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A liquid flow in an open container

P.B. Shola
Department of Mathematics, University of Ilorin, Ilorin, Nigeria

Abstract

This article considers a two dimensional slow flow of a highly viscous liquid in an open rectangular container, driven by the container's base which moves along its direction at a constant speed. Using perturbation technique, the first deviation from the associated hydrostatic solution for a right angled contact angle is obtained and presented.

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1.0 Introduction

A two dimensional, slow flow of a highly viscous liquid in an open rectangular container is considered. The liquid is driven by the container's base which moves steadily along its plane with no liquid slipping out of the lower corners of the container. The upper part of the liquid is exposed to the atmosphere and is being controlled by the force of gravity and surface tension. Consequently this part of the boundary (i.e the part of the fluid exposed to the atmosphere) is a free surface whose shape has to be determined as part of the solution to the problem. (see the diagram below).

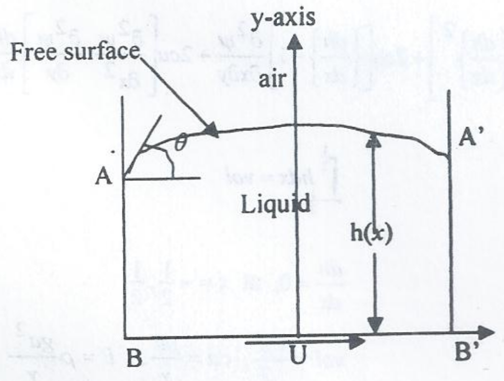

Figure 1: Flow in an open container

The historical background to this problem can be found in [1]. Numerical computation for the solution to the problem was considered for instance in [2]. An analytic approach ,through perturbation technique, was considered in [3] to obtain an approximate solution to the problem for the case when the capillary number is small and the velocity distribution of the container's base (which drives the fluid) is such that it causes no singularity in the flow. This paper relaxes the later restriction by considering the case when the container's base moves with a constant speed, U, along its direction. In this case there are singularities in the flow at the two lower corners of the container's base that have to be considered. The method of subtracting a function that has the same order of magnitude of singularity from the solution is used: Local solution at these corners that possess the same magnitude are found and subtracted from the solution, leaving a well behaved function for the solution. The contact angle, θ, is assumed to be zero. The contact angle in this case is taken to be the angle between the free surface and the horizontal at the point the free surface attaches itself to the vertical wall. The horizontal axis (i.e x-axis) is assumed to be along the base of the container while the y-axis is taken to be mid-way between the two vertical walls of the container. Like in [3], the flow functions (stream, pressure and free surface functions) are expanded in powers of the capillary number and the first two terms of the expansion obtained for these functions so that for a flow with a very small capillary number , these can be considered as an approximate solution to the problem,

2.0 Governing Equations

In dimensionless variables, with u, v denoting the horizontal and vertical components of the velocity, ψ and p denoting the stream function and pressure respectively and $h(x)$ denoting the height of the free surface from the x-axis at point x, the equation governing the flow is the biharmonic equation

$$\nabla^4\psi = 0 \tag{2.1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$, $u=\frac{\partial\psi}{\partial y}$, $v=\frac{\partial\psi}{\partial x}$ together with the following boundary conditions, (see [2],[3] for instance for detail) Boundary conditions at fixed boundaries ($x=\pm\frac{1}{2}, y=0$)

$$\psi = \frac{\partial\psi}{\partial x}=0, \text{ at } x=\pm\frac{1}{2} \tag{2.2}$$

$$\psi = \frac{\partial\psi}{\partial x}=1, \text{ at } y=0 \tag{2.3}$$

Boundary conditions at the free surface $y=h(x)$

$$\frac{\partial\psi}{\partial y}\frac{dh}{dx}+\frac{\partial\psi}{\partial x}=0, \tag{2.4}$$

$$4\frac{\partial^2\psi dh}{\partial x\partial y dx}+\left[\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial x^2}\right]\left[\left\{1-\frac{dh}{dx}\right\}^2\right]=0, \tag{2.5}$$

$$(cap+Bh)\left[+\left\{\frac{dh}{dx}\right\}^2\right]+2ca\left[\left\{\frac{dh}{dx}\right\}-1\right]\frac{\partial^2\psi}{\partial x\partial y}+2ca\left[\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y}\right]\frac{dh}{dx}=\frac{d^2h}{dx^2}\left[\left\{\frac{dh}{dx}\right\}\right]^{\frac{1}{2}} \tag{2.6}$$

Volume constraint

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} h dx = vol \tag{2.7}$$

Contact angle constraint

$$\frac{dh}{dx}=0, \text{ at } x=-\frac{1}{2},\frac{1}{2} \tag{2.8}$$

$$vol=\frac{vol}{a^2},\quad ca=\frac{\mu u}{\gamma},\quad B=\rho\frac{ga^2}{\gamma}$$

The pressure distribution is obtained from the equations

$$\frac{\partial\rho}{\partial x}=\frac{\partial\nabla^2\psi}{\partial y},\quad \frac{\partial\rho}{dy}=\frac{\partial\nabla^2\psi}{\partial x} \tag{2.9}$$

Expansion

Here we assume that the functions h, ψ, $p(x,y)$ can be expanded in the powers of the capillary number, ca, (for ca<<1) as follows

$$h(x)=h_0+cah_1(x)+ca^2h_2(x)+\cdots \tag{2.10}$$

$$\psi(x,y)=\psi_o(x,y)+ca\psi_1(x,y)+ca^2\psi_2(x,y)+\cdots$$

$$p=\frac{p_o}{ca}+p_1(x,y)+cap_2(x,y)+\cdots$$

$$h_o=vol,\quad p_oBh_o$$

First Approximation

Inserting the expansion above in equation (2.1) to (2.9) and equating the coefficient of ca^0 we have

$$\nabla^4\psi_0 = 0 \tag{2.11}$$

$$\psi_O = \frac{\partial \psi_O}{\partial x} = 0 \quad \text{at} \quad x = \pm\frac{1}{2} \tag{2.12}$$

$$\psi_O = 0, \qquad \psi_O = \frac{\partial \psi_O}{\partial x} = 1 \quad \text{at} \quad y = 0 \tag{2.13}$$

$$\frac{\partial \psi_O}{\partial x} = 0 \quad \text{at} \quad y = h_O \tag{2.14}$$

$$\frac{\partial^2 \psi_0}{\partial y^2}; \frac{\partial^2 \psi_0}{\partial x^2} \quad \text{at } y = h_0 \tag{2.15}$$

$$\frac{\partial p_1}{\partial x} = \frac{\partial \nabla^2 \psi_O}{\partial y}, \qquad \frac{\partial p_1}{\partial y} = \frac{\partial \nabla^2 \psi_O}{\partial x} \tag{2.16}$$

whereas matching the coefficient of *ca* in equations (2.6), (2.7) and (2.8) gives

$$\frac{d^2 h_1}{dx^2} \cdot Bh_1 = p_1 - 2\frac{\partial^2 \psi_O}{\partial x \partial y} \tag{2.17}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} h_1 dx = 0 \tag{2.18}$$

$$\frac{\partial h_1}{\partial x} = 0 \qquad \text{at } x = \pm\frac{1}{2} \tag{2.19}$$

Solution

The solution to the equations (2.11) to (2.19) above possesses singularities at the two lower corner points and for this reason the solution is expressed in the form

$$\psi_O = \varphi + \varphi_R + \varphi_R \tag{2.20}$$

where φ_R and φ_R are local solution at the corners and are each obtained as follows. A solution local to the corner point A, $(-\frac{1}{2}, 0)$ of the container is obtained from the biharmonic equation in polar form together with the boundary conditions along the vertical and the horizonal boundaries containing the point:

$$\nabla^4 \varphi_L = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]^2 \varphi_L = 0$$

$$\frac{1}{r}\frac{\partial \varphi_L}{\partial \theta} = 0 \ \text{ at } \theta = \tfrac{\pi}{2}, \quad \varphi_L(r,\theta) = \varphi_L = \left(r, \tfrac{\pi}{2}\right) = 0, \quad \frac{1}{r}\frac{\partial \varphi_L}{\partial \theta} = \frac{1}{2} \ \text{ at } \theta = 0$$

to be

$$\varphi_L = \frac{\frac{r}{2}}{1 - \frac{\pi^2}{2}}\left(\tfrac{\pi}{2}\left(\theta \cdot \tfrac{\pi}{2}\right)\sin\theta + \theta\cos\theta\right)$$

where r is the radial distance measured from the corner point A taken as the origin. This local solution possesses the required singularity in the biharmonic equation at the corner. Transforming this local solution into the Cartesian coordinates and shifting the origin from the corner point to the mid-point of the base of the container, the local solution transforms into

$$\varphi_L = \frac{\frac{1}{2}}{1-\frac{\pi^2}{4}}\left\{\left[\pi\frac{y}{2}+x+\frac{1}{2}\right]\tan^{-1}\frac{y}{x+\frac{1}{2}}\cdot\pi^2\frac{y}{4}\right\}$$

Using the same procedure the local solution at the other corner point $(\frac{1}{2}, 0)$ is obtained as

$$\varphi_L = \frac{\frac{1}{2}}{1-\frac{\pi^2}{4}}\left\{\left[\pi\frac{y}{2}+x\frac{1}{2}\right]\tan^{-1}\frac{y}{x-\frac{1}{2}}\cdot\pi^2\frac{y}{4}\right\}$$

Now substituting expression for ψ_0 (in 2.20) into the equation (2.11) to (2.15) above the following equations results

$$\nabla^4\varphi = 0$$

$$\varphi\left(-\tfrac{1}{2}, y\right) = f_1(y),\quad \varphi\left(\tfrac{1}{2}, y\right) = f_2(y),\quad \varphi(x,0) = 0,$$

$$\frac{\partial\varphi}{\partial x} = g_1(y)\ \text{at}\ x = -\tfrac{1}{2},\quad \frac{\partial\varphi}{\partial x} = g_2(y)\ \text{at}\ x = \tfrac{1}{2},\quad \frac{\partial\varphi}{\partial y} = 0\ \text{at}\ y = 0,\quad \frac{\partial\varphi}{\partial x} = G(x)\ \text{at}\ y = h_0$$

$$\frac{\partial^2\varphi}{\partial x^2}\cdot\frac{\partial^2\varphi}{\partial y^2} = S(x)\ \text{at}\ y = h_0$$

where

$$f_1 = \frac{\frac{1}{4}}{1-\frac{\pi^2}{4}}\left\{\left[\pi\frac{y}{2}-1\right]\tan^{-1}y+\pi^2\frac{y}{4}\right\} \qquad f_2 = \frac{\frac{1}{4}}{1-\frac{\pi^2}{4}}\left\{\left[\pi\frac{y}{2}+1\right]\tan^{-1}y\pi^2\frac{y}{4}\right\}$$

$$g_1 = \frac{\frac{1}{4}}{1-\frac{\pi^2}{4}}\left\{\frac{\left(\pi\frac{y}{2}-1\right)y}{1+y^2}\tan^{-1}y\right\}, \qquad g_2 = \frac{\frac{1}{4}}{1-\frac{\pi^2}{4}}\left\{\frac{\left(\pi\frac{y}{2}+1\right)y}{1+y^2}+\tan^{-1}y\right\}$$

$$S(x) = -\frac{\frac{1}{4}\left[h_0^2-\left(x+\frac{1}{2}\right)^2\right]\left[\pi\left(x+\frac{1}{2}\right)-2h_0\right]}{1-\frac{\pi^2}{4}\left[\left(x+\frac{1}{2}\right)^2+h_0^2\right]^2}\quad \frac{\frac{1}{4}\left[h_0^2-\left(x-\frac{1}{2}\right)^2\right]\left[\pi\left(x-\frac{1}{2}\right)-2h_0\right]}{1-\frac{\pi^2}{4}\left[\left(x-\frac{1}{2}\right)^2+h_0^2\right]^2}$$

$$G(x) = -\frac{\frac{1}{4}}{\left(1-\frac{\pi^2}{4}\right)}\left[\frac{h_0\left(\frac{1}{2}\pi h_0+x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2+h_0^2}+\tan^{-1}\frac{h_0}{x+\frac{1}{2}}\right]+\frac{\frac{1}{4}}{\left(1-\frac{\pi^2}{4}\right)}\left[\frac{h_0\left(\frac{1}{2}\pi h_0+x-\frac{1}{2}\right)}{\left(x-\frac{1}{2}\right)^2+h_0^2}+\tan^{-1}\frac{h_0}{x-\frac{1}{2}}\right]$$

These equations has the solution

$$\varphi = \varphi_1 + \varphi_2 \tag{2.21}$$

where

$$\varphi_1 = \sum_n \left\{ \left[A_n \sinh\alpha_n x - 2x\tanh\left(\left(\frac{\alpha_n}{2}\right)\cosh\alpha_n x\right) + b_n\left(\cosh\alpha_n x - 2x\coth\left(\frac{\alpha_n}{2}\right)\sinh\alpha_n x\right)\right]\sin\alpha_n y \right.$$
$$\left. + \left[E_n\left(\sinh\lambda_n y - \frac{y}{h_0}\tan\lambda_n h_0\cosh\lambda_n y + \frac{y\sinh^2\lambda_n h_0}{h_0\cosh\lambda_n h_0}\sinh[\lambda_n(y-h)]\right) + \frac{s_n}{2\lambda_n} y\sinh[\lambda_n(y-h_0)]\right]\sin\lambda_n x \right\} \quad (2.22)$$

with

$$\lambda_n = 2\pi n, \quad \alpha_n = \pi\frac{n}{h_0}, \quad s_n = 4\int_0^{\frac{1}{2}} S_n\sin\lambda_n x dx, \quad b_n = \frac{\sinh\frac{\alpha_n}{2}}{h_0(\alpha_n + \sinh\alpha_n)}\int_0^{h_0}(g_1(y) - g_2(y))\sin\alpha_n y dy \quad (2.23)$$

and

$$\varphi_2 = \sum\left\{\left[\left[\left\{ d_n\left[-\frac{1}{\alpha_n}\left(\frac{1}{2}\alpha_n\tanh\frac{\alpha_n}{2} + 1\right)\sinh\alpha_n + x\cosh\alpha_n\right] + \left\{ c_n\left[-\frac{1}{\alpha_n}\left(\frac{1}{2}\alpha_n\coth\frac{\alpha_n}{2} + 1\right)\cosh\alpha_n x\right]\right]\right.\right.\right.$$
$$+ x\sinh\alpha_n x\}\sin\alpha_n y + \left\{\frac{2H}{\sinh(2\lambda_n h_0)}\right\}[y\sinh(\lambda_n h_0)\cosh\lambda_n(y-h_0) - h_0\sinh\lambda_n y] \quad (2.24)$$
$$\left. + \left\{\frac{t_n}{\lambda_n\cosh(\lambda_n h_0)}[\lambda_n(y-h_0)\sinh\lambda_n y - \coth(\lambda_n h_0)\sinh\lambda_n y]\right\}\cos\lambda_n x\right\}$$

where

$$t_n = 4\int_0^{\frac{1}{2}} G(x)\sin\lambda_n x dx, \qquad d_n = \frac{\alpha_n\cosh\frac{\alpha_n}{2}}{h_0(\alpha_n - \sinh\alpha_n)}\int_0^{h_0}(f_2(y) - f_1(y))\sin\alpha_n y dy \quad (2.2.5)$$

The boundary conditions on φ_1 and φ_2 are chosen to satisfy the following

	φ_1	φ_2	φ
$\varphi(-1/2, y)$	0	$f_1(y)$	$f_1(y)$
$\varphi(1/2, y)$	0	$f_2(y)$	$f_2(y)$
$\partial\varphi/\partial x$ at $x = -½$	$g_1(y)$	0	$g_1(y)$
$\partial\varphi/\partial x$ at $x = ½$	$g_2(y)$	0	$g_2(y)$
$\partial\varphi/\partial x$ at $y = h_0$	0	$G(x)$	$G(x)$
$\partial\varphi/\partial x$ at $y = h_o$	0	0	0
$\partial^2\varphi/\partial y^2 - \partial^2\varphi/\partial x^2$ at $y = h_0$	0	0	0

The unknown constants A_n, E_n in φ_1 are determined from the system of linear equations

$$\sum_n A_n a_{mn} + E_{nm} e_m = R_m, \qquad A_m a_m + \sum_n E_n e_{mn} = R_m \quad (2.26)$$

where

$$a_n = \frac{h_0(\alpha_n - \sinh\alpha_n)}{2\cosh\frac{\alpha_n}{2}} \quad (2.27a)$$

$$a_{nm} = \int_0^{\frac{1}{2}} \left(\sinh \alpha_n x - 2x \tanh\left(\frac{\alpha_n}{2}\right) \cosh \alpha_n x \right) \alpha_n \sin \lambda_m x dx \tag{2.27b}$$

$$= \alpha_n \left[\mathrm{I}_2\left(\lambda_m, \alpha_n, \tfrac{1}{2}\right) \cdot 2\mathrm{I}_7\left(\lambda_m, \alpha_n, \tfrac{1}{2}\right) \tanh \frac{\alpha_n}{2} \right]$$

$$e_m = \frac{2\lambda_m h_0 - \sinh(2\lambda_m h_0)}{8h_0} \tag{2.27c}$$

$$R_m = \sum_n \left[b_n \int_0^{\frac{1}{2}} \left(\cosh \alpha_n x - 2x \coth \tfrac{1}{2}\alpha_n \sinh \alpha_n x\right) \alpha_n \sinh \lambda_m x dx \right] + \frac{s_m}{8\lambda_m} \sinh \lambda_m h_0 \tag{2.27d}$$

$$= \sum_n \left[b_n \alpha_n \left[\mathrm{I}_3\left(\lambda_m, \alpha_n, \tfrac{1}{2}\right)\right] - 2\left(\coth \tfrac{1}{2}\alpha_n\right) \mathrm{I}_5\left(\lambda_m, \alpha_n, \tfrac{1}{2}\right)\right] + \frac{s_m}{8\lambda_m} \sinh \lambda_m h_0$$

$$B_{mn} = (-1)^n \lambda_n \int_0^{h_0} \left[\sinh(\lambda_n y) - \left(\frac{y}{h_0}\right) \tanh(\lambda_0 h_0) \cosh(\lambda_n y) + \frac{y \sinh^2(\lambda_n h_0)}{h_0 \cosh(\lambda_n h_0)} \sinh[\lambda_n (y - h_0)] \right] \sin(\alpha_n y) dy \tag{2.27e}$$

$$= (-1)^n \lambda_n \Big[\mathrm{I}_2(\alpha_m, \lambda, h_0) - \left(\frac{1}{h_0}\right) \tan(\lambda_n h_0) \mathrm{I}_7(\alpha_m, \lambda, h_0) + \frac{\sinh^2(\lambda_n h_0)}{h_0 \cosh(\lambda_n h_0)} \{\cosh(\lambda_n h_0) \mathrm{I}_5(\alpha_m, \lambda, h_0)$$
$$- \sinh(\lambda_n h_0) \mathrm{I}_7(\alpha_m, \lambda, h_0)\}\Big]$$

$$R_m = \frac{b_m h_0 (\alpha_m + \sinh \alpha_m)}{2 \sinh \frac{\alpha_m}{2}} - \frac{1}{2} \sum_n (-1)^n S_n \int_0^{h_0} [(y \sinh[\lambda_n (y - h_0)])] \sin \alpha_m y dy + \int_0^{h_0} g_2(y) \sin \alpha_n y dy$$

$$= \frac{b_m h_0 (\alpha_m + \sinh \alpha_m)}{2 \sinh \frac{\alpha_m}{2}} - \frac{1}{2} \sum_n (-1)^n S_n [\cosh(\lambda_n h_0) \mathrm{I}_5(\alpha_m, \lambda_n, h_0) - \sinh(\lambda_n h_0) \mathrm{I}_7(\alpha_m, \lambda_n, h_0)] + \int_0^{h_0} g_2(y) \sin \alpha_m y dy$$

(2.27f)

where I_1 to I_{10} are as defined in the appendix. Taking n terms of the summation, the system of linear equations in equation (2.26) can be written in terms of matrix as

$$\begin{pmatrix} A & D_{11} \\ D_{12} & E \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \end{pmatrix} = \begin{pmatrix} \underline{b}_1 \\ \underline{b}_2 \end{pmatrix} \tag{2.28}$$

where

$$\underline{x}_1 = (A_1, A_2, \cdots, A_n)^T, \quad \underline{x}_2 = (E_1, E_2, \cdots, E_n)^T$$
$$\underline{b}_1 = (R_1, R_2, \cdots, R_n)^T, \quad \underline{b}_2 = (R_1, R_2, \cdots, R_n)^T$$
$$A = [a_i^*] = \begin{pmatrix} a_{11}^* & a_{12}^* \cdots & a_{1n}^* \\ \vdots & \vdots & \vdots \\ a_{n1}^* & a_{n2}^* \cdots & a_{mn}^* \end{pmatrix}, \qquad E = [e_i] = \begin{pmatrix} e_{11} & e_{12} \cdots & e_{1n} \\ \vdots & \vdots & \vdots \\ e_{n1} & e_{n2} \cdots & e_{nn} \end{pmatrix}$$

and D_{11} and D_{22} are *n* x *n* diagonal matrices with entries e_{il}^*, a_{ii} respectively on row *i* column *i*. To economize on the storage, this linear system of equations in equation (2.28), may be written equivalently as two systems of linear equations namely

$$(-A D_{22}^{-1}E + D_{11})\,\underline{x}_2 = \underline{b}_1 - A D_{22}^{-1}\,\underline{b}_2, \qquad \underline{x}_1 = - D_{22}^{-1}(\underline{b}_1 - E\,\underline{x}_2)$$

The unknown constants Cm^*, H_n^* in φ_2 (see equation. 2.24) are determined from the linear systems

$$-\sum_n H_n^* r_{nm} + C_m^* v_m = zm, \quad H_m^* um + \sum_n C_n^* k_{nm} = w_m \tag{2.29}$$

where

$$V_m = \frac{h_0(\alpha_m + \sinh\alpha_m)}{4\alpha_m \sinh\frac{\alpha_m}{2}}, \quad U_n = \frac{1}{2}\left\{\frac{1}{2} - \frac{h_0\lambda_n}{\sinh(2\lambda_n h_0)}\right\}$$

$$r_{mn} = \frac{2(-1)^n}{\sinh(2\lambda_n h_0)}\int_0^{h_0}[y\sinh(\lambda_n h_0)\cosh\lambda_n(y-h_0) - h_0\sinh\lambda_n y]\sin\alpha_m y\,dy$$

$$= \frac{2(-1)^n}{\sinh(2\lambda_n h_0)}[\sinh(\lambda_n h_0)\{\cosh(\lambda_n h_0)I_7(\alpha_m,\lambda_n,h_0) - \sinh(\lambda_n h_0)I_5(\alpha_m,\lambda_n,h_0)\} - h_0 I_2(\alpha_m,\lambda_n,0,h_0)]$$

$$Z_m = -\int_0^{h_0}[f_2(y) + f_2(y)]\sin\alpha_m y\,dy + \sum_n \frac{t_n(-1)^n}{\lambda_n\cosh\lambda_n h_0}\{\lambda_n(I_5(\alpha_m,\lambda_n,h_0) - h_0 I_2((\alpha_m,\lambda_n,h_0)) - \coth(\lambda_n h_0)I_2(\alpha_m,\lambda_n,h_0))\}$$

$$k_{mn} = -(\tfrac{1}{2}\alpha_n\coth\tfrac{1}{2}\alpha_n + 1)I_1(\lambda_m,\alpha_n,\tfrac{1}{2}) + \alpha_n I_8(\lambda_m,\alpha_n,\tfrac{1}{2})$$

$$W_n = \frac{t_m}{4\cosh\lambda_m, h_0}\left\{\lambda_m, h_0 + \coth\lambda_m h_0 - \sum_n[d_n\{-(\tfrac{1}{2}\alpha_n\tan\tfrac{1}{2}\alpha_n + 1)I_4(\lambda_4,\alpha_n,\tfrac{1}{2}) + \alpha_n I_6(\lambda_m,\alpha_n,\tfrac{1}{2})\}]\right.$$

This linear system of equations in equation (2.29), has the same structures as in equation (2.28), where in this case,

$$\underline{x}_1 = (H_1^*, H_2^*, \cdots, H_n^*)^T, \quad \underline{x}_2 = (C_1^*, C_2^*, \cdots, C_n^*)^T$$
$$\underline{b}_1 = (z_1, z_2, \cdots, z_n)^T, \quad \underline{b}_2 = (w_1, w_2, \cdots, w_n)^T$$

$$A = \left[a_{ij}^*\right] = \begin{pmatrix} -r_{11} & -r_{12}\cdots & -r_{1n} \\ \vdots & \vdots & \vdots \\ -r_{n1} & -r_{n2}\cdots & -r_{nn} \end{pmatrix} \qquad E = \left[e_{ij}\right] = \begin{pmatrix} k_{11} & k_{12}\cdots & k_{1n} \\ \vdots & & \vdots \\ k_{n1} & k_{n2}\cdots & k_{nn} \end{pmatrix}$$

(Taking n terms of the summation involved) and D_{11} and D_{22} are *n* x *n* diagonal matrices with entries v_n, u_i respectively on row *i* column *i*. The system can therefore be solved in the way mentioned above. The pressure at the interface is given by $p_1(x, h_0) = p_1^*(x, h_0) + c$, where

$$p_1(x,h_0)=\int\frac{\partial\nabla^2\psi_0}{\partial y}dx-\int\frac{\partial\nabla^2\psi_0}{\partial x}dy=\frac{2}{1-\frac{\pi^2}{4}}\left[\frac{\left(x+\frac{1}{2}\right)+\frac{\pi h_0}{2}}{\left(x+\frac{1}{2}\right)^2+h_0^2}+\frac{\left(x-\frac{1}{2}\right)+\frac{\pi h_0}{2}}{\left(x-\frac{1}{2}\right)^2+h_0^2}\right]+2\sum\left((-1)^n\right)\left[2\alpha_n\left(-2A_n\tanh\tfrac{1}{2}\alpha_n+d_n^*\right)\cosh\alpha_n x\right.$$

$$\left.+2\alpha_n\left(-2b_n\coth\tfrac{1}{2}\alpha_n+C_n^*\right)\sinh\alpha_n x\right]+\frac{2E_n\lambda_n}{h_0}\sinh\lambda_n h_0\cos\lambda_n x+\left\{\frac{2H\lambda_n}{\cosh\lambda_n h_0}+2t_n\lambda_n\tanh\lambda_n h_0\right\}\sin\lambda_n x\Big\}$$

The second approximation h_1 of the interface is obtained from

$$\frac{d^2h_1}{dx^2}-Bh_1=-p_1^*(x,h_0)-2\frac{\partial^2\psi_0}{\partial x\partial y}-c \quad \text{at} \quad y=h_0, \quad \text{as} \quad h_1(x)=J\cosh B^{\frac{1}{2}}x+K\sinh B^{\frac{1}{2}}\int_{\frac{1}{2}}^{x}T(s)\sin B(x-s)ds\,\cdot$$

$$\sum_n\left[\mathrm{I}_n\cosh(\alpha_n x)+\mathrm{I}_{2n}\sinh(\alpha_n x)+\mathrm{I}_{3n}\left(x\sinh(\alpha_n x)-\frac{2\alpha_n}{\alpha_n^2 B}\cosh\alpha_n x\right)+\mathrm{I}_{4n}\left(x\cosh\alpha_n x-\frac{2\alpha_n}{\alpha_n^2 B}\sinh\alpha_n x\right)\right.$$

$$\left.+\mathrm{I}_{5n}\cos\lambda_n x+\mathrm{I}_{6n}\sin\lambda_n x\right]\frac{c}{B}\,.$$

where $$T(x)=\frac{1}{1-\frac{\pi^2}{4}}\left[\frac{2\left[\left(x+\frac{1}{2}\right)+\frac{1}{2}\pi h_0\right]}{\left(x+\frac{1}{2}\right)^2+h_0^2}+\frac{2\left[\left(x-\frac{1}{2}\right)+\frac{1}{2}\pi h_0\right]}{\left(x-\frac{1}{2}\right)^2+h_0^2}+\frac{\left(x+\frac{1}{2}\right)h_0\left[2h_0-\left(x+\frac{1}{2}\right)\pi\right]}{\left[\left(x+\frac{1}{2}\right)^2+h_0^2\right]^2}+\frac{\left(x-\frac{1}{2}\right)h_0\left[2h_0-\left(x-\frac{1}{2}\right)\pi\right]}{\left[\left(x-\frac{1}{2}\right)^2+h_0^2\right]^2}\right]$$

$$\mathrm{I}_{1n}=\frac{(-1)^n\alpha_n}{\alpha_n^2-B}\left[2A_n\alpha_n-2d_n^*\left(\tfrac{1}{2}\tan\tfrac{1}{2}\alpha_n+1\right)+6\left(-2A_n\tanh\tfrac{1}{2}\alpha_n+d_n^*\right)\right]$$

$$\mathrm{I}_{2n}=\frac{(-1)^n\alpha_n}{\alpha_n^2-B}\left[2b_n\alpha_n-2d_n^*\left(\tfrac{1}{2}\coth\tfrac{1}{2}\alpha_n+1\right)+6\left(-2b_n\coth\tfrac{1}{2}\alpha_n+C_n^*\right)\right]$$

$$\mathrm{I}_{3n}=\frac{2(-1)^n\alpha_n^2}{\alpha_n^2-B}\left[-2A_n\tanh\tfrac{1}{2}\alpha_n+d_n^*\right] \qquad \mathrm{I}_{4n}=\frac{2(-1)^n\alpha_n^2}{\alpha_n^2-B}\left[-2b_n\coth\tfrac{1}{2}\alpha_n+C_n^*\right]$$

$$\mathrm{I}_{5n}=\frac{-1}{\lambda_n^2+B}\left[\frac{2E_n\lambda_n}{h_0}\left(\sinh\lambda_n h_0+\lambda_n h_0\cosh\lambda_n h_0\right)+s_n\lambda_n h_0\right]$$

$$\mathrm{I}_{6n}=\frac{-4}{\lambda_n^2+B}\left[\frac{H_n\lambda_n}{\sinh 2\lambda_n h_0}\left(\lambda_n h_0\cosh\lambda_n h_0+\sinh\lambda_n h_0\right)+\frac{\lambda_n t_n}{\sinh 2\lambda_n h_0}+t_n\lambda_n\tanh\lambda_n h_0\right]$$

The contact angle conditions $h'(½)=0$ fix the constants J and K:

$$K=\left[2B^{\frac{1}{2}}\cosh\tfrac{1}{2}B^{\frac{1}{2}}\right]^{-\frac{1}{2}}-\int_{-\frac{1}{2}}^{\frac{1}{2}}T(s)\cosh B^{\frac{1}{2}}\left(\tfrac{1}{2}-s\right)ds-2\sum_n\left[\mathrm{I}_{2n}\alpha_n\cosh\tfrac{1}{2}\right.$$

$$\left.+\mathrm{I}_{4n}\left(\tfrac{1}{2}\alpha_n\sinh\tfrac{1}{2}\alpha_n-\frac{3\alpha_n-B}{\alpha_n^2-B}\cosh\tfrac{1}{2}\alpha_n\right)+\mathrm{I}_{6n}\lambda_n\cosh\tfrac{1}{2}\lambda_n\right]$$

$$J=\left[2B^{\frac{1}{2}}\sinh\tfrac{1}{2}B^{\frac{1}{2}}\right]^{-\frac{1}{2}}\left[-\int_{-\frac{1}{2}}^{\frac{1}{2}}T(s)\cos shB^{\frac{1}{2}}\left(\tfrac{1}{2}-s\right)ds\right]+2\sum_{n}\left[I_{1n}\alpha_n\sinh\tfrac{1}{2}\alpha_n+I_{3n}\left(\tfrac{1}{2}\alpha_n\cosh\tfrac{1}{2}\alpha_n\right.\right.$$

$$\left.\left.-\frac{3\alpha_n^2-B}{\alpha_n^2-B}\sinh\frac{1}{2}\alpha_n\right)-I_{5n}\lambda_n\sin\frac{1}{2}\lambda_n\right]$$

The volume constraint condition (i.e. equation (2.18)) determines the constant c in $h_1(x)$ as

$$c=2KB^{\frac{1}{2}}\sinh\tfrac{B^{\frac{1}{2}}}{2}-B^{\frac{3}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{x}T(s)\sin B(x-s)ds\,dx-2B\sum_{n}\left[I_{1n}\alpha_n^{-1}\sinh\tfrac{1}{2}\alpha_n\right.$$

$$\left.+I_{3n}\left(\tfrac{1}{2}\alpha_n^{-1}\cosh\tfrac{1}{2}\alpha_n-\frac{3\alpha_n^2-B}{\alpha_n^2\left(\alpha_n^2-B\right)}\sinh\frac{1}{2}\alpha_n\right)-I_{5n}\lambda_n^{-1}\sin\tfrac{1}{2}\lambda_n\right]$$

3.0 Conclusion

An approximate solution valid for small capillary number is obtained, through perturbation technique, for the flow of an incompressible fluid in an open container generated by the container's base that moves steadily with a constant speed. This is achieved by expanding the flow quantities, such as the stream, pressure and the free surface functions, in terms of capillary number. Consequently for small capillary number, the leading terms of the expansion provides a reasonable approximation to the exact solution to the problem, for the case when the free surface (the upper boundary of the fluid) meets the vertical walls of the container at a right angle. However because the expressions above including the systems of linear equations contains exponential functions using many terms of the summations in those expressions and equations is the problem with the above procedure.

Appendix

$$I_1(p,q,b)=\int_0^b\cos(px)\cosh(qx)dx=\frac{p\sin(pb)+q\cos(pb)\tanh(qb)}{p^2+q^2}\cosh(qb)$$

$$I_2(p,q,b)=\int_0^b\sin(px)\sinh(qx)dx=\frac{-p\cos(pb)\tan(qb)-q\sin(pb)}{p^2+q^2}\cosh(qb)$$

$$I_3(p,q,a,b)=\int_0^b\sin(px)\cosh(qx)dx=\frac{-p\cos(pb)-q\sin(pb)\tan(qb)}{p^2+q^2}\cosh(qb)+\frac{p}{p^2+q^2}$$

$$I_4(p,q,a,b)=\int_0^b\cos(px)\sinh(qx)dx=\frac{p\sin(pb)\tanh h(qb)-q\cos(pb)}{p^2+q^2}\cosh(qb)+\frac{q}{p^2+q^2}$$

$$I_5(p,q,b)=\int_0^b x\sin(px)\sinh(qx)dx=\frac{\cosh(qb)}{p^2+q^2}\left[\left\{qb-\frac{p^2-q^2}{p^2+q^2}\tanh(qb)\right\}\sin(pb)\right.$$

$$\left.+\left\{-pb\tanh(qb)+\frac{2pq}{p^2+q^2}\right\}\cos(pb)\right]-\frac{2pq}{\left(p^2+q^2\right)^2}$$

$$I_6(p,q,b)=\int_0^b x\cos(px)\cosh(qx)dx=\frac{\cosh(qb)}{p^2+q^2}\left[\left\{qb\tanh h(qb)-\frac{p^2-q^2}{p^2+q^2}\right\}\cos(pb)\right.$$

$$+\left\{-pb+\frac{2pq}{p^2+q^2}\tanh(pb)\right\}\sin(pb)\Bigg]-\frac{p^2-q^2}{\left(p^2+q^2\right)^2}$$

$$I_7(p,q,b)=\int_0^b x\sin(px)\cosh(qx)dx=\frac{\cosh(qb)}{p^2+q^2}\Bigg[\left\{qb\tanh(qb)-\frac{p^2-q^2}{p^2+q^2}\right\}\sin(pb)$$

$$+\left\{-pb+\frac{2pq}{p^2+q^2}\tanh(qb)\right\}\cos(pb)\Bigg]$$

$$I_8(p,q,b)=\int_0^b x\cos(px)\sinh(qx)dx=\frac{\cosh(qb)}{p^2+q^2}\Bigg[\left\{qb-\frac{p^2-q^2}{p^2+q^2}\tan(qb)\right\}\cos(pb)$$

$$+\left\{-pb\tanh(qb)+\frac{2pq}{p^2+q^2}\tanh(qb)\right\}\sin(pb)\Bigg]$$

$$I_9(p,q,a,b)=\int_0^b x\sin(qx)dx=\left[\frac{b}{q}-\frac{1}{q^2}\tan(qb)\right]\cosh(qb)$$

$$I_{10}(p,q,b)=\int_0^b x\cosh(qx)dx=\left[\frac{b}{q}\tanh(qb)-\frac{1}{q^2}\right]\cosh(qb)+\frac{1}{q^2}$$

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