Existence of Solution of a Time-Dependent Radial Porous Medium Combustion

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Abstract

A non-linear time-dependent partial differential equation describing combustion in a radial porous medium is studied. Existence of a solution is established.

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1.0 Introduction

Porous medium combustion occurs in a number of situations including, the burning of coal, burning of cigarettes, use of catalytic converters as exhaust filters and the smouldering of polyurethane. A model for combustion in a porous medium was developed by Lawson and Norbury [3]. Norbury and Stuart [4] extended the model to a three-dimension situation in which the chemical process is

$$\text{solid} + O_2 \rightarrow \text{heat} + CO_2 + \text{ash}$$

Their model represents conservation of mass and energy for both the gas and solid species, while the fluid flow is governed by Darcy's law and the ideal gas law. Subsequently, in case of radial porous medium combustion, Koriko [2] used a number of asymptotic considerations including large activation energy (of Frank-Kamenetskii [1]) to arrive at a simplified model of the form,

$$U_t = U_{xx} + \frac{(1+\delta_0)}{x}U_x + \frac{\delta_1 e^U}{\delta_2 e^U + 1}. \qquad (1.1)$$

with boundary and initial conditions; $U(1,t) = U(l,t) = 0$ and $U(x,0) = 0$ where, δ_0, δ_1, δ_2 are non-zero constants. Oyelami [5] examined the existence of solution of differential - difference equation using the method and upper solution. In this paper, we shall establish the existence of (1.1) using the method of Sattinger [6]. Before going to the main results, we shall define some terms.

2.0 Existence of Solution

Consider the following parabolic equation,

$$U_t = \nabla U + f(x,t,U), \quad x \in R^n, t > 0$$
$$U(x,0) = f_0(x), \; x = (x_1, x_2, \ldots, x_n) \qquad (2.1)$$

$f_0(x)$ is bounded for $x \in R^n$ and has a countable number of continuities. $f(x)$ satisfies the uniform Lipschitz condition.

$$|\psi(x,t,U_1) - \psi(x,t,U_2)| \le M(|U_1 - U_2|), \; (x,t) \in G$$

where $G = \{(x,t) / x \in R^n, 0 \le t \le T\}$.

Problem (2.1) represents a generalisation of problem (1.1). Consider the parabolic boundary value problems of the type

$$LU + f(x,t,U) - U_t = 0$$
$$\text{In } \Gamma_T = D_x(0,T)$$
$$U = g \quad \text{on} \quad \partial\Gamma_T$$

or

$$\frac{\partial u}{\partial v} + \beta u = g(x,t) \quad \text{on } \partial D_x(0.T)$$
$$U(x,0) = g(x) \quad \text{on } D \cap \{t = 0\}$$

where L is a second order uniformly elliptic operator

$$L = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} U_{x^i x^j} + \sum_{j=1}^{n} b_i U_{x^i}$$
$$(x = x^1, .., x^n)$$

Here $\frac{\partial}{\partial \upsilon}$ denotes the outward conformal derivative and we assume $\beta \geq 0$ everywhere on the boundary $\partial D_x(0,T)$.

The coefficients of L are assumed to be smooth (say Holder Continuous) and the matrix a_{ij} is to be uniformly positive definite on Γ_T. Any undifferentiated terms in U may be grouped with the term f(x, U) including linear terms. We assume the boundary of D is smooth say $C^{2+\alpha}$, though this condition could be somewhat relaxed. $f(x, U, t)$ is assumed smooth throughout, say f is C' in U and holder continuous in x and t.

2.1 **Definition**

A function $u_0(x, t)$ is an upper solution of (2.1) on Γ_T if

$$Lu_0 + f(x, u_0) - \frac{\partial u_0}{\partial t} < 0 \quad \text{in } \Gamma_T$$
$$Bu_0 \geq g \text{ on } \partial\Gamma_T$$

Similarly, $v_0(x, t)$ is called a lower solution of (2.1) if

$$Lv_0 + f(x, v_0) - \frac{\partial v_0}{\partial t} \geq 0 \quad \text{in } \Gamma_T$$
$$Bv_0 < g \text{ on } \partial\Gamma_T$$

3.0 Main Results

3.1 Theorem

The equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{(1+\delta_0)}{x}\frac{\partial u}{\partial x} + \frac{\delta_1 e^u}{\delta_2 e^u + 1} = \frac{\partial u}{\partial t}$$

with boundary and initial conditions

$$U(1,t) = U(l,t) = 0 \text{ and } U(x,0) = 0$$

has a solution if $\delta_1 = \delta_2$ for all $t \geq 0$.

Proof.

We are to prove theorem (1.1) by the method of upper and lower solutions. Consider $\underline{u}(x,t) = 0$. We shall show that $\underline{u}(x,t) = 0$ is a lower solution. Obviously, $u(1,t) = u(\ell,t) = 0$. Since,

$$\underline{u}_t = \underline{u}_x = \underline{u}_{xx} = 0$$

This, implies that,

$$0 \geq \frac{\delta_1}{\delta_2 + 1}$$

hence, $\underline{u}(x, t) = 0$ is a lower solution. Also consider,

$$\bar{u}(x,t) = \frac{1}{2}(x-1)(1-x)$$

It is easily seen that the function defined above satisfies our boundary conditions. Furthermore, it does not need to depend explicitly on *t*. We shall show that $\bar{u}(x,t)$ as defined above is an upper solution

$$\bar{u}(1,t) = \bar{u}(\ell,t)$$

also,

$$\bar{u}_x = \tfrac{1}{2}(1 - 2x + \ell)$$

$$\bar{u}_{xx} = -1$$
$$\bar{u}_t = 0$$

Now evaluating (1.1) using the above information with the fact that,

$$X_{\max} = \frac{1+l}{2} \quad and \quad X_{\min} = l$$

we have

$$-1 + \frac{(l+\delta_0)}{2}(1-l-1+l) - 0 \leq -\frac{-\delta_l}{\delta_2+1}$$

$$-1 \leq \frac{-\delta_l}{\delta_2+1}, \qquad 1 \geq \frac{\delta_l}{\delta_2+1}$$

This is satisfied when $\delta_2 = \delta_1$

Remark: Our result is also valid for this case $\delta_2 > \delta_1$

3.2 **Theorem**

Our equation (1.1) *has a solution for* $0 \leq t \leq \delta_l/\delta_2$

Proof

Consider,

$$u(x, t) \cong 0$$

We shall show that $\underline{u}(x, t)$ is a lower solution.

Consider,

$$\underline{u}(1,t) = \underline{u}(l,t)\underline{u}(x,0) = 0$$

$$\underline{u}_{xx} + \frac{(1+\delta_0)}{x} + \underline{u}_x - \underline{u}_t = -\frac{\delta_1 e^{\underline{u}(1,t)}}{\delta_2 e^{\underline{u}(1,t)} + 1}$$

$$0 \geq -\frac{\delta_1}{\delta_2+1}$$

Hence, $\underline{u}(x,t) = 0$ is a lower solution. Also, consider,

$$\underline{u}(x,t) = -\log\left(1 - \frac{\delta_1 t}{\delta_2}\right) \qquad \bar{u}(x,0) = 0$$

$$\bar{u}(1,t) = \bar{u}(\ell,t) = -\log\left(1 - \frac{\delta_1 t}{\delta_2}\right) \leq -0$$

when, $t < ½$

$$\bar{u}_x = \bar{u}_{xx} = 0, \qquad \bar{u}_t = \frac{1}{1 - \frac{\delta_1 t}{\delta_2}} \frac{\delta_1}{\delta_2}$$

Simplifying, we obtain,

$$\bar{u}_t = \frac{\delta_1}{\delta_2 - \delta_1 t}$$

$$\bar{u}_{xx} + \frac{(1+\delta_0)}{x}\bar{u}_x - \bar{u}_t \leq -\frac{\delta_1 e^{\bar{u}(x,0)}}{\delta_2 e^{u(x,0)} + 1}, \qquad \frac{-\delta_1}{\delta_2 - \delta_1 t} \leq \frac{-\delta_1}{\delta_2 + 1}$$

$$\frac{\delta_1}{\delta_2 - \delta_1 t} \geq \frac{\delta_1}{\delta_2 + 1}$$

The left hand side attains its maximum when t = 0. that is,

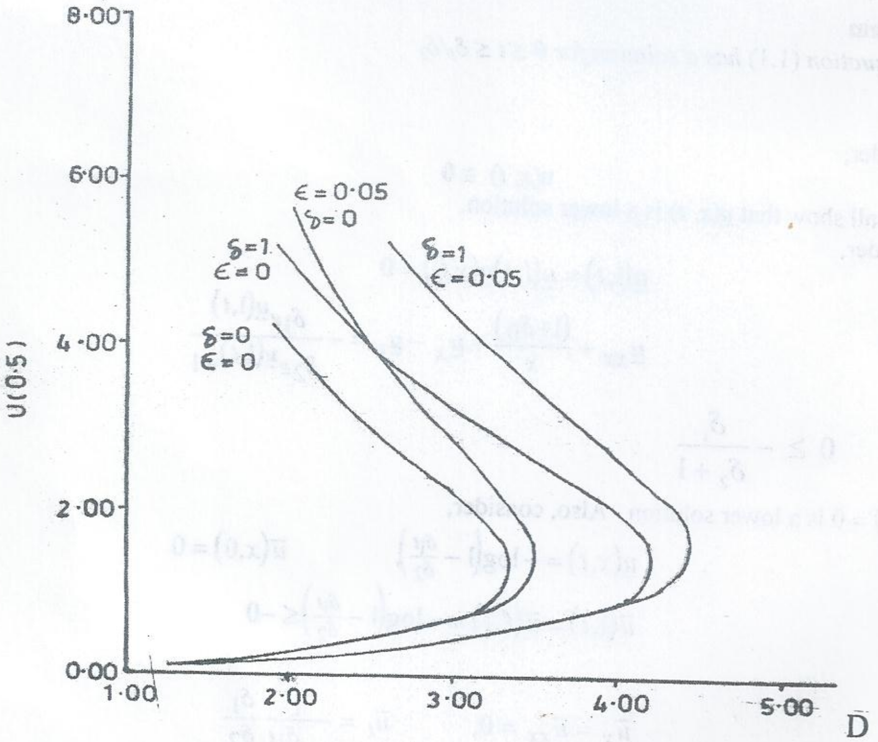
$$\frac{\delta_1}{\delta_2} \geq \frac{\delta_1}{\delta_2 + 1}$$

Therefore,

$$u(x,t) = -\log\left(1 - \frac{\delta_1 t}{\delta_2}\right) \text{ is an upper solution.}$$

4.0 Conclusion

Since we have been able to establish both lower and upper solutions, we conclude that there exits a solution. The graphical display is the numerical solution of steady state situation.



The graph of maximum U against D

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