{1}{2}\alpha + \left(1-\frac{1}{2}\alpha r\right)\xi n^{-2}\right]^{-1}\left[\left(1+\xi n^{-2}\right)^4 + A^2\left(1+\xi n^{-2}\right)^2\right] \tag{2.11}$$

where $A^2 = -K(\xi)\left(1+\xi n^{-2}\right)^2$. In other words, we are considering sub-critical bifurcation for which imperfection sensitivity is observed. We expand (w,f) in powers of ε viz: $\begin{pmatrix} w \\ f \end{pmatrix} = \sum_{m=1}^{\infty}\varepsilon^m \begin{pmatrix} w_m \\ f_m \end{pmatrix}$

(2.12)

and substitute into the governing equation (2.5). We then have the following sequence of equations by equating powers of ε

$$0(\varepsilon): \begin{array}{l} L_1(f_1,w_1) = 0 \\ L_2(f_1,w_1) = \lambda\left[\frac{1}{2}\alpha w_0 - \xi\left(1-\frac{1}{2}\alpha r\right)w_0''\right]\sin x \end{array} \tag{2.13}$$

$$0(\varepsilon^2): \begin{array}{l} L_1(f_2,w_2) = -\left(1+\xi n^{-2}\right)^2 H\left[\frac{1}{2}S(w_1,w_1) + S(w_0\sin x, w_1)\right] \\ L_2(f_2,w_2) = -K(\xi)H\left[S(w_1,f_1) + S(w_0\sin x, f_1)\right] \end{array} \tag{2.14}$$

$$0(\varepsilon^3): \begin{array}{l} L_1(f_3,w_3) = -\left(1+\xi n^{-2}\right)^2 H\left[S(w_1,w_2) + S(w_0\sin x, w_2)\right] \\ L_2(f_3,w_3) = -K(\xi)H\left[S(w_1,f_2) + S(w_2,f_1) + S(w_0\sin x, f_2)\right] \end{array} \tag{2.15}$$

etc.

with boundary conditions $w_i = w_{i,xx} = f_i = f_{i,xx} = 0, \quad i = 1, 2, \cdots$ (2.16)

where

$$L_1(f,w) = \overline{\nabla}^4 f - \left(1+\xi n^2\right)^2\left(w_{xx} + \xi r w_{yy}\right) \tag{2.17}$$
$$L_2(f,w) = \overline{\nabla}^4 w - K(\xi)\left(f_{xx} + \xi r f_{yy}\right) + \lambda\left[\tfrac{1}{2}\alpha w_{xx} + \xi\left(1 - \tfrac{1}{2}\alpha r\right)w_{yy}\right]$$

Let Δ^2 be the average of the mean square of the deflection

$$\Delta^2 = \frac{1}{\pi}\int_0^{\pi}\left\langle w^2(x,y)\right\rangle dx \tag{2.18}$$

Substituting for w leads to

$$\Delta^2 = \varepsilon^2\Delta_{11} + 2\varepsilon^3\Delta_{12} + \varepsilon^4\left(2\Delta_{13} + \Delta_{22}\right) + O\left(\varepsilon^5\right) \tag{2.19}$$

where

$$\Delta_{ij} = \frac{1}{\pi}\int_0^{\pi}\left\langle w_i(x,y)w_j(x,y)\right\rangle dx \qquad i,j = 1,2,\cdots \tag{2.20}$$

Using the same procedure as in [5] we can show that

$$\Delta_{12} = 0 \quad \text{and} \quad \frac{\Delta_{22}}{\Delta_{13}} \to 0 \quad \text{as } \lambda \to \lambda_c$$

Thus,

$$\Delta^2 = \varepsilon^2\Delta_{11} + 2\varepsilon^3\Delta_{12} + 2\Delta_{13}\varepsilon^4\left(1 + \frac{\Delta_{22}}{2\Delta_{13}}\right) \tag{2.21}$$

we have using the above results that as $\lambda \to \lambda_c$

$$\Delta^2 \approx \varepsilon^2\Delta_{11} + 2\varepsilon^4\Delta_{13} \tag{2.22}$$

We note that (2.22) is a relation between Δ^2, λ, and ε since the Δ_{ij}'s are functions of λ. But (2.22) does not converge for Δ^2 greater than the critical mean square so that setting $\frac{d\lambda}{d\Delta^2} = 0$ will not yield the necessary result. We therefore apply Poincare's scheme and reverse the series. From this procedure we have

$$\varepsilon^2 = \alpha_1(\lambda)\Delta^2 + \alpha_2(\lambda)\Delta^4 + \cdots \tag{2.23}$$

which on substitution in equation (2.22) and equating powers of Δ^2 and performing elementary operations give

$$\alpha_1 = \frac{1}{\Delta_{11}}, \quad \alpha_2 = \frac{-2\Delta_{13}}{\Delta_{11}^3} \tag{2.24}$$

Maximizing λ with respect to Δ^2 using (2.23) and (2.24) we arrive at the buckling equation

$$8\varepsilon^2\frac{\Delta_{13}(\lambda)}{\Delta_{11}(\lambda)} \approx 1 \tag{2.25}$$

Note that we have truncated the series (2.22) at the Δ^4 term to arrive at this result. We need asymptotic expressions for Δ_{11} and Δ_{13} for which we need to compute w_1 and w_3 using the definition given by (2.20). We need therefore consider equation (2.13) and assume a solution of the form

$$w_1(x,y) = \psi_1(y)\sin x \quad \text{and} \quad f_1(x,y) = \phi_1(y)\sin x \tag{2.26}$$

this on substitution leads to the result

$$\left(\xi n^{-2}\frac{\partial^2}{\partial y^2} - 1\right)^2\phi_1 - \left(1 + \xi n^{-2}\right)^2\left(\xi r\frac{\partial^2}{\partial y^2} - 1\right)\psi_1 = 0 \tag{2.27}$$

$$\left(\xi n^{-2}\frac{\partial^2}{\partial y^2}-1\right)^2\psi-K(\xi)\left(\xi r\frac{\partial^2}{\partial y^2}-1\right)\phi_1+\lambda\left[-\frac{1}{2}\alpha+\xi\left(1-\frac{1}{2}\alpha r\right)\frac{\partial^2}{\partial y^2}\right]\psi_1=\lambda\left[\frac{1}{2}\alpha w_0-\xi\left(1-\frac{1}{2}\alpha r\right)w_0''\right]$$

Following the same procedure as in Amazigo [1], we have that

$$S\psi_1(\omega)=\lambda^2\left[\frac{1}{2}\alpha+\xi n^{-2}\left(1-\frac{1}{2}\alpha r\right)\omega^2\right]^2\left(1+\xi n^2\omega^2\right)^4Q^2(\omega)S_{00}(\omega)$$

$$S\phi_1(\omega)=\lambda^2\left[\frac{1}{2}\alpha+\xi n^{-2}\left(1-\frac{1}{2}\alpha r\right)\omega^2\right]^2\left(1+\xi n^2\omega^2\right)^4Q^2(\omega)S_{00}(\omega) \tag{2.28}$$

where

$$Q(\omega)=\left[\left\{\left(1+\xi\omega^2n^{-2}\right)^2-\lambda\left[\frac{1}{2}\alpha+\xi n^{-2}\left(1-\frac{1}{2}\alpha r\right)\omega^2\right]\right\}\left(1+\xi\omega^2n^{-2}\right)^2 -K(\xi)\left(1+\xi n^{-2}\right)^2\left(1+\xi rn^{-2}\right)^2\right]^{-1} \tag{2.29}$$

$$S_\psi(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}R_\psi(z)e^{-i\omega z}dz \tag{2.30}$$

with

$$R_u(z)=\langle u_1(y+z)u_1(y)\rangle \tag{2.31}$$

Substituting for w_1 in (20) gives

$$\Delta_{11}=\frac{1}{2}\left\langle\psi_1^2(y)\right\rangle=\frac{1}{2}R_{\psi 1}(0)=\frac{1}{2}\int S_{\psi 1}(\omega)d\omega \tag{2.32}$$

If we let

$$B_m\equiv\int F(\omega)[Q(\omega)]^m d\omega \qquad m\geq 2 \tag{2.33}$$

where $F(\omega)$ is any smooth integrable function analytic in the strip $|Im\omega|\prec b$ for some b with $F(\pm 1)\neq 0$. We show (see Appendix) that

$$B_m\approx\frac{\pi(m-1)(2m-3)!}{2^{2m-2}[(m-1)!]^2}\frac{F(1)+F(-1)}{[P(1)]^{\frac{1}{2}}[(\lambda_c-\lambda)g(1)]^{m-\frac{1}{2}}} \qquad \lambda\to\lambda_c \tag{2.34}$$

where

$$P(\omega)\left(\xi n^{-2}\right)^2\left|\begin{matrix}\left(\omega^2-1\right)\xi n^{-2}+\frac{3}{2}\left(1+\xi n^{-2}\right) & \left(\omega^2-1\right)\xi n^{-2}+2\left(1+\xi n^{-2}\right)\\ \left(\omega^2-1\right)\xi n^{-2}+2\left(1+\xi n^{-2}\right) & \lambda_c\end{matrix}\right| \tag{2.35}$$

and

$$g(\omega)=\left(1+\xi\omega^2n^{-2}\right)^2\left[\frac{1}{2}\alpha+\xi\left(1-\frac{1}{2}\alpha r\right)\omega^2n^{-2}\right] \tag{2.36}$$

Use of this result gives

$$\Delta_{11}\approx\frac{\lambda^2\pi S_{00}(1)}{4\xi n^{-2}(\lambda_c-\lambda)^{1/2}}\left[\frac{\left[\frac{1}{2}\alpha+\xi\left(1-\frac{1}{2}\alpha r\right)\xi n^{-2}\right]\left(1+\xi n^{-2}\right)^2}{4\left(1+\xi n^{-2}\right)-\frac{3}{2}\lambda_c}\right]^{\frac{1}{2}} \qquad \lambda\to\lambda_c \tag{2.37}$$

$\left\langle w_0^2\right\rangle$ is independent of λ and hence 0(1) as $\lambda\to\lambda_c$ and $\left\langle\psi_1^2\right\rangle=0\left[(\lambda_c-\lambda)^{-1/2}\right]$. By the above result $\psi_1(y)+w_0(y)\approx\psi_1(y)$ as $\lambda\to\lambda_c$. Thus for subsequent analysis $w_0(y)$ can be dropped and the (f_2,w_2) problems becomes

$$L_1(f_2,w_2)=-\tfrac{1}{2}\left(1+\xi n^{-2}\right)^2 HS(w_1,w_1)$$
$$L_2(f_2,w_2)=-K(\xi)HS(w_1,f_1) \tag{2.38}$$

Substituting for f_1, w_1 using (2.26) gives

$$L_1(f_2,w_2)=-\tfrac{1}{2}\left(1+\xi n^{-2}\right)^2 H\left[\psi_1''\psi_1+\psi_1^2-\left(\psi_1''\psi_1-\psi_1^2\right)\cos 2x\right]$$
$$L_2(f_2,w_2)=-\tfrac{1}{2}K(\xi)H\{(\psi_1\phi_1''+\psi_1''\phi_1+2\psi_1'\phi_1')-(\psi_1\phi_1''+\psi_1''\phi_1-2\psi_1'\phi')\cos 2x\} \tag{2.39}$$

We assume the solution of this problem in the form

$$\begin{matrix} w_2(x,y) \\ f_2(x,y) \end{matrix} = \sum_{m\ odd} \begin{pmatrix} P_m(y) \\ Q_m(y) \end{pmatrix} \sin mx \tag{2.40}$$

We substitute into equation (2.39) to have

$$\mathrm{M}_1^m(\mathrm{P}_m,Q_m)=\left(1+\xi n^{-2}\right)^2\left(-P_m\psi_1''\psi_1+\mathrm{T}_m\psi_1^2\right)$$
$$\mathrm{M}_2^m(\mathrm{P}_m,Q_m)=-K(\xi)P_m(\psi_1''\phi_1+\psi_1\phi_1'')+2\mathrm{T}_m\psi_1'\phi' \tag{2.41}$$

where

$$\mathrm{M}_1^m(\mathrm{P}_m,Q_m)\equiv\left(\xi\frac{d^2}{dy^2}-m^2\right)^2 Q_m(y)+\left(1+\xi n^{-2}\right)^2\left(m^2+\xi r\frac{d^2}{dy^2}\right)R_m(y)$$

$$\mathrm{M}_2^m(\mathrm{P}_m,Q_m)\equiv\left\{\left(\xi\frac{d^2}{dy^2}-m^2\right)^2-\lambda\left[\tfrac{1}{2}\alpha m^2+\xi n^{-2}\left(1-\tfrac{1}{2}\alpha r\right)\frac{d^2}{dy^2}\right]\right\}R_m(y)+K(\xi)\left(m^2+\xi r\frac{d^2}{dy^2}\right)^2 Q_m(y) \tag{2.42}$$

and
$$P_m=\frac{8H}{\pi m\left(m^2-4\right)},\qquad T_m=\frac{4H\left(m^2-2\right)}{\pi m\left(m^2-4\right)} \tag{2.43}$$

By definition and using representation for w_1 and w_2, we have

$$\Delta_{12}=\tfrac{1}{2}\langle\psi_1(y)R_m(y)\rangle \tag{2.44}$$

Using the same procedure as in [1, 9] we have

$$\Delta_{12}=0 \quad\text{and}\quad \Delta_{22}=\tfrac{1}{2}\Sigma\left\langle R_m^2(y)\right\rangle \tag{2.45}$$

A lengthy derivation shows that $$\Delta_{22}=\mathrm{O}\left((\lambda_c-\lambda)^{-3}\right) \tag{2.46}$$

If we did for the (w_2,f_2) problem w_0 is dropped we have the (w_3,f_3) problem to be

$$L_1(f_3,w_3)=-\left(1+\xi n^{-2}\right)^2 HS(w_1,w_2)$$
$$L_2(f_3,w_3)=-K(\xi)HS(w_1,f_2)+(w_2,f_1) \tag{2.47}$$

We substitute for w_1, f_1, w_2, f_2, using (2.26) and (2.40). The solution to the resulting equations can be found in the form

$$\begin{matrix} w_3 \\ f_3 \end{matrix}=\begin{pmatrix} h_1(y) \\ \chi_1(y) \end{pmatrix}\sin x+\sum_{m\ odd}\begin{pmatrix} h_m(y) \\ \chi_m(y) \end{pmatrix}\sin mx \tag{2.48}$$

Since we are interested in Δ_{13} we exhibit only the equation χ_1 and h_1 which are

$$\begin{aligned} M_1^{(1)}(\chi_1,h_1) &= -\left(1+\xi n^{-2}\right)^2 \sum P_m\left(\psi_1 R''_m + m^2\psi''_1 R_m\right) - 2\left(1+\xi n^{-2}\right)^2 \sum \chi_m \psi'_1 R'_m \\ M_2^{(1)}(\chi_1,h_1) &= -K(\xi)\sum P_m\left(\psi_1 Q''_m + m^2 Q_m\psi''_1 + \phi_1 R''_m + m^2 R_m\phi''_1\right) \\ &\quad - 2K\left(\xi\sum \chi_m\right)\left(R'_m\phi'_1 + \psi'_1 Q'_m\right) \end{aligned} \tag{2.49}$$

where $M_1^{(1)}$ and $M_2^{(1)}$ are defined in equation (2.42) and

$$\chi_m = \frac{4mH}{\pi\left(m^2-4\right)} \tag{2.50}$$

Using (2.48) and (2.26) in the definition (2.20) gives

$$\Delta_{13} = \frac{1}{2}\langle\psi_1(y)h_1(y)\rangle \tag{2.51}$$

The expression for Δ_{13} is therefore given as

$$\Delta_{13} = -\frac{1}{2}\iint\left[I_1(w_1,w_2;m) + I_2 + J_3 + J_4 + K_5 + K_6 + L_7(w_1,w_2;m)dw_1dw_2\right] \tag{2.52}$$

where

$$\begin{aligned} I_1(w_1,w_2;m) &= K\left[KQ_1^{(1)}(w_1)H_m(w_1+w_2) + \left(1+\xi n^{-2}\right)^2 Q_1^{(2)}(w_2)Q_m^{(1)}(w_1+w_2)\right]\times \\ &\quad \left[S_{\phi\phi}(w_1)S_\phi(w_2) + S_\phi(w_1)S_{\phi\psi}(w_2)\right]\left[-P_m\left(w_1^2+w_2^2\right) + 2T_m w_1 w_2\right]\times \\ &\quad \left[m^2 P_m w_2^2 + (w_1+w_2)^2 - 2\chi_m w_2(w_1+w_2)\right] \end{aligned} \tag{2.53}$$

$$\begin{aligned} I_2(w_1,w_2;m) &= \left(1+\xi n^{-2}\right)\left[KQ_1^{(1)}(w_1)H_m^{(1)}(w_1+w_2) + \left(1+\xi n^{-2}\right)^2 Q_1^{(2)}(w_2)Q_m^{(2)}(w_1+w_2)\right]\times \\ &\quad \left[S_\phi(w_1)S_\phi(w_2)\right]\left[-P_m\left(w_1^2+w_2^2\right) + 2T_m w_1 w_2\right]\left[m^2 P_m w_2^2 + P_m(w_1+w_2)^2 - 2\chi_m w_2(w_1+w_2)\right] \end{aligned} \tag{2.54}$$

$$\begin{aligned} J_3(w_1,w_2;m) &= K^2 Q_1^{(1)}(w_1)Q_m^{(1)}(\omega_1+\omega_2)\left[S_{\phi\psi}(\omega_1)S_{\phi\psi}(\omega_2)S_\psi(\omega_1)S_\phi(\omega_2)\right]\times \\ &\quad \left[-P_m\left(\omega_1^2+\omega_2^2\right) + 2T_m\omega_1\omega_2\right]\left[m^2 P_m\omega_2^2 + P_m(\omega_1+\omega_2)^2 - 2\chi_m\omega_2(\omega_1+\omega_2)\right] \end{aligned} \tag{2.55}$$

$$\begin{aligned} J_4(\omega_1,\omega_2;m) &= K\left(1+\xi n^{-2}\right)Q_1^{(1)}(\omega_1)Q_m^{(2)}(\omega_1+\omega_2)\left[S_\psi(\omega_1)S_{\phi\psi}(\omega_2)\right]\times \\ &\quad \left[-P_m\left(\omega_1^2+\omega_2^2\right) + 2T_m\omega_1\omega_2\right]\left[m^2 P_m\omega_2^2 + P_m(\omega_1+\omega_2)^2 - 2\chi_m\omega_2(\omega_1+\omega_2)\right] \end{aligned} \tag{2.56}$$

$$\begin{aligned} K_5(\omega_1,\omega_2;m) &= -Km^2 P_m(P_m+T_m)\omega_1^2\omega_2^2 Q_1^{(1)}(\omega_1)S_\psi(\omega_1)\left[2KH_m^2(0)S_{\phi\phi}(\omega_2)\right. \\ &\quad \left. + \left(1+\xi n^{-2}\right)H_m^{(1)}(0)S_\psi(\omega_2)\right] \end{aligned} \tag{2.57}$$

$$\begin{aligned} K_6(\omega_1,\omega_2;m) &= -Km^2 P_m(P_m+T_m)\omega_1^2\omega_2^2 Q_1^{(1)}(\omega_1)S_{\psi\phi}(\omega_1)\left[2KQ_m^2(0)S_{\phi\phi}(\omega_2)\right. \\ &\quad \left. + \left(1+\xi n^{-2}\right)Q_m^{(1)}(0)S_\psi(\omega_2)\right] \end{aligned} \tag{2.58}$$

$$\begin{aligned} L_7(\omega_1,\omega_2;m) &= -\left(1+\xi n^{-2}\right)m^2(P_m+T_m)\omega_1^2\omega_2^2 Q_1^{(2)}(\omega_1)S_\psi(\omega_1)\left[2KQ_m^2(0)S_{\phi\phi}(\omega_2)\right. \\ &\quad \left. + \left(1+\xi n^{-2}\right)Q_m^{(1)}(0)S_\psi(\omega_2)\right] \end{aligned} \tag{2.59}$$

with $Q_m^{(1)}(\omega) = \left(\omega^2\xi n^{-2} + m^2\right)^2 Q_m(\omega), \quad m = 1,3,5,\cdots \quad Q_m^{(1)}(\omega) = -K(\xi n)^2 m^2 Q_m(\omega)$

$$H_m^{(1)}(\omega)=\left[\left(\omega^2\xi n^{-2}+m^2\right)^2-\lambda\left[\frac{1}{2}\alpha m^2-\xi n^{-2}\left(1-\frac{1}{2}\alpha r\omega^2\right)\right]\right]Q_m(\omega)$$

$$H_m^{2}(\omega)=-\left(1+\xi n^{-2}\right)m^2Q_m(\omega) \tag{2.60}$$

$$Q_m(\omega)=\left\{\left[\left(\xi n^{-2}\omega^2+m^2\right)^2\right]\left[\left(\xi n^{-2}\omega^2+m^2\right)^2-\lambda\left[\frac{1}{2}\alpha m^2+\left(1-\frac{1}{2}\alpha r\right)\xi n^{-2}\omega^2\right]\right]\right.$$
$$\left.-\left(1+\xi n^{-2}\right)^2Km^4\left(1+\xi r n^{-2}\right)^2\right\}^{-1}$$

P_m, T_m, and χ_m are given by (2.43) and (2.50). We note that by comparing (2.60) and (2.31) $Q_1(\omega)=Q(\omega)$. $P(\omega)$ and $g(\omega)$ are defined by (2.35) and (2.36) respectively. The result (2.52) is used to evaluate the expression for Δ_{13} asymptotically. The calculation gives

$$\Delta_{13}=\frac{3\pi^2\left(1+\xi n^{-2}\right)^2\left[\frac{1}{2}\alpha+\psi\left(1-\frac{1}{2}\alpha r\right)n^{-2}\right]\lambda^4S_{00}(1)}{8\xi^2n^{-4}\left[4\left(1+\xi n^2\right)-\frac{3}{2}\lambda_c\right](\lambda_c-\lambda)^4}(-b) \tag{2.61}$$

where b is defined in [6]. Comparison of (2.47) and (2.61) leads to

$$\frac{\Delta_{22}}{\Delta_{13}}=O(\lambda_c-\lambda)\ \text{as}\ \lambda\to\lambda_c \tag{2.62}$$

Substituting for Δ_{11} and Δ_{13} using (2.37) and (2.61) into the buckling equation (2.25) give,

$$\left(1-\frac{\lambda}{\lambda_c}\right)^{\frac{5}{4}}=2\left[\frac{\lambda_c\left(1+\xi n^{-2}\right)\left[\frac{1}{2}\alpha+\xi\left(1-\frac{1}{2}n^{-2}\right)\right]}{4\left(1+\xi n^2\right)-\frac{3}{2}\lambda_c}\right]^{\frac{1}{4}}\left(\frac{3\pi S_{00}(1)}{\xi n^{-2}}\right)^{\frac{1}{2}}(-b)^{\frac{1}{2}}\varepsilon\frac{\lambda}{\lambda_c} \tag{2.63}$$

for $b<0$ the shell is imperfection sensitive.

3.0 Conclusion

We have derived an asymptotic formula for the imperfection sensitivity of toroidal shell segment under random imperfection. We note that while for the modal imperfection [6] the loss in buckling strength is of the order $\varepsilon^{\frac{2}{3}}$ for the random imperfection it is of the order $\varepsilon^{\frac{4}{5}}$ as derived in (2.63) above.

Appendix

Asymptotic Evaluation of the integral $B_m=\int F\left(\omega[Q(\omega)]^m d\omega\right)\quad m\geq 2$

with

$$Q(\omega)=\left\{\left(1+\xi\omega^2n^{-2}\right)^2\left[\left(1-\frac{1}{2}\xi n^{-2}\omega^2\right)-\lambda\left[\frac{1}{2}\alpha+\xi\left(1-\frac{1}{2}\alpha r\right)\omega^2\right]-K(\xi)\left(1+\xi n^{-2}\right)^2\right]\right\}^{-1} \tag{A.1}$$

we find that us (2.10) and (2.22) respectively we have that

$$\frac{1}{Q(\omega)}=\left(\omega^2-1\right)^2P(\omega)+(\lambda_c-\lambda)g(\omega) \tag{A.2}$$

where $P(\omega)$ and $g(\omega)$ are defined by (2.35) and (2.36). Thus

$$B_M = \int \frac{F(\omega)}{P^m(\omega)}\left[\left(\omega^2 - 1\right)^2 + (\lambda_c - \lambda)\frac{g(\omega)}{P(\omega)}\right]^{-m} d\omega \qquad (A.3)$$

the poles of the integrand $\left[\left(\omega^2 - 1\right)^2 + (\lambda_c - \lambda)\frac{g(\omega)}{P(\omega)}\right]^{-m}$ in the upper half plane occur at

$$\omega_{1,2} = \pm 1 + \frac{1}{2}\left[(\lambda_c - \lambda)\frac{g(1)}{P(1)}\right]^{\frac{1}{2}} + O(\lambda_c - \lambda) \qquad (A.4)$$

The analyticity of $\frac{F(w)}{P^m(w)} |\operatorname{Im}\omega| \prec \alpha$ for some α makes it possible for us to shift the integral to give

$$B_M = \int_{-\infty + w_1}^{\infty + w_1} \frac{F(\omega)}{P^m(\omega)}\left[\left(w^2 - 1\right)^2 + (\lambda_c - \lambda)\frac{g(w)}{P(w)}\right]^{-m} dw + 2\pi i[\textit{residue at } w_1, w_2] \qquad (A.5)$$

where $\frac{1}{2}(\lambda_c - \lambda)g(1)[P(1)]^{-1} \prec \alpha_1 \prec \alpha$.

Taking α_1 fixed the integral is bound and hence $O(1)$ as $\lambda \to \lambda_c$. Evaluating the residues yields the result (2.34).

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