Vibration under a moving load of a non-uniform Rayleigh beam on variable elastic foundation

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Abstract.

The problem of the vibrations under a concentrated moving mass of a non-uniform Rayleigh beam resting on a variable elastic foundation is investigated. The technique is based on the Generalized Galerking Method and Struble's asymptotic technique. Numerical results in plotted curves are presented. The results show that the response amplitudes of the non-uniform Rayleigh beam decrease as the rotatory inertia correction factor r^0 increases. Similarly, for fixed value of r^0, the displacements of a non-uniform Rayleigh beam resting on a variable elastic foundation decease as the foundation moduli F increases. Furthermore, the critical speed for the moving mass problem is reached prior that of the moving force problem for both illustrative examples considered.

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1.0 Introduction

It is well known fact that a rolling load on a bridge, a girder or a railway produces a greater transverse deflection and greater stresses than does the same load acting statically. This effect of moving loads on bridge is of great practical importance, and many researchers in engineering, physics and applied mathematics have worked on the solution of the problem [1,2,3].

In a beam, bar or arches problems like this, the position of the load is continuously changing, thus the effect of the mass of the load is taken to be of paramount importance. One-dimensional structures such as this are often of variable cross-sections and as such their properties vary with respect to the spatial coordinates along the span of the structure. Practical problems involving variable cross-sections and the effects of the mass of the moving load such as this vis a vis its counterpart involving uniform one-dimensional structure is relatively rare in literature [4]. Apart from the earlier work of Kolousek [5] on non-uniform structure under moving loads, worthy of mention is the work of Morgaevskii [4] who studied the problem of a non-uniform arch subjectaed to a moving load.

Recently, Oni [6] considered the response of a non-uniform thin beam resting on a constant elastic foundation to several moving masses. For the close form solution of the problem, he used the versatile technique of Galerkin to reduce the complex governing fourth other partial differential equation with variable and singular coefficients to a set of ordinary differential equations with variable coefficients. This set was later simplified and solved using modified asymptotic method of struble. It is remarked here that the method in this work is suitable only for simply supported end conditions.

In this paper, the problem of the vibrations under concentrated moving mass of a non-uniform Rayleigh beams on a variable elastic foundation is investigated. The method in [7], involving the Generalized Galerkin Method which is suitable for other variants of classical boundary conditions, is employed to solve the governing differential equation. The analysis is carried out for various of rotatory inertia and subgrade moduli.

2.0 Formulation of Problem

Let us consider a moving load, $P(x,t)$ of mass M acting downward on a non-uniform Raleigh beam and moving to the right at a constant velocity v as indicated in the figure below. As the beam is non-uniform, its properties such as moment of inertia J, and the mass per unity length of the beam ξ vary along the span L of the beam.

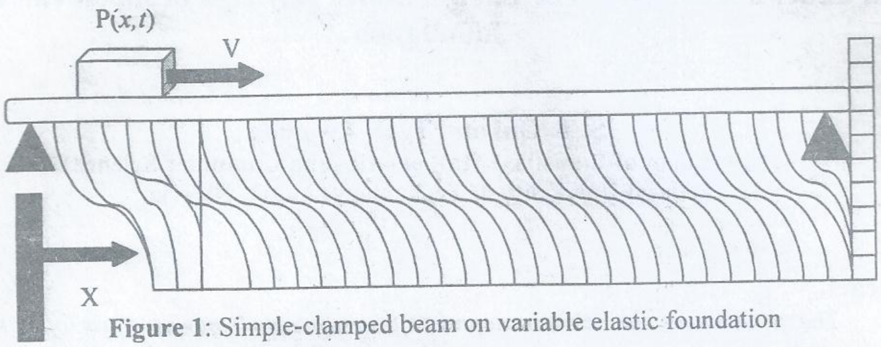

Figure 1: Simple-clamped beam on variable elastic foundation

Assuming that at time $t = 0$, the load is at the left-hand support, it is observed that at any later time t, the distance of the load from the left support will be vt. The equation of motion with damping neglected is given by

$$\frac{\partial^2}{\partial x^2}\left[EJ(x)\frac{\partial^2 Z(x,t)}{\partial x^2}\right]+\xi_b(x)\frac{\partial^2 Z(x,t)}{\partial t^2}-\frac{\partial}{\partial x}\left[\xi_b(x)r^0\left(\frac{\partial^3 Z(x,t)}{\partial x\partial t^2}\right)\right]+F(x)Z(x,t)=P(x,t) \quad (2.1)$$

where E is the Young's Modulus, $Z(x,t)$ is the transverse displacement, $F(x)$ is the variable elastic foundation, r^0 is the measure of rotatory inertia effect and x, t are respectively spatial and time coordinates.

In this system, when the effect of the mass of the moving load on the beam is considered, $P(x,t)$ takes the form

$$P(x,t)=M\partial(x-Vt)\left[g-\left(\frac{\partial^2}{\partial t^2}+\frac{2v\partial^2}{\partial x\partial t}+v^2\frac{\partial^2}{\partial x^2}\right)Z(x,t)\right] \quad (2.2)$$

An example of variable elastic foundation in [9] is adopted namely:

$$F(x)=F\left(4x-3x^2+x^3\right) \quad (2.3)$$

where F is the foundation modulus. Also, let $J(x)$ and $\xi(x)$ take the form [6]

$$J(x)=J_0\left(1+\sin\frac{\pi x}{L}\right)^3 \quad (2.4)$$

and

$$J(\xi)=\xi_0\left(1+\sin\frac{\pi x}{L}\right) \quad (2.5)$$

At this juncture, the boundary conditions for our dynamical system are arbitrary and the initial conditions without any loss of generality are taken to be

$$Z(x,0)=0=\frac{\partial Z}{\partial t}(x,0) \quad (2.6)$$

3.0 Operational Simplification

Substituting equation (2.2), (2.3), (2.4) and (2.5) into (2.1), simplifying and arranging yields

$$\frac{EJ_0}{4}\frac{\partial^2}{\partial x^2}\left[\left(10+15\sin\frac{\pi x}{L}-6\cos\frac{2\pi x}{L}-\sin\frac{3\pi x}{L}\right)\frac{\partial^2 Z}{\partial x^2}(x,t)\right]$$

$$+\xi_0\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^2 Z(x,t)}{\partial t^2}-r^0\xi_0\frac{\partial}{\partial x}\left[\left(1+\sin\frac{\pi x}{L}\right)\frac{\partial^2 Z}{\partial x\partial t^2}(x,t)\right]$$

$$+M\partial(x-vt)\left(\frac{\partial^2}{\partial t^2}+\frac{2v\partial^2}{\partial x\partial t}+v^2\frac{\partial^2}{\partial x^2}\right)Z(x,t)+F\left(4x-3x^2+x^3\right)Z(x,t)=Mg\partial(x-vt) \quad (3.1)$$

Evidently, an exact closed form solution of the above partial differential equation (3.1) does not exist. Consequently, an approximate solution is sought. Thus, the Galerkin technique described in [8] is employed.

This versatile technique requires that the solution of equation (3.1) takes the form:

$$Z_m(x,t)=\sum_{i=1}^{m} Y_i(t)U_i(x) \tag{3.2}$$

where $U_i(x)$ is a function chosen such that the pertinent end conditions are satisfied. Substituting equation (3.2) into (3.1) and simplifying one obtains

$$\sum_{i=1}^{m}\frac{EJ_0}{4}\left[\left[G_1(1,x)\frac{d^4U_i(x)}{dx^4}+\frac{6\pi}{L}G_2(1,x)\frac{d^3U_i(x)}{dx^3}+\frac{3\pi^2}{L^2}G_3(1,x)\frac{d^2U_i(x)}{dx^2}\right]Y_i(t)\right.$$
$$+\xi_0\left(1+\sin\frac{\pi x}{L}\right)U_i(x)\ddot{Y}_i(t)-\xi_0 r^0\left[\left(1+\sin\frac{\pi x}{L}\right)\frac{d^2U_i(x)}{dx^2}+\frac{\pi}{L}\cos\frac{\pi x}{L}\frac{dU_i(x)}{dx}\right]\ddot{Y}_i(t) \tag{3.3}$$
$$+M\partial(x-vt)\left[U_i(x)\ddot{Y}_i(t)+2v\frac{dU_i(t)}{dx}\dot{Y}_i(t)+v^2\frac{d^2U_i(x)}{dx^2}Y_i(t)\right]F\left(4x-3x^2+x^3\right)U_i(x)(x)Y_i(t)$$
$$\left.-Mg\partial(x-vt)\right]=0$$

where
$$G_1(1,x)=\frac{10+15\sin\pi x}{L}-\frac{6\cos 2\pi}{L}-\frac{\sin 3\pi x}{L} \tag{3.4}$$

$$G_2(1,x)=\frac{5\cos\pi x}{L}-\frac{4\sin 2\pi x}{L}-\frac{\cos 3\pi x}{L} \tag{3.5}$$

$$G_3(1,x)=\frac{3\sin 3\pi x}{L}-\frac{8\cos 2\pi x}{L}-\frac{5\sin\pi x}{L} \tag{3.6}$$

To obtain $Y_i(t)$ from equation (3.3), it is sequired that the expression on the left hand side of equation (3.3) be orthogonal to tahe function $U_j(x)$. Thus

$$\int_0^L\left\{\sum_{i=1}^{m}\left\{\frac{EJ_0}{4}\left[G_1(1,x)U_i^{iv}(x)+\frac{6\pi}{L}G_2(1,x)U_i^{iii}(x)+\frac{3\pi^2}{L^2}G_3(1,x)U_i^{11}(x)\right]Y_i(t)\right.\right.$$
$$+\xi_0\left(1+\sin\frac{\pi x}{L}\right)U_i(x)\ddot{Y}_i(t)-r^0\xi_0\left[\left(1+\sin\frac{\pi x}{L}\right)U_i^{11}(x)+\frac{\pi}{L}\cos\frac{\pi x}{L}U_i^{1}(x)\right]\ddot{Y}_i(t)$$
$$+M\partial(x-vt)\left[U_i(x)\ddot{Y}_i(t)+2vU_i^{1}(x)\dot{Y}_i(t)+v^2U_i^{11}(x)Y(t)_i\right]+F\left(4x-3x^2+x^3\right)U_i(x)Y_i(t) \tag{3.7}$$
$$\left.\left.-Mg\delta(x-vt)\right\}\right\}U_{ic}(x)dx=0$$

Simplifying and rearranging the above equation yields

$$\sum_{i=1}^{m}\left\{[R_{0A}(i,j)]\ddot{Y}_i+[R_{0A}]Y(t)+\frac{M}{\xi_0}\left[R_1(t)\ddot{Y}_i(t)+2vR_2(t)\dot{Y}_i(t)+v^2R_3(t)Y_i(t)\right]\right\}=\frac{Mg}{\xi_0}U_j(v,t) \tag{3.8}$$

where

$$R_{0A}(i,j)=I_1+I_2-r^0\left(I_3+I_4+\frac{\pi I_5}{L}\right)$$
$$R_{0B}(i,j)=\frac{EJ_0}{4\xi_0}\left[10I_6 15I_7-6I_8-I_9+\frac{6\pi}{L}(5I_{10}+4I_{11}-I_{12})+\frac{3\pi^2}{L^2}(3I_{13}+8I_{14}-5I_{15})\right] \tag{3.9}$$
$$+\frac{F}{\xi_0}(4I_{16}-3I_{17}+I_{18})$$

$$\left.\begin{aligned}
&I_1 = \int_0^L U_i(x)U_j(x)dx && I_7 = \int_0^L \sin\tfrac{\pi x}{L}U_i^{iv}(x)U_j(x)dx && I_{13} = \int_0^L \sin\tfrac{\pi x}{L}U_i''(x)U_j(x)dx\\
&I_2 = \int_0^L \sin\tfrac{\pi x}{L}U_i(x)U_i(x)dx && I_8 = \int_0^L \cos 2\tfrac{\pi x}{L}U_i^{iv}(x)U_j(x)dx && I_{14} = \int_0^L \cos 2\tfrac{\pi x}{L}U_i''(x)U_j(x)dx\\
&I_3 = \int_0^L U_i''(x)U_j(x)dx && I_9 = \int_0^L 3\sin\tfrac{\pi x}{L}U_i^{iv}(x)U_j(x)dx && I_{15} = \int_0^L \sin\tfrac{\pi x}{L}U_i''(x)U_j(x)dx\\
&I_4 = \int_0^L \sin\tfrac{\pi x}{L}U_i''(x)U_j(x)dx && I_{10} = \int_0^L \cos\tfrac{\pi x}{L}U_i''(x)U_j(x)dx && I_{16} = \int_0^L xU_i(x)U_j(x)dx\\
&I_5 = \int_0^L \cos\tfrac{\pi x}{L}U_i''(x)U_j(x)dx && I_{11} = \int_0^L \sin 2\tfrac{\pi x}{L}U_i''(x)U_j(x)dx && I_{17} = \int_0^L x^2U_i(x)U_j(x)dx\\
&I_6 = \int_0^L U_i^{iv}(x)U_j(x)dx && I_{12} = \int_0^L \cos 3\tfrac{\pi x}{L}U_i''(x)U_j(x)dx && I_{18} = \int_0^L x^3U_i(x)U_j(x)dx
\end{aligned}\right\} \quad (3.10)$$

$$\left.\begin{aligned}
R_1(t) &= \int_0^L \delta(x-vt)U_i(x)U_j(x)dx\\
R_2(t) &= \int_0^L \delta(x-vt)U_i'(x)U_j(x)dx
\end{aligned}\right\} \quad (3.11)$$

$$R_3(t) = \int_0^L \delta(x-vt)U_i''(x)U_j(x)dx \quad (3.12)$$

t is remarked at this juncture that the Dirac-delta function is an even function and so can be expressed as

$$\delta(x-vt) = \frac{1}{L} + \frac{2}{L}\sum_{n=1}^{\infty}\cos\frac{n\pi vt}{L}\cos\frac{n\pi t}{L} \quad (3.13)$$

Using (3.13) in equation (3.8) and rearranging yields

$$\sum_{i=1}^{m}\left\{[R_{0A}(i,j)]\ddot{Y}_i + [R_{0B}(i,j)]\dot{Y}_i(t) + \frac{M}{L\xi_0}\left[H_A(i,j,n,t)\ddot{Y}_i(t) + H_B(i,j,n,t)\dot{Y}_i(t) + v^2GH(i,j,n,t)Y_i(t)\right]\right\}$$

$$= \frac{Mg}{\xi_0}U_i(v,t) \quad (3.14)$$

where

$$H_A(i,j,k,t) = \Omega_{1A}(i,j)\left(2\sum_{n=1}^{\infty}\frac{\cos n\pi vt\Omega_{1A}(i,j)}{L\Omega_{1A}(i,j)} + 1\right) \quad (3.15)$$

$$H_B(i,j,k,t) = 2V\Omega_{2A}(i,j)\left(2\sum_{n=1}^{\infty}\frac{\cos n\pi vt\Omega_{2A}(i,j)}{L\Omega_{2A}(i,j)} + 1\right) \quad (3.16)$$

$$H_C(i,j,k,t) = V^2\Omega_{3A}(i,j)\left(2\sum_{n=1}^{\infty}\frac{\cos n\pi vt\Omega_{3A}(i,j)}{L\Omega_{3A}(i,j)} + 1\right) \quad (3.17)$$

$$\Omega_{1A}(i,j) = \int_0^L U_i(x)U_j(x)dx; \qquad \Omega_{1B} = \int_0^L \cos\frac{n\pi x}{L}U_i(x)U_j(x)dx; \quad (3.18)$$

$$\Omega_{2A}(i,j) = \int_0^L U_i'(x)U_j(x)dx; \qquad \Omega_{2B} = \int_0^L \cos\frac{n\pi x}{L}U_i'(x)U_j(x)dx \quad (3.19)$$

$$\Omega_{3A}(i,j) = \int_0^L U_i''(x)U_j(x)dx; \qquad \Omega_{3B} = \int_0^L \cos\frac{n\pi x}{L}U_i''(x)U_j(x)dx \quad (3.20)$$

In order to solve equation (3.14) for all variants of classical end support conditions, a suitable form of function $U_i(x)$, the beam function, is chosen as

$$U_i(x)=\sin\frac{\alpha_i x}{L}+A_i\cos\frac{\alpha_i x}{L}+B_i\sinh\frac{\alpha_i x}{L}+C_i\cosh\frac{\alpha_i x}{L} \quad (3.21)$$

so that

$$U_j(x)=\sin\frac{\alpha_j x}{L}+A_j\cos\frac{\alpha_j x}{L}+B_j\sinh\frac{\alpha_j x}{L}+C_j\cosh\frac{\alpha_j x}{L} \quad (3.22)$$

where the constants $A_i, A_j, B_i, B_j, C_i, C_j$ and the mode frequencies α_i, α_j are determined by using i desired ends support conditions. Thus substituting (3.21) and (3.22) into equation (3.14) yields

$$R_{0A}^*(i,j)\ddot{Y}_i(t)+R_{0B}^*(i,j)\dot{Y}_i(t)+\lambda\left(H_A^*(i,j,n,t)\ddot{Y}_i(t)+H_B^*(i,j,n,t)\dot{Y}_i(t)\right)H_C^*(i,j,n,t)Y_i(t) = \frac{Mg}{\xi_0}\left(\sin\frac{\alpha_j vt}{L}+A_j\cos\frac{\alpha_j vt}{L}+B_j\sinh\frac{\alpha_j vt}{L}+C_j\cosh\frac{\alpha_j vt}{L}\right) \quad (3.23)$$

where

$$\lambda=\frac{M}{L\xi_0} \quad (3.24)$$

and

$R_{0A}(i,j)$, $R_{0B}(i,j)$, $H_A(i,j,n,t)$, $H_B(i,j,n,t)$, and $H_C(i,j,n,t)$ *become respectively* $R_{0A}^*(i,j)$, $R_{0B}^*(i,j)$, $H_A^*(i,j,n,t)$, $H_B^*(i,j,n,t)$, $H_C^*(i,j,n,t)Y_i(t)$ after the substitution of (3.21) and (3.22) and summation sign neglected.

Evidently, an exact solution to equation (3.23) is not possible. Consequently, a modification of Struble's technique described in [3] is employed. Thus, equation (3.23) is rearranged to take the form :

$$\ddot{Y}_i(t)+\frac{\lambda H_A^*(i,j,n,t)\dot{Y}(t)}{R_{0A}^*(i,j)+\lambda H_A^*(i,j,n,t)}+\frac{R_{0B}^*(i,j)+\lambda H_C^*(i,j,n,t)}{R_{0A}^*(i,j)+\lambda H_A^*(i,j,n,t)}Y(t) = \frac{\lambda gL}{R_{0A}^*(i,j)+\lambda H_A^*(i,j,n,t)}\left(\sin\frac{\alpha_j vt}{L}+A_j\cos\frac{\alpha_j vt}{L}+B_j\sinh\frac{\alpha_j vt}{L}+C_j\cosh\frac{\alpha_j vt}{L}\right) \quad (3.25)$$

This technique seeks the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass. An equivalent free system operator defined by the modified frequency then replaces equation (3.25). To this end, the right-hand side of (3.25) is set to zero and a parameter $\varepsilon_0 \prec 1$ is considered for any arbitrary mass ratio λ defined as

$$\varepsilon_0=\frac{\lambda}{1+\lambda} \quad (3.26)$$

which implies

$$\lambda=\varepsilon_0+O\left(\varepsilon_0^2\right) \quad (3.27)$$

consequently

$$\frac{1}{R_{0A}^*(i,j)+\lambda H_A^*(i,j,n,t)}=\frac{1}{R_{0A}^*(i,j)}\left[1-\frac{\varepsilon_0 H_A^*(i,j,n,t)}{R_{0A}^*(i,j)}\right]+O\left(\varepsilon_0^2\right) \quad (3.28)$$

$$\left|\frac{\varepsilon_0 H_A^*(i,j,n,t)}{R_{0A}^*(i,j)}\right| \prec 1 \quad (3.29)$$

Which implies all the coefficients of $Y_i(t)$ and its derivatives in equation (3.25) can be written in terms of the parameter ε_0. When ε_0 is set to zero in equation (3.25), a situation corresponding to the case in which the inertia effct of the load is regarded as neglisgible is obtained. In such a case the solution is of the form

$$Y_i(t) = A^0 \cos(\beta_{MF}t - \varphi_i) \tag{3.30}$$

where A^0 and ϕ_1 are constants and

$$\beta_{MF}^2 = \frac{R^*_{OB(i,j)}}{R^*_{OA(i,j)}} \tag{3.31}$$

Since $\varepsilon_0 \prec 1$ for any arbitrary mass ratio λ, Stcuble's technique requires that the asymptotic solution of the homogeneous part of (3.25) be of the form

$$Y_t(t) = A(i,j)\cos(\beta_{MF}t - \psi(i,j)) \tag{3.31*}$$

When equation (3.31*) is substituted into the homogeneous part of (3.25) one arrives at

$$-2\beta_{MF}A(i,j)\sin[\beta_{MF}t - \psi(i,j)] + 2\beta_{MF}A(i,j)Q(i,j)\cos[\beta_{MF}t - \psi(i,j)] - A(i,j)\beta_{MF}^2\cos[\beta_{MF}t - \psi(i,j)]$$
$$-\frac{2v\varepsilon_0\Omega_{2B}(i,j)}{R^*_{0A}(i,j)}\left[\frac{2\Omega_{2B}(i,j)\cos\frac{n\pi vt}{L}}{\Omega_{2B}(i,j)} + 1\right]A(i,j)\beta_{MF}\sin[\beta_{MF}t - \psi(i,j)]$$
$$+\left\{N_a(i,j) - N_b(i,j)\left[\Omega_{1A}(i,j) + 2\Omega_{1B}(i,j)\cos\frac{n\pi vt}{L}\right]\right\}A(i,j)\cos[\beta_{MF}t - \psi(i,j)]$$
$$+\frac{v^2\varepsilon_0\Omega_{3B}(i,j)}{R_{0A}(i,j)}\left[\frac{\Omega_{3A}(i,j)}{\Omega_{3B}(i,j)} + 2\cos\frac{n\pi vt}{L}\right]A(i,j)\cos[\beta_{MF}t - \psi(i,j)] = 0 \tag{3.32}$$

to $O(\varepsilon_0)$ only, where

$$N_a(i,j) = \frac{R^*_{0B}(i,j)}{R^*_{0A}(i,j)}; \quad N_b(i,j)\frac{\varepsilon_0 R^*_{0B}(i,j)}{R^{*2}_{0A}(i,j)} \tag{3.33}$$

The variational equations are obtained by equating the coefficients of $\sin[\beta_{MF}t - \psi(i,j)]$ and $\cos[\beta_{MF}t - \psi(i,j)]$ terms on both sides of the above equation. Thus neglecting those terms that do not contribute to the variational equations, equation (3.32) reduces to

$$+2\beta_{MF}A(i,j)\dot{\psi}(i,j)\cos[\beta_{MF}t - \psi(i,j)] - \varepsilon_0 A(i,j)\frac{\Omega_{1A}(i,j)}{R_{0A}(i,j)}\beta_{MF}^2\cos[\beta_{MF}t - \psi(i,j)]$$
$$+\frac{v^2\varepsilon_0 A(i,j)\Omega_{3B}(i,j)}{R_{0A}(i,j)}\cos[\beta_{MF}t - \psi(i,j)] - 2\beta_{MF}A(i,j)\sin[\beta_{MF}t - \psi(i,j)]$$
$$-2\varepsilon_0 v\beta_{MF}\frac{\Omega_{2A}(i,j)A(i,t)}{R_{0A}(i,j)}\sin[\beta_{MF}t - \psi(i,t)] = 0 \tag{3.34}$$

From above, the variational equations are obtained respectively as

$$2\beta_{MF}A(i,j)\dot{\psi}(i,j) - \varepsilon_0 A(i,j)\frac{\Omega_{1A}(i,j)}{\Omega_{0A}(i,j)}\beta_{MF}^2 + \frac{v^2\varepsilon_0 A(i,j)\Omega_{3B}(i,j)}{R_{0A}(i,j)} = 0 \tag{3.35)}$$

and

$$\beta_{MF}A(i,j) + \varepsilon_0 v\beta_{MF}\frac{\Omega_{2A}(i,j)A(i,j)}{R_{0A}(i,j)} = 0 \tag{3.36}$$

rearranging the first order differential equations (3.35) and (3.36), one obtains

$$\dot{\psi}(i,t)=\frac{\varepsilon_0\left[\beta_{MF}^2\Omega_{1A}(i,j)-v^2\Omega_{3A}(i,j)\right]}{2\beta_{MF}R_{0A}(i,j)} \tag{3.37}$$

$$A(i,j)=\frac{\varepsilon_0 v\Omega_{2a}(i,j)A(i,t)}{R_{0A}(i,j)} \tag{3.38}$$

Equation (3.37) and (3.38) when solved respectively yields

$$\psi(i,j)=\frac{\varepsilon_0\left[\beta_{MF}^2\Omega_{1A}(i,j)-v^2\Omega_{3A}(i,j)\right]}{2\beta_{MF}R_{0A}(i,j)}i+\phi_c \tag{3.39}$$

where ϕ_c is a constant and $A(i,t)=f_0e^{-s_0t}$ (3.40)

where f_0 is a constant and $s_0=\dfrac{\varepsilon_0 v\Omega_{2A}(i,j)}{R_{0A}(i,j)}$ (3.41)

Thus, when the inertial effect of the moving load is taken into consideration, the first approximation to the homogeneous system is $\mathrm{Y}_i(t)=A(i.t)\cos\left[\varpi_i t-\phi_c\right]$ (3.42)

where $\varpi_i=\beta_{MF}-\dfrac{\varepsilon_0\left[\beta_{MF}^2\Omega_{1A}(i,j)-v^2\Omega_{3A}(i,j)\right]}{2\beta_{MF}R_{0A}^*(i,j)}$ (3.43)

is the modified natural frequency representing the frequency of the free system due to the presence of the moving mass. Therefore, the differential operator which acts on $Y_i(t)$ can be replaced by the equivalent free system operator defined by the modified frequency ϖ_i, that is,

$$\frac{d^2\mathrm{Y}_i(t)}{dt^2}+\mathrm{Y}_i(t)\varpi_i^2=Q_G\left(\sin\frac{\alpha_j vt}{L}+A_j\cos\frac{\alpha_j vt}{L}+B_j\sinh\frac{\alpha_j vt}{L}+C_j\cosh\frac{\alpha_j vt}{L}\right) \tag{3.44}$$

where

$$Q_G=\frac{\varepsilon_0 gL}{R_{OA}(i,j)} \tag{3.45}$$

Using Laplace transformation technique and the convolution theory, expression for $\mathrm{Y}_i(t)$ is obtained. Thus; in view of (3.2), one obtains

$$\begin{aligned}Z_m(x,t)=\sum_{i=1}^{m}\frac{Q_G}{\varpi_i\left(\varpi^4-\lambda_j^4\right)}&\Big\{\left(\varpi_i^2-\lambda_i^2\right)\Big\{C_j\varpi_j\left[\cosh\lambda_j t-\cos\varpi_i t\right]+\mathrm{B}_j\left[\varpi_i\sinh\lambda_j t-\lambda_j\sin\varpi_i t\right]\Big\}\\&+\left(\varpi_i^2+\lambda_i^2\right)\Big\{A_j\varpi_i\left[\cosh\lambda_i t-\cos\varpi_i t\right]-\left[\lambda_j\varpi_i t-\varpi_i\sin\lambda_j t\right]\Big\}\Big\}\\&\times\left(\sin\frac{\alpha_i}{L}+A_i\cos\frac{\alpha_i}{L}+B_i\sinh\frac{\alpha_i}{L}+C_i\cosh\frac{\alpha_i}{L}\right)\end{aligned} \tag{3.46}$$

where $\lambda_j=\dfrac{\alpha_j v}{L}$, which represents the response to a moving mass of anon-uniform Rayleigh beam resting on a variable elastic foundation. The corresponding moving force solution is

$$\begin{aligned}Z_m(x,t)=\sum_{i=1}^{m}\frac{P_G}{\beta_{MF}\left(\beta_{MF}^4-\alpha_j^4\right)}&\Big\{\left(\beta_{MF}^2-\alpha_j^2\right)\Big\{C_j\beta_{MF}\left[\cosh\alpha_j t-\cos\beta_{MF}t\right]\Big\}\\&+\mathrm{B}_j\left[\beta_{MF}\sinh\alpha_j t-\alpha_j\sin\beta_{MF}t\right]\Big\}+\left(\varpi_{MF}^2-\alpha_j^2\right)\Big\{A_j\varpi_{MF}\left[\cos\alpha_j t-\cos\varpi_{MF}t\right]\\&-\left[\alpha_j\sin\varpi_{MF}t-\varpi_{MF}\sin\alpha_j t\right]\Big\}\Big\}\times\left[\sin\frac{\lambda_{ix}x}{L}+A_i\cos\frac{\lambda_{ix}}{L}+B_i\sinh\frac{\lambda_{ix}}{L}+C_i\cosh\frac{\lambda_{ix}}{L}\right]\end{aligned} \tag{3.46*}$$

where
$$P_G = \frac{Mg}{\xi_0 R_0^* A} \tag{3.46**}$$

4.0 Illustrative Examples

In order to illustrate our results in the foregoing analysis, in what follows we provide some examples;

(a) simply supported Rayleigh beam
(b) clamped-clamped Rayleigh beam

4.1 Simply Supported Rayleigh Beam

For a Rayleigh beam having simple supports at both ends, the boundary conditions are
$$Z(0,t)=0=Z(L,t);\ Z_{xx}(0,t)=0=Z_{xx}(L,t) \tag{4.1}$$
It follows for normal models that
$$U_i(0)=0=U_i(L);\ U_{i,xx}(0)=U_{i,xx}(L) \tag{4.2}$$
which imples
$$U_j(0)=0=U_j(L);\ U_{j,xx}(0)=U_{j,xx}(L) \tag{4.3}$$
Applying (4.2) and (4.3)
$$A_i=0,\ B_i=0 \text{ and } C_i=0 \text{ and } \lambda_i=i\pi \tag{4.4}$$
Similarly,
$$A_j=0,\ B_j=0 \text{ and } C_j=0 \text{ and } \lambda_j=j\pi \tag{4.5}$$
Substituting equation (4.4) and (4.5) into equation (3.25), rearranging and following arguments similar to those in previous section, Struble's technique is used to obtain
$$\gamma_{mm}=\gamma_{mf}-\frac{\varepsilon_0\left(L^2\gamma_{mf}^2+V^2 i^2\pi^2\right)}{4L\gamma_{mf}\left(A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}\right)} \tag{4.6}$$
where
$$A^*=\frac{1}{2L}\left[L+\left(L-i\pi^2\right)\left(2B^*-1\right)\right];\quad B^*=\frac{1}{2}+\frac{2j\left(1+i^2-j^2\right)}{\pi\left[(1+j)^2-j^2\right]\cdot\left[(1-j)^2-i^2\right]} \tag{4.6*}$$
and
$$\gamma_{mf}^2=\frac{1}{2\xi_0\left(A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}\right)}\left\{\frac{EJi^2\pi^3}{L^3}[i\pi(5i-6)]+\frac{30\left(1-j^2-i^2\right)\left(1+2ij-i^2\right)}{\left[(1+j)^2-i^2\right]\left[(1-j)^2-i^2\right]}\right.$$
$$\left.+\frac{3\left(9-j^2-i^2\right)\left(9-4ij+i^2\right)}{\left[(3+j)^2-i^2\right]\left[(3-j)^2-i^2\right]}\right\}+\frac{FL}{\pi^4(i-j)^4(1+j)^4}\left[3L^2(i-j)^4(-1)^{i+j}\left[\pi^2(2-L)(i+j)^2+2L\right]\right. \tag{4.7}$$
$$\left.-3L^2(1+j)^4(-1)^{j-i}\left[\pi^2(2-L)(j-i)^2+2L\right]-32j\pi^2 i\left(j^2-i^2\right)^2+12L^3\left(j^4+6i^2j^2+i^4\right)\right]$$
as the modified frequency of the free system due to the presence of the moving mass of this model. γ_{mf} is the frequency of the corresponding moving force problem. Thus, the moving mass problem takes the form
$$\frac{d^2Y_i(t)}{dt^2}+\gamma_{mm}^2 Y_i(t)=\frac{L\varepsilon_0 g}{\left(A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}\right)}\sin\frac{j\pi vt}{L} \tag{4.8}$$

Equation (4.8) when solved in conjunction with the initial conditions one obtains expression for $Y_i(t)$. In view of (3.2),

$$Z_m(x,t)=\sum_{i=1}^{m}\frac{L\varepsilon_0 g}{\gamma_{mm}\left[A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}\right]\left[\gamma_{mm}^2-\left(\frac{j\pi v}{L}\right)^2\right]}\left[\gamma_{mm}\sin\frac{j\pi vt}{L}-\frac{j\pi v}{L}\sin\gamma_{mm}t\right]\sin\frac{i\pi x}{L} \qquad (4.9)$$

which represents the transverse-displacement response to a moving mass of a simply supported non-uniform Rayleigh beam resting on a variable winkler elastic foundation. The corresponding moving force solution is

$$Z_m(x,t)=\sum_{i=1}^{m}\frac{P_{0f}}{\gamma_{mf}\left[\gamma_{mf}^2-\left(\frac{j\pi v}{L}\right)^2\right]}\left[\gamma_{mf}\sin\frac{j\pi vt}{L}-\frac{j\pi v}{L}\sin\gamma_{mf}t\right]\sin\frac{i\pi x}{L} \qquad (4.9^*)$$

where

$$P_{0f}=\frac{gM}{\varepsilon_0 L\left[A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}\right]} \qquad (4.9^{**})$$

4.2 **Clamped-clamped Rayleigh Beam**

For a clamped-clamped Rayleigh beam, the boundary conditions are

$$Z(0,t)=0=Z(L,t);\ Z_x(0,t)=0=Z_x(L,t) \qquad (4.10)$$

which implies that for normal modes

$$U_i(0)=0=U_i(L);\ U_{i,x}(0)=U_{i,x}(L) \qquad (4.11)$$

which follows hat

$$U_j(0)=0=U_j(L);\ U_{j,x}(0)=U_{j,x}(L) \qquad (4.12)$$

Applying (4.11) to (3.21) one obtains

$$A_i=\frac{\sinh\alpha_i-\sin\alpha_i}{\cos\alpha_i-\cosh\alpha_i}=\frac{\cos\alpha_i-\cosh\alpha_i}{\sin\alpha_i+\sinh\alpha_i}=-C_i \qquad (4.13)$$

and

$$B_I=-1 \qquad (4.14)$$

from (4.13), the frequency equation can simply be obtained as

$$\cos\alpha_i\cosh\alpha_i=1 \qquad (4.15)$$

Expressions for A_j, B_j, and C_j and corresponding frequency equation are obtained by simple interchange of i and j in (4.13). (4.14) and (4.15). Thus, the general solutions of the associated moving mass problem are obtained by substituting relevant results in (4.13). (4.14) and (4.15) into (3.46).

5.0 Remarks on Analytical Solutions

The displacement response of a non-uniform Rayleigh beam may increase without bound. This is an interesting phenomenon in a dynamic system such as this. From equation (4.9*), it is evident that the simply supported non-uniform Rayleigh beam tranversed by a moving force will be in a state of resonance when

$$\gamma_{mf}=\frac{j\pi v}{L} \qquad (5.1)$$

while equation (4.9) shows that the same beam transversed by a moving mass encounters a resonant effect at

$$\gamma_{mm}=\frac{j\pi v}{L} \qquad (5.2)$$

where $$\gamma_{mm} = \gamma_{mf}\left[1-\varepsilon_0\left(\frac{1}{4\left(A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}\right)}+\frac{v^2 i^2\pi^2}{4L^2\gamma^2_{mf}\left(A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}\right)}\right)\right] \tag{5.3}$$

It is then easily shown that $$\frac{\gamma_{mf}\left[A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}-\frac{\varepsilon_0}{4}\left(1+\frac{r^0\pi^2 i^2}{L^2\gamma^2_{mf}}\right)\right]}{A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}}=\frac{j\pi v}{L} \tag{5.4}$$

clearly $$A^*+\frac{r^0\pi^2 i^2 B^*}{L^2} \succ A^*+\frac{r^0\pi^2 i^2 B^*}{L^2}-\frac{\varepsilon_0}{L}\left(1+\frac{v^2\pi^2 i^2}{L^2\gamma^2_{mf}}\right) \tag{5.5}$$

Consequently, for the same natural frequency, the critical speed for the moving mass problem is smaller than that of the moving force problem. Thus, the resonance is reached earlier in the moving mass system than in the moving force system. Furthermore, from equation (3.46*), it is evident that for other classical boundary conditions, the non-uniform Rayleigh beam on a variable Winkler elastic foundation and transversed by a moving force encounters a resonance effect when $$\beta_{mf} = \frac{\alpha_j v}{L} \tag{5.6}$$

while equation (3.46) shows that the same beam under the action of a moving mass reaches the state of resonance whenever $$\varpi_i = \frac{\alpha_j v}{L} \tag{5.7}$$

Consequently, in view of (3.43)

$$\beta_{mf}\frac{R^*_{0A}(i,j)-\frac{\varepsilon_0}{2}\left(\Omega_{1A}(i,j)-\frac{v^2\Omega_{3A}(i,j)}{\beta^2_{MF}}\right)}{R^*_{0A}}=\frac{\alpha_j v}{L} \tag{5.8}$$

From equations (5.6) and (5.8), the results and analysis similar to those of the simply supported end conditions are obtained for clamped-clamped end conditions.

6.0 Discussions of Numerical Results

In this section, calculations of practical interests in dynamics and Engineering design are presented for both illustrative examples.

An elastic beam of length 12.123 m has been considered. Furthermore, v = 8.12m/s, $\frac{M}{L\mu} = 0\cdot 25$,

$EJ = 6\cdot 068\times 10^6 m^3/s^2$, the values of foundation moduli F are ranged between $0N/m^3$ *and* $1{,}000{,}000\, N/m^3$ and values of rotatory inertial correction factor varies between 0 and 50. The results are displayed graphically in the Figures 1 to 10.

Figures 1 and 2 show the deflection of the simply supported non-uniform Rayleigh beam under the action of a travelling moving force and a travelling moving mass respectively for various values of rotatory inertia correction factor r^0. Clearly, as the rotatory inertia factor increases, the response amplitudes decrease. Furthermore, Figure 3 displays for fixed r^0 the displacement response of the simply supported beam under the action of a moving forces for various values of foundation moduli F. It is shown that as F increases, the displacement response decreases. The same analysis is obtained in Figure 4 when the same beam is under the

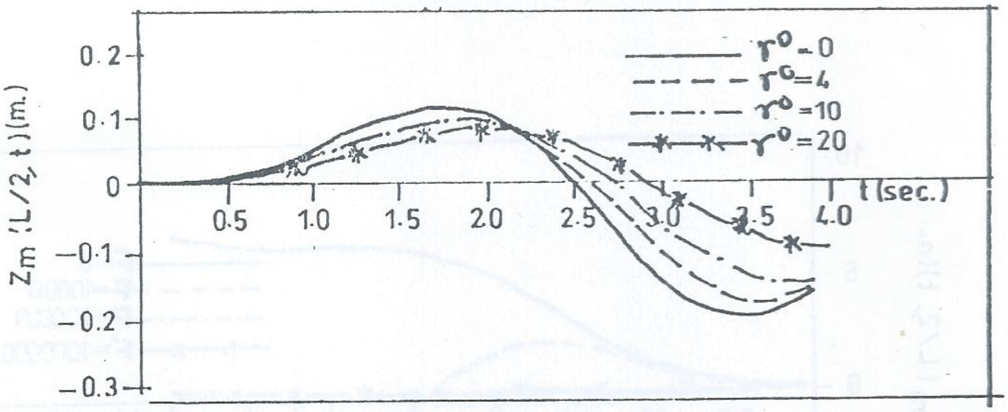

Figure 1: Deflection profile of moving force for simply supported non-uniform Rayleigh beam for various values of r^0

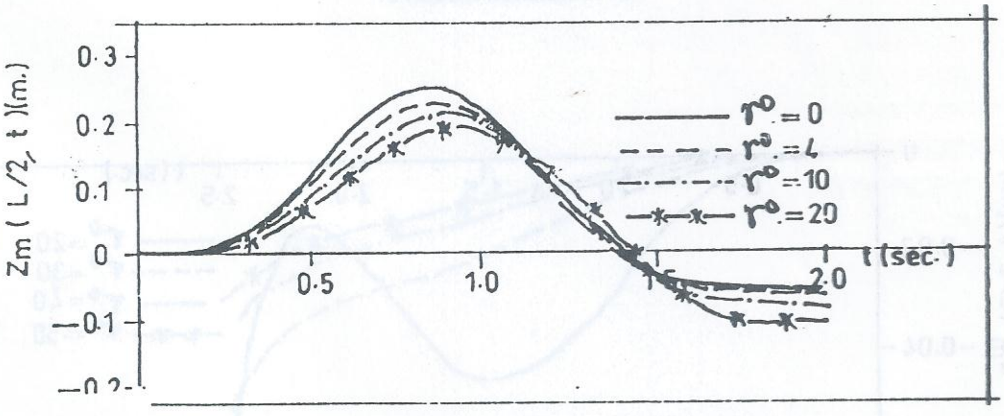

Figure 2: Deflection profile of moving mass for simply supported non-uniform Rayleigh beam for various values of r^0

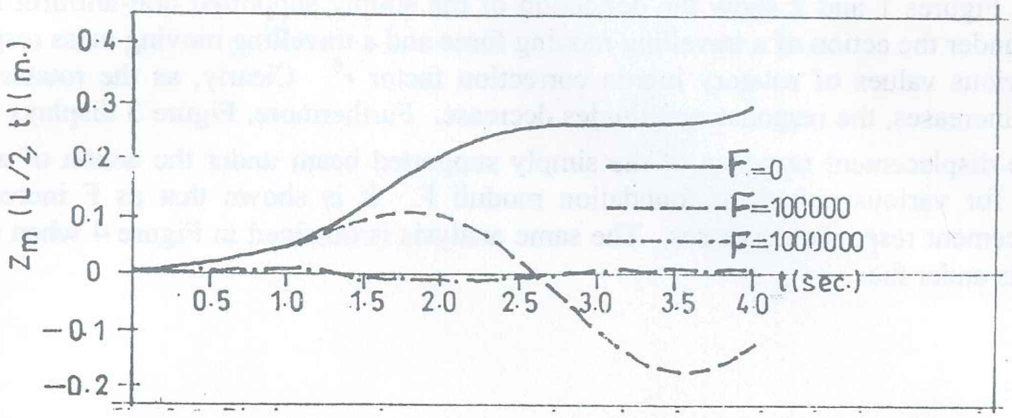

Figure 3: Displacement response of moving force for simply supported non-uniform Rayleigh beam for values of foundation moduli F.

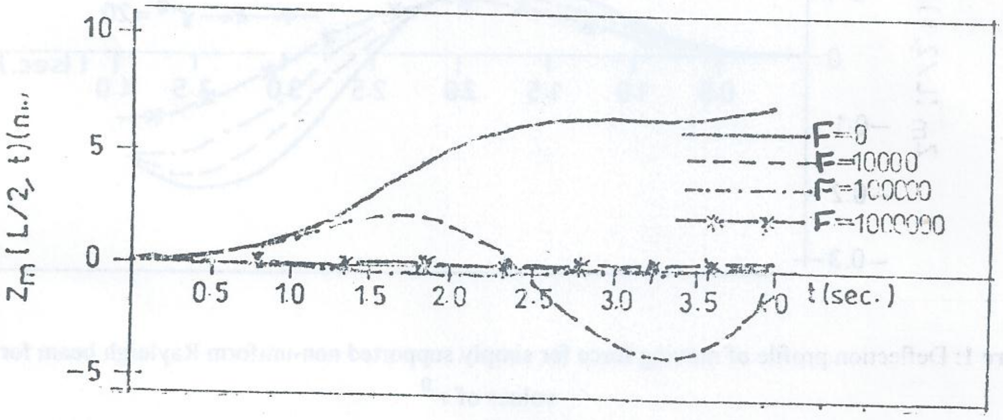

Figure 4: Displacement response of moving mass for simply supported non-uniform Rayleigh beam for values of foundation moduli F.

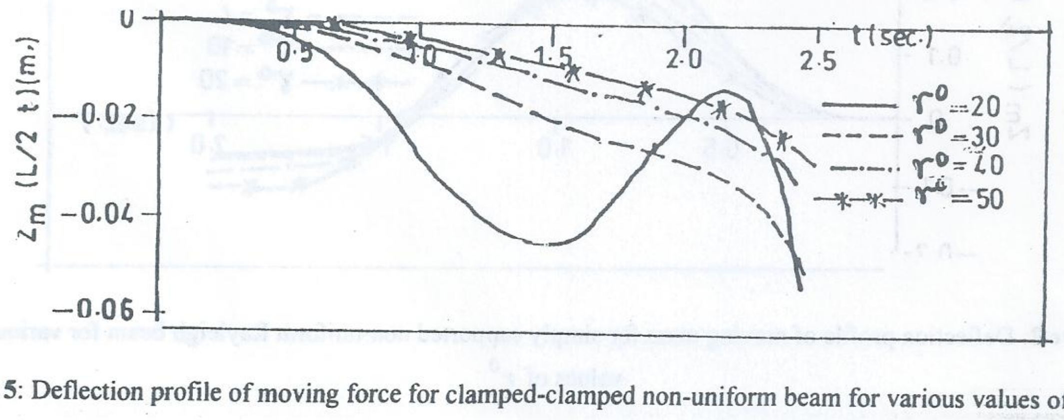

Figure 5: Deflection profile of moving force for clamped-clamped non-uniform beam for various values of r^0

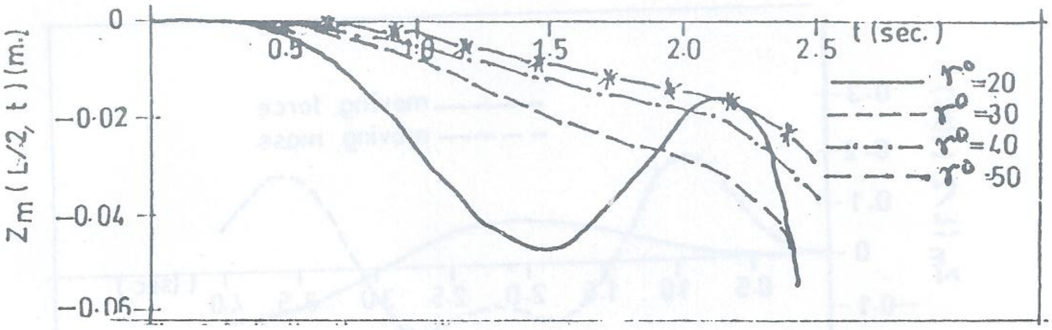

Figure 6: Deflection profile of moving force for clamped-clamped non-uniform beam for various values of r^0

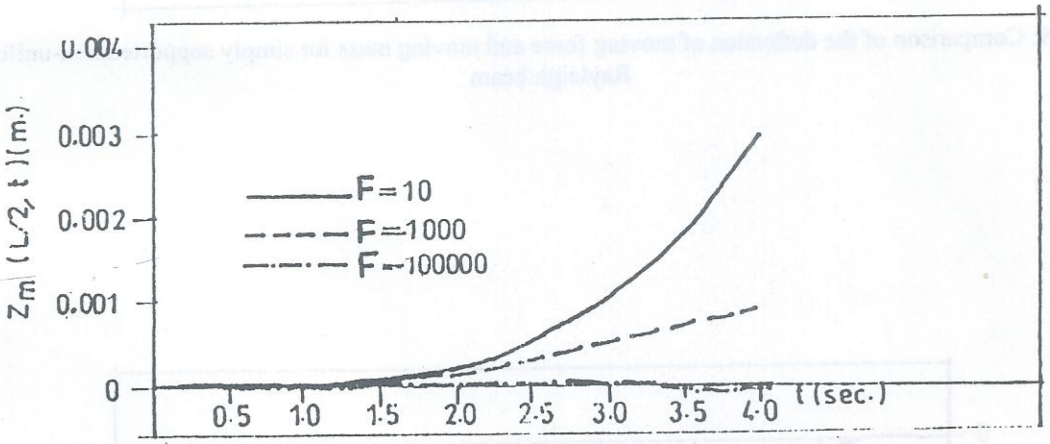

Figure 7: Displacement response of moving force for clamped-clamped non-uniform Rayleigh beam for various values of foundation moduli F.

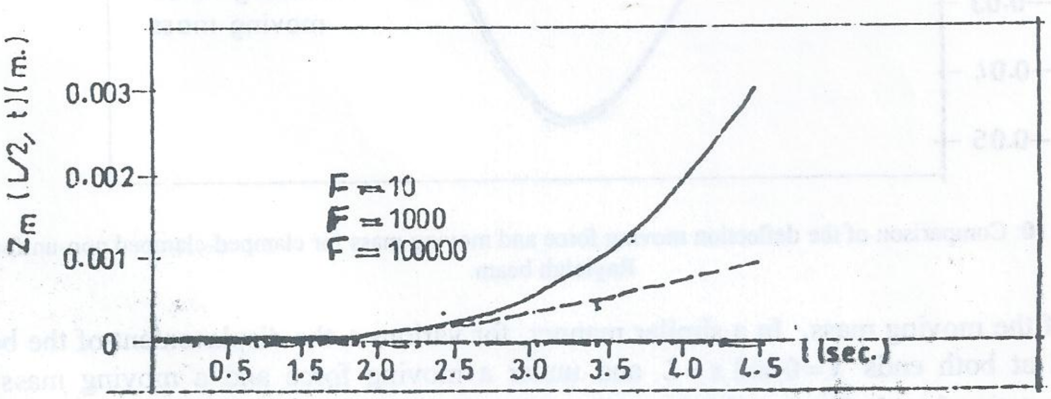

Figure 8: Displacement response of moving mass for clamped-clamped non-uniform Rayleigh beam for various values of foundation moduli F.

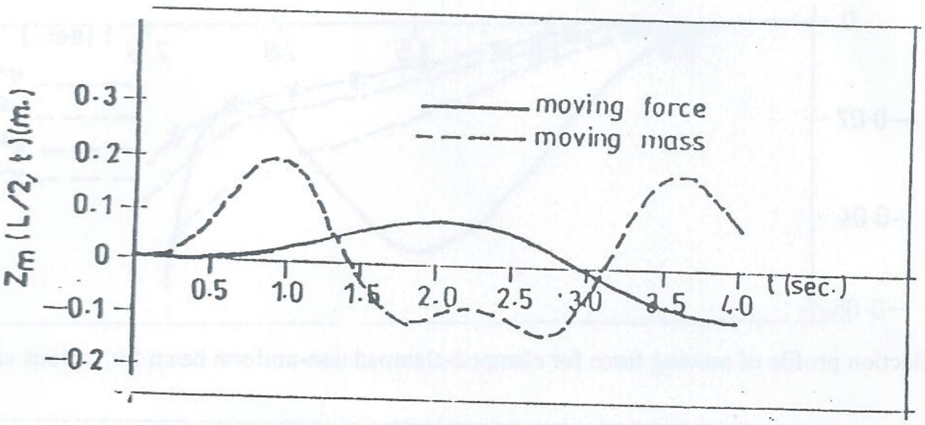

Figure 9: Comparison of the deflection of moving force and moving mass for simply supported non-uniform Rayleigh beam

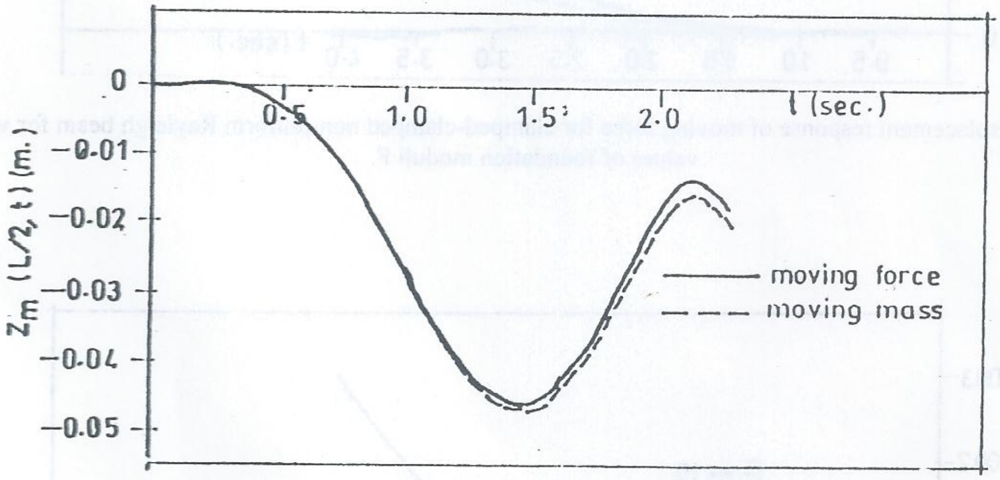

Figure 10: Comparison of the deflection moving force and moving mass for clamped-clamped non-uniform Rayleigh beam.

action of the moving mass. In a similar manner, for various t, the displacement of the beam clamped at both ends $x = 0$ and $x = L$ and under a moving force and a moving mass for various rotatory inertia factor and a fixed foundation modulus F are respectively plotted in

Figures 5 and 6. Results similar to Figures 6.1 and 6.2 are obtained. However, higher values of rotatory inertia factor are required for a more noticeable effect than those of simply supported end conditions. In particular, values of r between 0 and 20 give more less the same response amplitudes. Also, Figures 7 and 8 show the deflection profile of the clamped-clamped beam traversed by a moving force for various values of F and for fixed value of r^0, it is evident that as F increases, the response amplitude decreases. Similar results are obtained when the same clamped-clamped beam is tranversed by a moving mass as shown in Figure 8. For the purpose of comparison the displacement curves of the moving force and moving mass for both cases of simply supported and clamped-clamped non-uniform Rayleigh beam with $r^0 = 4$, F = 100000N/m are illustsrated in Figures 9 and 0. It can be noted that the response amplitude for a moving mass is greater than that of moving force. This result similarly holds for other choice of values of r^0 and F. This shows that it could be tragic to rely on the moving force solution as an approximation to the moving mass problem.

7.0 Conclusion

The problem of the vibrations under a moving load of a non-uniform Rayleigh beam on a variable elastic foundation has been solved. The technique involves the use of the elegant Galarkin's method to reduce the governing equation forth order partial differential equations. For the solutions of these equations, a modification of the Struble's technique is employed. Numerical analysis was carried out and the work exhibits the following interesting features.

(i) For both simply supported and clamped-clamped end conditions, as the rotatory inertia correction factor increases, the response amplitudes of the non-uniform Rayleigh beam decrease.

(ii) Higher values of rotatory inertia factor r^0 are required for a more noticeable effect in the case of clamped-clamped end conditions than those of simply supported end conditions for both moving force and moving mass problems.

(iii) For fixed r^0, the displacements of a non-uniform Rayleigh beam resting on a variable elastic foundation decrease as the foundation moduli F increases whether beam is simply supported or clamped at both ends.

(iv) For fixed r^0 and F the response amplitude for the moving mass problem is greater than that of the moving force problem for both illustrative boundary conditions.

(v) As r increases, the critical speed of the non-uniform Raleigh beam increases.

Finally, in both illustrative examples, for the same natural frequency, the critical speed for the moving mass problem is smaller than that of the moving force problem.

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