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On the purely seasonal ARIMA Process

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Abstract

Because of their common occurrence in the analysis of seasonal time series, this paper is devoted to the study of time series models whose non-zero autocorrelations and partial autocorrelations are only at the multiples of the seasonal lag. After the introduction of the concepts of stationarity, invertibility and moments, we studied the autocorrelation function (acf) *and partial autocorrelation function* (pacf) *of the seasonal ARIMA* (0,d,0) x (1,D,1)s *time series model, where the subindex s refers to the seasonal period. A numerical example is given to illustrate the methods.*

Key words: *Sessional time series, stationarity, invertibility, moments, autocorrelation function, partial autocorrelation function*

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1.0 Introduction

Let $Y_t,\ t \in Z$ and $e_t,\ t \in Z$ be two stochastic processes defined on some probability space (Ω, f, p), where $Z = (\cdots,-1,0,1,\cdots)$. $Y_t,\ t \in Z$ is said to be a multiplicative seasonal ARIMA (p,d,q) x (P,D,Q)S model with respect to the processes $e_t,\ t \in Z$, if

$$\phi_P(B)\Phi_P\left(B^s\right)(1-B)^d\left(1-B^s\right)^D Y_t = \theta_q(B)\Theta_Q\left(B^s\right)e_t \tag{1.1}$$

where

$$\phi_p(B) = 1-\phi_1 B-\phi_2 B^2-\cdots-\phi_p B^p \tag{1.2}$$

$$\theta_q(B) = 1-\theta_1 B-\theta_2 B^2-\cdots-\theta_q B^q \tag{1.3}$$

$$\Phi_P\left(B^s\right) = 1-\Phi_1 B^s-\Phi_2 B^{2s}-\cdots-\Phi_p B^{Ps} \tag{1.4}$$

$$\Theta_Q\left(B^s\right) = 1-\Theta_1 B^s-\Theta_2 B^{2s}-\cdots-\theta_Q B^{Qs} \tag{1.5}$$

Equations (1.2) and (1.3) are polynomials of B with no common roots; while (1.4) and (1.5) are polynomials of B^s with no common roots. The roots of these polynomials lie outside of the unit circle, and $\{e_t\}$ is zero mean white noise process with constant variance, $\sigma^2 > 0$.

Model (1.1) contains within periods and between periods relationships. Within periods relationships represent the correlation among $\cdots Y_{t-2}, Y_{t-1}, Y_t, Y_{t+1}, Y_{t+2}, \cdots$ and between periods relationships represent correlation among $\cdots Y_{t-2s}, Y_{t-s}, Y_t, Y_{t+s}, Y_{t+2s}, \cdots$. The within periods component

$$\phi_p(B)(1-B)^d Y_t = \theta_q(B)a_t \tag{1.6}$$

is called the nonseasonal part of (1.1), where obviously the series $\{a_t\}$ will not be white noise. Similarly, the between periods components

$$\Phi_P(B^s)(1-B^s)^D Y_t = \Theta_Q(B^s)b_t \tag{1.7}$$

is called the seasonal part, where again the series $\{b_t\}$ will not be white noise.

When a series contains only the seasonal component, $\phi_P(B) = \theta_q(B) = 1$ and equation (1.1) educes to

$$\Phi_P(B^s)(1-B)^d (1-B^s)^D Y_t = \Theta_Q(B^s)e_t \tag{1.8}$$

where $(1-B)^d$ is the regular differencing to remove the stochastic trend (if any) in the series and $(1-B^s)^D$ is the seasonal differencing operator (when the mean of a realization shifts according to a seasonal pattern, seasonal differencing often induces a constant.) In practice, $d = 1,2$, and $D = 1,2$. In short, second degree (or higher) regular and/or seasonal differencing is virtually never needed.

The frequency with which data are recorded determines the value assigned to *s*, the length of the periodic interval. This implies that $s \geq 1$ and the common values of *s* for variation within a year are *s* = 1 for yearly data; *s* = 2 for data collected two times in a year; *s* = 3 for data collected three times in a year; *s* = 4 for quarterly data showing seasonal effects within years; *s* = 6 for bi-monthly data and *s* = 12 for monthly data showing seasonal effects within years.

In considering of equation (1.8), we have restricted attention to P = Q = 1. Assuming *s* = 12, would we earnestly expect that the value of the series, say in January, to be related to the value in January of the next 24 months, next 36 months and so on? Based on this argument, we consider model (1.8) for P = Q = 1 to obtain

$$(1-\phi B^s)X_t = (1-\theta B^s)e_t \tag{1.9}$$

where $\phi = \Phi_1, \theta = \Theta_1$ and

$$X_t = (1-B)^d (1-B^s)^D Y_t \tag{1.10}$$

X_t are formed from the original series Y_t by differencing to remove stochastic nonstationarity in the within period and between period means. In our terminology model (1.9) is the multiplicative seasonal ARIMA (0,d,0) x (1,D.1)s time series model; which is a purely seasonal series model with ;one seasonal autoregressive (AR) coefficient and one seasonal moving average (MA) coefficient.

Wei (1989) and Parkratz (1983) have considered stationary seasonal processes with one seasonal AR coefficient and one seasonal MA coefficient at lag *s*. The results show that for the purely seasonal process with one AR coefficient at lag *s*, the theoretical acf decays exponentially at lags *s*, 2*s*, 3*s*, ... either all on the positive side or alternating in sign starting from the negative sign. The theoretical acf for the purely seasonal process with one MA coefficient at lag *s*, has a spike at lag so followed by a cut off to zero at lags 2*s*, 3*s*, ... These results are identical to the nonseasonal AR (1) and MA (1) acf's and pacf's except the coefficients for the seasonal processes occur at multiples of lag *s* (*s*, 2*s*, 3*s*, ...) instead of at lags 1, 2, 3, ...

In this paper we consider the problem of stationarity and invertibility, the theoretical acf and pacf of the process satisfying equation (1.9). Our results will be shown to be identical to the ARMA (1, 1) process except that the coefficients of the acf's and pacf's for the purely seasonal process (1.9) occur at multiples of lag *s* (*s*, 2*s*, 3s, ...) instead of at lags 1, 2, 3, ... An empirical example is included to illustrate the properties of the model.

More generally, we can write the purely seasonal process (1.9) as follows

$$\left(1-\phi B^{s}\right)X_{t} = \theta_{0} + \left(1-\theta B^{s}\right)e_{t} \tag{1.11}$$

The parameter θ_0 represents the deterministic trend when $E(X_t) \neq 0$. Clearly, the acf and pacf of (1.9) and (1.11) are the same.

2.0 Stationarity and Invertibility Conditions

As noted by Box and Jenkins (1976), one advantage of the multiplicative form (1.1) is that it simplies the checking of stationarity and invertibility conditions. Within a multiplicative model these conditions apply separately to the seasonal and nonseasonal coefficients.

The stationarity requirement applies only to the AR coefficients and we treat the nonseasonal and seasonal AR components separately since they are multiplied. Thus, the stationarity condition for the purely seasonal model (1.9) is that the roots of

$$1-\phi B^{s} = 0 \tag{2.1}$$

lie outside the unit circle. Simiarly, the invertibility requirement applies only to the MA coefficients and we treat the nonseasonal and seasonal components separately. Thus, the invertibility condition for the purely seasonal modeal (1.9) is that the roots of

$$1-\theta B^{s} = 0 \tag{2.2}$$

lie outside the unit circle.

Computation of the roots of (2.1) and (2.2) give the conditions $|\phi| \prec 1$ and $|\theta| \prec 1$ as stated and proved in Theorem 2.1.

Theorem 2.1

Let $\{e_t'\}$ be the zero mean white noise process with constant variance, $\sigma^2 \succ 0$. Then the process satisfying $\left(1-\phi B^{s}\right)X_{t} = \left(1-\theta B^{s}\right)e_{t}$ is stationary if $|\phi| \prec 1$ and invertible $|\theta| \prec 1$.

Proof.

For stationarity, we wish to find the root (*s* in number) of $1-\phi B^{s} = 0$. In effect we wish to solve the equation $\phi B^{s} = 1$ or $B^{s} = \frac{1}{\phi}$, where *s* is a given positive integer and $\phi \neq 0$. By DeMoivere's Theorem, the general expression for the roots (Amazigo et.al, 2000) is

$$B = \phi^{\frac{-1}{s}}\left[\cos\left(\frac{2k\pi}{s}\right) + i\sin\left(\frac{2k\pi}{s}\right)\right], \quad k = 0,1,2,\cdots,s-1$$

For stationarity, the roots lie outside the unit circle. That is,

$$|B| = \left|\phi^{\frac{-1}{s}}\right|\left|\cos\left(\frac{2k\pi}{s}\right) + i\sin\left(\frac{2k\pi}{s}\right)\right| = \left|\phi^{\frac{-1}{s}}\right|,$$

since $\left|\cos\left(\frac{2k\pi}{s}\right) + i\sin\left(\frac{2k\pi}{s}\right)\right| = 1$, and hence $\frac{1}{\left|\phi^{\frac{1}{s}}\right|} \succ 1$ and this implies $\left|\phi^{\frac{1}{s}}\right| \prec 1 \Rightarrow |\phi|^{\frac{1}{s}} \prec 1 \Rightarrow |\phi| \prec 1$.

Similarly, the invertibility condition follows. That is, $|\theta| \prec 1$

3.0 Covariance Structure

In obtaining the expressions for the covariance, we have made the assumption that the random variables e_t are Gaussian with $E(e_t) = 0$. $E(e_t^2) = \sigma^2 \prec \infty$. We also used the fact that by expression (1.9), e_t is assumed to be independent of $X_u, u \prec t$. Based on these assumptions it can easily be checked that

$$E(X_t) = \mu_x = 0 \tag{3.1}$$

and the autocovariances are given by

$$R(k) = E[(X_{t-\mu_x})(X_{t-k-\mu_x})] = E(X_t X_{t-k}) = \begin{cases} \sigma^2\left[\dfrac{1+\theta^2-2\phi\theta}{1-\phi}\right], & k=0 \\ \sigma^2\left[\dfrac{(\phi-\theta)(1-\phi\theta)}{1-\phi^2}\right], & k=s \\ \phi^{a-1}R(S), & k=as;\ a=1,2,3,\cdots \\ 0,\ elsewhere & \end{cases} \tag{3.2}$$

provided $|\phi| \prec 1$, $|\theta| \prec 1$. From equation (3.2), the acf is given by

$$\rho_k = R(k)/R(0) = \begin{cases} 1; & k=0 \\ \dfrac{(\phi-\theta)(1-\phi\theta)}{1+\phi^2-2\phi\theta}, & k=s \\ \phi^{a-1}\rho_s, & k=as;\ a=1,2,3,\cdots \\ 0,\ elsewhere & \end{cases} \tag{3.3}$$

Observe that when s = 1, the acf (3.3) is that of the nonseasonal ARIMA (1,1). For $s \geq 2$, the coefficients which appear at lags 1, 2, 3, ... in ARIMA (1,1) now occur at multiples of lag *s* (*s*, 2*s*, 3*s*, ...). At other lags, the acf's are theoretically zero. Like the ARIMA (1,1) the acf (3.3) decays exponentially from the starting value ρ_s, which depends on ϕ and θ. The sign of ρ_s is determined by the sign of $\phi - \theta$ and dictates from which side of zero the exponential decay takes place as shown in Figure 1. In regions 1, 5 and 6 all the autocorrelations are on the positive side while in regions 2, 3 and 4, the autocorrelations start from the negative side and decays with alternating signs at multiples of lag *s*.

The pacf is another tool for identification, but the pacf for seasonal models are more complicated. The fundamental fact about the purely seasonal model (1.9) is that it is identical to ARIMA (1,1) model at multiples of lag *s* (*s*, 2*s*, 3*s*, ...). We therefore expect the pacf to have nonzero coefficients at one or more multiples of lag *s* (*s*, 2*s*, 3*s*, ...). The pacf is given by

$$\phi_{kk} = \begin{cases} \rho_1, & k=1 \\ |A_k^*|/|A_k|, & k \succ 1 \end{cases} \tag{3.4}$$

where

$$A_k = \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_{k-1} \\ \rho_1 & 1 & \cdots & \rho_{k-2} \\ \vdots & \vdots & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \cdots & 1 \end{bmatrix} \tag{3,5}$$

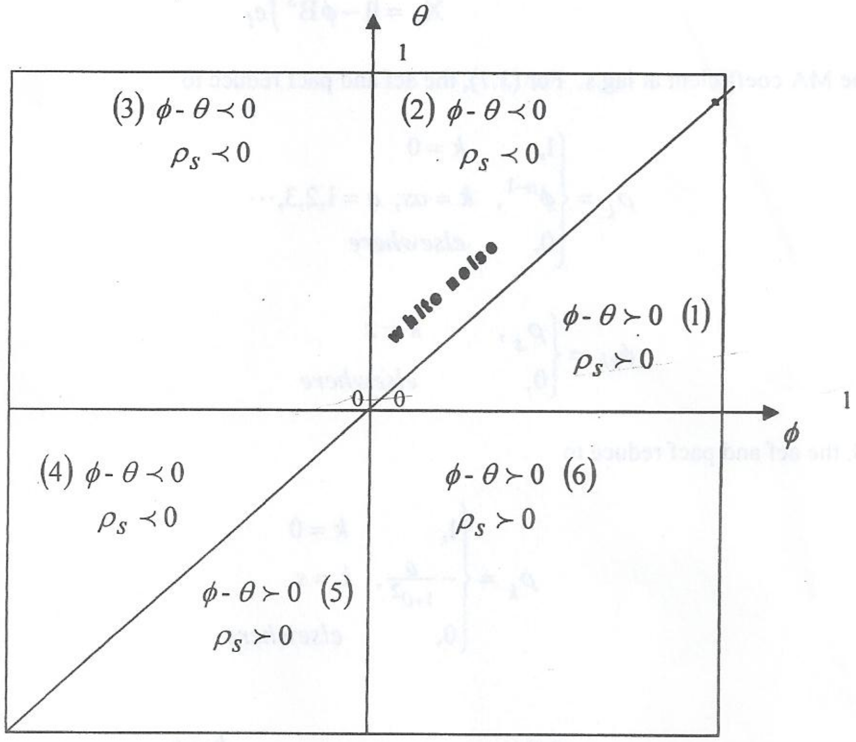

Figure 1: acf for various (0,d,0) x (1,D,1)s models

and A_k^* is the matrix composed of the first k -1 columns of A_k with the k[th] column replaced by $(\rho_1, \rho_2, \cdots, \rho_k)^T$. Using (3.4), the pacf of time series models (seasonal and nonseasonal) with acf (3.3) is zero every where except at the multiples of lag *s* (*s*. 2*s*, 3*s*, …) and is easily shown to be

$$\phi_{kk} = \begin{cases} \rho_s, & k = s \\ \dfrac{\rho_s(\phi - \rho_s)}{1-\rho_s^2}, & k = 2s \\ \dfrac{\rho_s(\phi-\rho_2)^2}{\left(1-\rho_s^2\right) - \rho_s^2\left(1-2\phi\rho_s+\phi^2\right)}, & k = 3s; \\ \dfrac{\rho_s(\phi-\rho_2)^3}{\left(1-\rho_s^2\right)^2 - \rho_s^2\left(1-2\phi\rho_s+\phi^2\right)^2}, & k = 4s; \end{cases} \tag{3.6}$$

For correct identification of models with acf (3.3), we look at the first few values of ϕ_{kk}, particularly the first few multiples of lag *s* (*s*, 2*s*, 3*s*, 4*s*) with special characteristic $\phi_{ss} = \rho_s$ and see if any are significantly different from zero.

Two of the common purely seasonal processes are

$$\left(1 - \phi B^s\right)X_t = e_t \tag{3.7}$$

with one AR coefficient at lag *s* and

$$X_t = \left(1-\phi B^s\right)e_t \tag{3.8}$$

with one MA coefficient at lag s. For (3.7), the acf and pacf reduce to

$$\rho_k = \begin{cases} 1, & k=0 \\ \phi^{a-1}, & k=as;\ a=1,2,3,\cdots \\ 0, & elsewhere \end{cases} \tag{3.9}$$

$$\phi_{kk} = \begin{cases} \rho_s, & k=s \\ 0, & elsewhere \end{cases} \tag{3.10}$$

For 3.8, the acf and pacf reduce to

$$\rho_k = \begin{cases} 1, & k=0 \\ -\frac{\theta}{1+\theta^2}, & k=s \\ 0, & elsewhere \end{cases} \tag{3.11}$$

$$\phi_{kk} = \begin{cases} \rho_s, & k=s \\ -\frac{\rho_s^2}{\left(1-\rho_s^2\right)}, & k=2s \\ \frac{\rho_s^3}{\left(1-2\rho_s^2\right)}, & k=3s \\ \frac{-\rho_s^4}{\left(1-3\rho_s^2+\rho_s^4\right)}, & k=4s \end{cases} \tag{3.12}$$

Equations (3.9) and (3.11) are the results of Wei (1989) and Parkratz (1983) discussed in Section 1. Equation (3.12) was established by Iwueze (2002) for time series models whose acf has a spike at lag s followed by a cutoff to zero at lags $2s$, $3s$, … .

Equation (3.3) helps us select suitable starting values for ϕ and θ in our estimation procedure. These are

$$\hat{\phi} = \frac{\hat{\rho}_{2s}}{\hat{\rho}_s} \tag{3.13}$$

and θ is given by the solution of

$$\hat{\rho}_s = \frac{\left(\hat{\phi}-\hat{\theta}\right)\left(1-\hat{\phi}\hat{\theta}\right)}{1+\hat{\theta}^2-2\hat{\phi}\hat{\theta}} \tag{3.14}$$

such that $\left|\hat{\theta}\right| \prec 1$. When $E(X_t) \neq 0$ (model (1.11)), our initial estimate of θ_0 is

$$\hat{\theta}_0 = (1-\hat{\phi})\overline{X} \tag{3.15}$$

4.0 Empirical Example

In this Section, we go through the full cycle of identification, estimation and diagnostic checking using real time series realization. At the identification stage estimated acf's are calculated from the available data. These are compared with (3.3) and (3.6) and a tentative model is chosen based on this comparison. The parameters of this model are estimated and the estimation – stage residuals $(\hat{e}_t)$ are then analysed with a residual acf to see if they are consistent with the hypothesis that the random shocks (e_t) are independent. If we reject this hypothesis, the structure within the residual acf may help us to tentatively identify another model.

Computations of estimated acf, pacf and estimation of model parameters are done with MINITAB.

4.1 Umudike Mean Monthly Temperature (°C): an example

This example examines the mean monthly temperature (°C) at Umudike, Nigeria from January 1974 to December 1987, which is listed at Table 1. In order to assess the forecasting performance of a model, we use only the first 156 observations of the series for model construction. The estimated acf and pacf of Y_t and $\nabla_{12}Y_t = (1-B^{12})Y_t$ are shown in Table 2.

The estimated acf Y_t at the seasonal lags (12, 24, 36) fail to die out quickly. This confirms the nonstationary character of the seasonal pattern and calls for seasonal differencing. The estimated acf for the seasonally differenced series $(\nabla_{12}Y_t)$ has a significant spike at lag 12. According to the theoretical acf of (3.11), this calls for a single seasonal MA term and we expect θ to be positive (Figure 1). The decaying pattern at lags 12, 24 and 36 in the estimated pacf of $X_t = \nabla_{12}Y_t$ suggests that the single MA term may be appropriate at the seasonal lag. Since $\overline{x} = 0\cdot0486,\ S_x = 1\cdot0127,\ n = 144,$ the t-value of $\overline{x} = \dfrac{0\cdot0486}{\left(\dfrac{1\cdot0127}{\sqrt{144}}\right)} = 12\dfrac{(0\cdot0486)}{1\cdot0127} = 0\cdot58$, which is not significant, and thus deterministic trend θ_0 is not needed. This analysis leads us to tentatively choose model (3.8) for X_t. Thus, we start with a tentative model of the form

$$(1-B^{12})Y_t = (1-\theta B^{12})e_t \tag{4.1}$$

Using MINITAB, the estimation of the above model gives $\hat{\theta} = 0\cdot7315$ with the standard error of 0.602. The residual acf (Table 3(a)) for the fitted model (4.1) has a significant spike at lag 12, and hence the model (4.1) is inadequate.

The fact that the residual pacf of model (4.1) as shown in Table 3(a) has only one significant spike lag 12 suggest the modification of the model to (1.9). That is,

$$(1-\phi B^{12})(1-B^{12})Y_t = (1-\theta B^{12})e_t \tag{4.2}$$

Parameter estimation gives

$$\underset{(0\cdot1191)}{(1+0\cdot1895B^{12})}(1-B^{12})Y_t = \underset{(0\cdot0986)}{(1-0\cdot6293B^{12})}e_t \tag{4.3}$$

and $\sigma_e^2 = 0\cdot5863$, where values in parenthesis below the parameter estimates are the associated standard errors. Residual acf and pacf of this modified model (4.2) as shown in Table 3(b) indicate no model inadequąacy.

Table 1: Buys-Ballot (1847) table for Umudike mean monthly temperature (^{0}C)

Year	MONTH													
	JAN	FEB	MAR	APR	MAY	JUNE	JULY	AUG.	SEPT	OCT.	NOV.	DEC	TOTAL	AV
1974	19	22	23	22	22	22	22	22	22	22	22	18	**258**	**21.50**
1975	18	22	23	22	22	22	22	22	22	22	22	18	**257**	**21.42**
1976	19	22	22	23	23	22	22	22	22	22	22	21	**262**	**21.83**
1977	22	22	23	23	22	22	22	22	22	22	22	19	**263**	**21.92**
1978	19	23	23	23	22	22	22	22	22	22	21	21	**262**	**21.83**
1979	21	23	23	23	23	23	22	23	22	23	23	19	**268**	**22.33**
1980	22	23	23	23	23	23	21	22	22	22	23	21	**268**	**22.33**
1981	19	22	23	23	23	22	22	22	22	22	21	21	**262**	**21.83**
1982	22	22	22	23	22	23	22	22	22	22	22	22	**266**	**22.17**
1983	19	23	24	24	23	23	22	22	22	23	23	22	**270**	**22.50**
1984	21	23	23	23	23	23	22	23	22	22	22	20	**267**	**22.25**
1985	23	22	23	23	22	22	22	22	22	22	23	20	**266**	**22.17**
1986	21	23	22	23	23	23	22	22	22	22	23	19	**265**	**22.08**
1987	21	23	23	24	23	23	22	22	23	23	23	22	**272**	**22.67**
TOTAL	**286**	**315**	**320**	**322**	**316**	**315**	**307**	**310**	**309**	**311**	**312**	**283**	**3706**	
AV	**20.43**	**22.50**	**22.86**	**23.00**	**22.57**	**22.50**	**21.93**	**22.14**	**22.07**	**22.21**	**22.29**	**20.21**		**22.06**

Table 2: Sample acf $\left(\hat{\rho}_k\right)$ and pacf $\left(\hat{\phi}_{kk}\right)$ for Umudike mean monthly temperature (°C)

(a) $\hat{\rho}_k$ for $\{Y_t\}: \overline{Y} = 22\cdot013,\ S_y = 1\cdot119$ and $\hat{\rho}_k$ for $X_t = \left(1 - B^{12}\right)Y_t : \overline{X} = 0\cdot049,\ S_x = 1\cdot013$

k	Y_t	$(1-B^{12})Y_t$	k	Y_t	$(1-B^{12})Y_t$	k	Y_t	$(1-B^{12})Y_t$
1	0.29	0.03	14	-0.03	0.12	27	-0.17	-0.03
2	-0.07	-0.12	15	-0.12	0.00	28	-0.15	0.00
3	-0.10	0.08	16	-0.15	0.00	29	-0.05	0.06
4	-0.12	-0.07	17	-0.10	-0.15	30	-0.03	-0.02
5	-0.02	0.10	18	-0.03	-0.06	31	0.00	0.17
6	0.00	0.05	19	-0.07	-0.17	32	-0.08	0.05
7	-0.02	0.07	20	-0.11	-0.14	33	-0.11	-0.04
8	-0.06	0.18	21	-0.12	0.02	34	-0.06	-0.04
9	-0.14	-0.05	22	-0.05	-0.06	35	0.23	-0.12
10	-0.03	0.12	23	0.28	0.07	36	0.43	0.13
11	0.30	-0.06	24	0.46	0.04	37	0.24	0.21
12	0.52	-0.54	25	0.16	-0.14	38	-0.10	-0.04
13	0.24	0.06	26	-0.08	-0.08	39	-0.17	-0.01

(b) $\hat{\phi}_{kk}$ for $\{Y_t\}$ and $\{X_t\}$

k	Y_t	$(1-B^{12})Y_t$	k	Y_t	$(1-B^{12})Y_t$	k	Y_t	$(1-B^{12})Y_t$
1	0.29	0.03	14	0.01	-0.01	27	-0.06	0.11
2	-0.17	-0.12	15	-0.02	0.05	28	0.00	-0.04
3	-0.03	0.09	16	-0.05	-0.06	29	0.04	-0.01
4	-0.11	-0.09	17	-0.09	-0.03	30	0.01	-0.09
5	0.04	0.13	18	-0.04	-0.04	31	0.10	0.09
6	-0.03	0.01	19	-0.11	-0.11	32	0.01	0.05
7	-0.03	0.11	20	-0.07	0.01	33	0.05	0.00
8	-0.06	0.16	21	-0.04	-0.05	34	-0.11	-0.04
9	-0.12	-0.03	22	-0.08	-0.09	35	0.00	-0.13
10	0.04	0.17	23	0.11	0.03	36	0.12	-0.05
11	0.30	-0.13	24	0.19	-0.29	37	0.13	0.12
12	0.40	-0.53	25	-0.04	-0.04	38	-0.10	-0.07
13	0.08	-0.01	26	-0.06	0.02	39	0.00	-0.01

The values of the Q statistic (Box and Jenkins (1976)) are not significant as shown in Table 4. Since the model (4.3) is adequate, we can use it to forecast the future mean monthly temperature values. For a given forecast origin, forecast can be calculated directly from the difference equation form as is the case in all ARIMA models. Expanding (4.2), we get the difference equation.

$$Y_t = (1+\phi)Y_{t-12} - \phi\left(Y_{t-24} - \theta e_{t-12} + e_t\right) \tag{4.4}$$

Inserting the estimated values of ϕ and θ gives the forecast form

$$Y_t = 0\cdot8105Y_{t-12} + 0\cdot1895\left(Y_{t-24} - 0\cdot6293\hat{e}_{t-12}\right) \tag{4.5}$$

Table 3: Residual acf and pacf from the fitted models (4.1) and (4.2)

(a) Residual acf and pacf from model (4.1)

k	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	k	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	k	$\hat{\rho}_k$	$\hat{\phi}_{kk}$
1	0.03	0.03	14	0.05	0.02	27	-0.06	0.01
2	-0.06	-0.06	15	-0.01	-0.01	28	-0.03	-0.03
3	0.08	0.09	16	-0.02	-0.06	29	-0.01	-0.01
4	-0.02	-0.03	17	-0.13	-0.12	30	-0.04	-0.06
5	0.07	0.09	18	-0.06	-0.07	31	0.10	0.09
6	0.06	0.04	19	-0.13	-0.13	32	0.01	0.01
7	0.05	0.06	20	-0.08	-0.04	33	-0.04	-0.03
8	0.16	0.15	21	-0.03	-0.05	34	-0.05	-0.07
9	-0.02	-0.03	22	-0.05	0.00	35	-0.03	-0.04
10	0.09	0.11	23	0.03	0.04	36	0.07	0.04
11	-0.03	-0.07	24	0.01	0.03	37	0.16	0.14
12	-0.17	-0.16	25	-0.04	0.05	38	-0.05	-0.09
13	0.04	0.00	26	-0.08	-0.03	39	-0.05	-0.10

(b) Residual acf and pacf from model (4.2)

k	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	k	$\hat{\rho}_k$	$\hat{\phi}_{kk}$	k	$\hat{\rho}_k$	$\hat{\phi}_{kk}$
1	0.03	0.03	14	0.05	0.03	27	-0.07	0.01
2	-0.06	-0.03	15	0.01	-0.01	28	-0.04	-0.04
3	0.09	0.10	16	-0.04	-0.07	29	-0.02	0.00
4	0.00	-0.01	17	-0.13	-0.12	30	-0.04	-0.04
5	0.08	0.08	18	-0.06	-0.08	31	0.08	0.10
6	0.08	0.06	19	-0.12	-0.11	32	0.00	0.02
7	0.05	0.06	20	-0.06	-0.04	33	-0.04	-0.02
8	0.16	0.15	21	-0.05	-0.05	34	-0.05	-0.06
9	-0.01	-0.02	22	-0.04	-0.01	35	-0.01	-0.03
10	0.09	0.09	23	0.00	0.03	36	0.03	0.00
11	-0.03	-0.07	24	-0.05	-0.01	37	0.16	0.14
12	-0.08	-0.09	25	-0.03	0.06	38	-0.05	-0.08
13	0.02	-0.02	26	-0.08	-0.03	39	-0.05	-0.09

Table 4: Summary of Q statistics from the fitted model (4.2)

K	*Q*	*df*	$\chi^2_{0.05}\ df$	Decision
12	10.0	10	18.3	Not significant
24	18.1	22	33.9	Not significant
36	23.2	34	48.6	Not significant
48	36.8	46	65.2	Not significant

df = degree of freedom

To derive the forecast variance, since the model is nonstationary but invertible, we first rewrite the model in the following AR representation

$$\pi(BY_t = e_t) \tag{4.6}$$

where

$$\pi(B) = 1 - \pi_1 B - \pi_2 B^2 \cdots = \frac{\left(1-\phi B^{12}\right)\left(1-B^{12}\right)}{1-\theta B^{12}} \tag{4.7}$$

By equating the coefficients of B on the both sides of (4.7), we have

$$\pi_j = \begin{cases} 1+\phi-\theta, & j=12 \\ \theta^{a-2}(\theta-\phi)(1-\theta), & j=12a;\ a=2,3,\cdots \\ 0, & elsewhere \end{cases} \tag{4.8}$$

The ψ weights that are needed for calculating the forecast variance can be obtained from the MA representation

$$Y_t = \psi(B)e_t \tag{4.9}$$

where

$$\pi(B)\psi(B) = 1 \tag{4.10}$$

and

$$\pi(B) = 1 - \psi_1 B - \Psi_2 B^2 - \cdots \tag{4.11}$$

From (4.10), the Ψ_j weights are

$$\Psi_j = \begin{cases} -\pi_1, & j=1 \\ \sum_{i=1}^{j=1} \pi_i \psi_{j-i} - \pi_j, & j=2,3,4\cdots \end{cases} \tag{4.12}$$

$$= \begin{cases} 0, & j \neq 12a;\ a=1,2,3,\cdots \\ -\pi_{12}, & j=12 \\ \sum_{i=1}^{a-1} \pi_{12i}\psi_{12(a-j)} - \pi_{12a}, & j=12a;\ a=2,3,4,\cdots \end{cases} \tag{4.13}$$

where the π_j weights are given in (4.8). Inserting the estimated values of ϕ and θ in (4.8) and (4.12), we obtain the numerical values of the π_j and ψ_j weights respectively.

Using equation (4.5), the first 12 forecasts, $\hat{Y}_{156}(\ell),\ \ell = 1,2,\cdots,\ 12\left(\hat{Y}_{156}(\ell)\right)$ is the ℓ – step ahead from the time origin t = 156, were calculated and are given in Table 5. Using the ψ_j weights the 95% forecast limits are also given in Table 5 along with the future observed values. These forecast track the seasonal pattern in this series rather well, and only one of the 12 confidence intervals fail to contain the observed future values even though the process is nonstationary.

Table 5: Forecasts for Umudike mean monthly temperature (^{0}C)

Year	Month	Forecast value	95% Confidence Limits		Actual Value	Forecast Error	Percentage Forecast Error
			Lower	Upper			
1987	Jan	21.33	19.79	22.87	21	-0.33	-1.57
	Feb.	22.55	21.01	24.09	23	0.45	1.96
	March	22.86	21.32	24.40	23	0.14	0.61
	April	23.11	21.57	24.65	24	0.89	3.71
	May	22.59	21.05	24.13	23	0.41	1.78
	June	22.62	21.08	24.16	23	0.38	1.65
	July	21.97	20.43	23.51	22	0.03	0.14
	Aug.	22.21	20.67	23.75	22	-0.21	-0.95
	Sept.	22.00	20.46	23.54	23	1.00	4.35
	Oct.	22.14	20.60	23.68	23	0.86	3.74
	Nov.	22.59	21.06	24.13	23	0.41	1.78
	Dec.	20.67	18.73	21.81	22	1.33	6.05

5.0 Conclusion

We have studied the stationarity and invertibility conditions and the covariance structure of the purely seasonal ARIMA process with one seasonal AR term and one seasonal MA term (model 1.9)). An expression for the first few nonzero values of the pacf was obtained.

The fundamental fact about our model (1.9) is that the coefficients of the acf and pacf which appear at lags 1,2,3, ... in ARIMA (1,d,1) now occur at multiples of lag s (s, $2s$, $3s$, ...).

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