A Simple Algorithm for the Inventory Problem with a Linear Increasing Demand

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Abstract

We consider the inventory system where a product has linear increasing demand over a finite horizon and no shortages are allowed. A new direct search method is proposed. The search method is implemented by a computer program written in MATLAB and implemented on a PC. Numerous test problems taken from the literature have been solved using the computer program. The results are accurate and the computer processing time for all the test problems was found to be less than one second. Unlike previous method, the proposed method is easy to implement.

Keywords: *Inventory, linear increasing demand, replenishment points.*
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1.0 Introduction

This paper examines the inventory system with linear increasing demand over a finite horizon without shortages. A simplified and efficient algorithm for determining the optimal inventory replenishment policy for this system is presented. The new algorithm can easily be incorporate into a computer program. Indeed, the algorithm was tested using a computer program written in MATLAB. The motivation for this work is the increase in the use of intelligent inventory management system (Prasad et al., 1996; Kobbacy and Liang, 1999; and Kobbacy and McDonnell, 2001) and the observation that only heuristic procedure has been implement in intelligent inventory system for this class of important inventory problem. Although, Donaldson (1977) proposed an analytical method for this inventory problem, subsequent authors claim that the analytical method is "too computationally complex" (Amrani and Rand, 1990; Yang and Rand, 1993; Teng, 1994, 1996; Hariga, 1995; and Yang et al. 1999). This characterisation has produced a burgeoning literature on heuristic methods for this inventory problem; see Hariga (1995), Zhao et al. (2001), Omosigho (2001) and references therein.

In the present paper, we propose a simplified algorithm based on the exact analytic method of Donaldson (1977). This simplified algorithm is easy to use and can be converted to a computer program. The rest of the paper is organized as follows. In the next section, the problem is stated, followed by the development of a simplified version of the analytical model. An algorithm is then presented for solving the problem. Finally, numerical results based on the algorithm are presented along with conclusions.

2.0 Model Assumptions and Statement of Problem

We consider the inventory system where the demand is given by

$$f(t) = a + bt, \ a > 0, b > 0, \ 0 \le t \le H;$$

H is finite and no shortages are allowed. The ordering cost and the holding cost per unit per unit time are c_1 and c_2 respectively. The initial and final inventory levels are both zero. We determine the replenishment policy that minimizes the total relevant inventory cost over the planning horizon $[0, H]$.

3.0 Model Development and Solution

Suppose the number of replenishment is n. Later, we shall explain how to obtain n. Let $t_i, i = 0, 1, 2, ..., n-1$, be the optimal reorder points that minimizes the total relevant inventory cost over the planning horizon $[0, H]$. Let $t_0 = 0$ and $t_n = H$. It is known, (Donaldson, 1977), that the total relevant inventory cost, $W(n)$, over $[0, H]$ is given by

$$W(n) = nc_1 + c_2 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} (u - t_i) f(u)\,du \tag{3.1}$$

For each fixed n, Donaldson(1977) showed that the necessary condition for optimal t_i, $i = 1, 2, \ldots, n-1$, is given by

$$(t_i - t_{i-1}) f(t_i) = \int_{t_i}^{t_{i+1}} f(u)\,du \quad i = 1, 2, \ldots, n-1. \tag{3.2}$$

In what follows, we shall first consider the case $f(t) = bt, b > 0$, and then extend the result to the case $f(t) = a + bt, \quad a > 0, \ b > 0$.

4.0 Some Preliminary Results

Proposition (Donaldson, 1977)

(a) If $f(t) = bt$, and $z_i = t_i / t_{i-1}$, $i = 2, 3, \ldots, n-1$, then equation (3.2) reduces to

$$z_{i+1} = \{3 - [2/z_i]\}^{1/2} \tag{4.1}$$

(b) If the sequence $\{g_i\}$, $i = 0, 1, \ldots$, is defined by:

$$g_i = \begin{cases} 0 & if \quad i = 0 \\ t_i / t_1 & if \quad i = 1, 2, \ldots \end{cases} \tag{4.2}$$

then $g_1 = 1$ and

$$g_i = g_{i-1} z_i \quad i = 2, 3, \ldots \tag{4.3}$$

(c) If

$$t_n = H, \text{ then } t_i = g_i H / g_n \tag{4.4}$$

Remark

In using equation (4.1), it is usual to set $z_1 = \infty$ (Donaldson, 1977) with the convention that $\alpha / z_1 = 0$ for any real number α.

5.0 The Main Results

Proposition

If $t_n = H$, and $f(t) = bt$, $b>0$, then the total relevant inventory cost, $W(n)$ given by equation (3.1), *becomes*

$$W(n) = nc_1 + \frac{bc_2 H^3}{3 g_n^3} \sum_{i=0}^{n-1} [g_{i+1}^3 - 1.5 g_i g_{i+1}^2 + 0.5 g_i^3] \tag{5.1}$$

Proof

If $f(t) = bt$, $b>0$, then equation (3.1) becomes $W(n) = nc_1 + bc_2 \sum_{i=0}^{n-1} \left(\frac{t_{i+1}^3}{3} - \frac{t_i t_{i+1}^2}{2} + \frac{t_i^3}{6} \right)$. Using

$t_i = g_i H / g_n$ gives the required result.

Proposition

$W(1) > W(2)$ *whenever* $c_1 / c_2 > bH^3 / 3\sqrt{3}$.

Proof.

By equation (5.1), $W(1) = c_1 + (bc_2H^3/3)$ and $W(2) = 2c_1 + \frac{bc_2H^3}{3\sqrt{3}}[\sqrt{3}-1]$. Using these values gives the result.

Remark

Observe that $\lim_{n\to\infty} W(n) = \infty$. *Hence if* $W(1) > W(2)$ *then* $W(n)$ *given by* (5.1) *is a locally convex function of* n. *On the other hand, if* $W(1) < W(2)$ *then* $W(n)$ *is an increasing function of* n *and in this case only one replenishment is optimal. From the foregoing analysis,* $W(n)$ *is either a* $\cup$-shaped *curve or satisfies* $W(1) < W(2) < W(3) < \dots$ *and equation* (5.1) *provides a convenient way for calculating the value of* $W(n)$ *for each* n *without knowing the values of* t_i, $i = 1, 2, \dots, n-1$. *Therefore, we can use function evaluation method to determine the optimal number of replenishment,* n. *We therefore propose a direct search method, based on evaluating the cost function given by equation* (5.1), *for the determination of* n. *In the enumeration process, we compare three successive values of* $W(n)$. *If* $n^* > 1$ *is the optimal number of replenishments, then* n^* *must satisfy:*

$$[W(n^*-1) - W(n^*)][W(n^*) - W(n^*+1)] < 0 \tag{5.2}$$

In the next section, this procedure is presented in an algorithm.

6.0 Algorithm

Case 1: $f(t) = bt,\ b > 0,\ 0 \le t \le H$

Step1: Initialization

Given c_1, c_2, b and H, set $g_0 = 0,\ g_1 = 1,\ n = 2,\ g_2 = \sqrt{3}$ and $z_2 = \sqrt{3}$.

Determine $W(1)$ and $W(2)$ using $W(1) = c_1 + bc_2H^3/3$ and $W(2) = 2c_1 + \frac{bc_2H^3}{3\sqrt{3}}[\sqrt{3}-1]$.

If $W(1) < W(2)$, report that only one replenishment is optimal and stop.

If $W(1) > W(2)$ go to step 2.

Step 2: Calculation of total relevant inventory cost

Increase n by 1 and calculate:

(a) z_n using equation (3),

(b) g_n using $g_n = g_{n-1}z_n$,

(c) $W(n)$ by substituting $g_0, g_1, \dots, g_{n-1}, g_n$ and other relevant values in equation (5.1).

Step 3: Optimality test (Compare three successive values of $W(n)$).

Let $n^* = n-1$, be a positive integer. If n^* satisfies equation (5.1) then this value is the optimal number of replenishment and go to step 4, otherwise return to step 2.

Step 4: Determination of replenishment points and replenishment quantities.

Let n^* be the optimal number of replenishment obtained in step 3. Determine t_i, $i = 0, 1, 2, \dots, n^*-1$, from $t_i = g_iH/g_n$. The order quantity at each replenishment point t_i, is given by

$$Q_i = 0.5b(t_{i+1} - t_i)(t_{i+1} + t_i) \tag{6.1}$$

Case 2: $f(t) = a + bt,\quad a > 0,\quad b > 0.$

The inventory policy for the system with demand given by $f(t) = a + bt, a > 0, b > 0$, ordering cost $= c_1$, holding cost $= c_2$, and time horizon, h is obtained by modifying the procedure first enunciated by Donaldson (1977). The modified approach is based on the following results:

Proposition

Let k *be a positive integer and* $0 < u_1 < u_2 < \dots < u_{k-1}$ *be a partition of the interval* $[0, k]$. *Suppose*

$(0 = u_0, u_1, u_2, \ldots, u_{k-1})$ *are replenishment points with the associated cost* $W(k)$ *and* $(0 = u_0, u_2, \ldots, u_{k-1})$ *are replenishment points with the associated cost* $W(k-1)$ *then,*

$$W(k) - W(k-1) = c_1 - c_2 u_1 (u_2 - u_1)(a + 0.5b[u_2 + u_1]) \tag{6.2}$$

Proof:

If the replenishment points are $(u_0, u_1, u_2, \ldots, u_{k-1})$ then, by equation (3.1), the associated total relevant inventory cost is

$$W(k) = kc_1 + c_2 \sum_{i=0}^{k-1} \int_{u_i}^{u_{i+1}} (s - u_i) f(s) ds \tag{6.3a}$$

Similarly, if the replenishment points are $(u_0, u_2, \ldots, u_{k-1})$ then the associated total relevant inventory cost is

$$W(k-1) = (k-1)c_1 + c_2 \int_{u_0}^{u_2} (s - u_0) f(s) ds + c_2 \sum_{i=2}^{k-1} \int_{u_i}^{u_{i+1}} (s - u_i) f(s) ds \tag{6.3b}$$

Subtracting equation (6.3b) from equation (6.3a) gives

$$W(k) - W(k-1) = c_1 + c_2 \left(\left(\sum_{i=0}^{2} \int_{u_i}^{u_{i+1}} (s - u_i) f(s) ds\right) - \int_{u_0}^{u_2} (s - u_0) f(s) ds \right)$$

$$= c_1 + c_2 \left(\int_{u_1}^{u_2} (s - u_1) f(s) ds - \int_{u_1}^{u_2} s f(s) ds \right) = c_1 - c_2 \int_{u_1}^{u_2} u_1 f(s) ds = c_1 - c_2 u_1 (u_2 - u_1)(a + 0.5b[u_2 + u_1])$$

which is equation (6.2).

To solve the inventory problem with $f(t) = a + bt, a > 0, b > 0$, ordering cost $= c_1$, holding cost $= c_2$, and time horizon $= h$, we solve the auxiliary problem given by $s(T) = bT$, $b > 0$, (where $T = t + a/b$), ordering cost $= c_1$, holding cost $= c_2$, and time horizon, $H = h + a/b$, as described above. Suppose the optimal replenishment points for the auxiliary problem are $T_i, i = 0, 1, \ldots, n-1$. Consider $t_i = T_i - a/b \geq 0$. Suppose there exist i^* such that $t_{i^*} = 0$, then the replenishment points for the inventory problem with $f(t) = a + bt, \quad a > 0, \quad b > 0$ are given by

$$t_j = T_j - a/b, j = i^*, i^* + 1, \ldots, n-1.$$

However, if there exist i^* such that $t_{i^*-1} < 0 < t_{i^*} < t_{i^*+1} < t_{i^*+2} < \ldots < t_{n-1} = H$. Let $u_0 = 0$, $u_1 = t_{i^*}$, $u_2 = t_{i^*+1}, \ldots, u_{k-1} = t_{n-1} = H$ The replenishment points for the real problem are either $u_0, u_1, u_2, \ldots, u_{k-1}$ or $u_0, u_2, \ldots, u_{k-1}$, i.e. k or $k-1$ replenishment points. To choose the optimal replenishment points, we evaluate

$$c_1 - c_2 u_1 (u_2 - u_1)(a + 0.5b[u_2 + u_1]).$$

If $c_1 - c_2 u_1 (u_2 - u_1)(a + 0.5b[u_2 + u_1]) > 0$ then the replenishment points are $u_0, u_2, \ldots, u_{k-1}$. And if $c_1 - c_2 u_1 (u_2 - u_1)(a + 0.5b[u_2 + u_1]) < 0$ then the replenishment points are $u_0, u_1, u_2, \ldots, u_{k-1}$. The use of equation (6.2) is different from the approach suggested by Donaldson (1977) who proposed the evaluation of W(k) and W(k-1).

When $f(t) = a + bt$, the quantity ordered at each replenishment point u_i, is given by

$$Q_i = (u_{i+1} - u_i)[a + 0.5b(u_{i+1} + u_i)] \quad (6.4)$$

7.0 Numerical Results and Discussion

The new procedure presented in this paper, is a straightforward iterative method, which can easily be implemented using a computer program. Indeed a computer program was written in MATLAB and used to solve numerous test problems. Table 1 shows the parameters and results for some of the problems solved using the program. The results are accurate and require low computer time. For each problem in Table 1, the associated replenishment points can be obtained using $t_i = g_i H / g_n$ when $a = 0$ or the alternative method when $a \neq 0$. Compared with the results published by Yang and Rand (1993), the results obtained by the proposed method are accurate. However, the proposed method has the advantage that it is easy to implement compared with previous methods. Figure1 shows the graph of $W(n)$ for the inventory problem with $f(t) = t, c_1 = 9,\ c_2 = 0.5$ and $H = 1$. The problem is designed to illustrate the case where the optimal number of replenishment is one. Previous authors ignored this case.

Table 1: Numerical results for 15 test problems. Problems 1 to 4 are from Donaldson (1977). Problems 5 to 15 are from Yang and Rand (1993). Demand is given by $f(t) = a + bt$. H is time horizon, c_1 and c_2 are ordering and holding costs respectively. $W(n^*)$ is total relevant inventory cost divided by c_2, n^* is the optimal number of replenishments

No.	a	b	H	c_1	c_2	$W(n^*)$	n^*	CPU time (seconds)
1	0	100	3	9	2/3	187.89	7	0.06
2	200	100	1	9	2/3	82.98	3	0.22
3	6	1	11	9	0.1	510.91	3	0.16
4	0	900	9	9	2	62.63	7	0.11
5	0	1600	3	42	0.56	1744.9	12	0.06
6	0	900	2	9	2	172.89	20	0.11
7	0	100	4	100	2	561.3	6	0.05
8	6	1	11	30	1	293.11	5	0.27
9	6	1	11	50	1	380.95	4	0.22
10	6	1	11	60	1	420.95	4	0.22
11	6	1	11	70	1	450.91	3	0.22
12	6	1	11	90	1	510.91	3	0.28
13	100	150	1	30	2	75.57	3	0.22
14	100	150	1.5	30	2	121.28	4	0.28
15	100	150	2	30	2	173.97	6	0.27

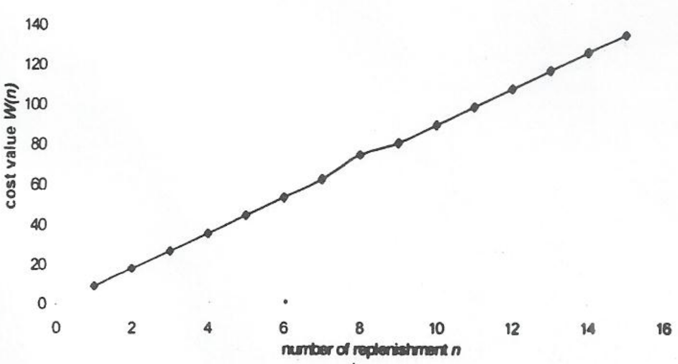

Figure 1: Cost Function $W(n)$ when $f(t) = 1, c_1 = 9, c_2 = 0.5, H = 1$.

Conclusion

We have presented a simple algorithm for the inventory problem with linear increasing demand over a finite horizon without shortage. The program written for the scheme requires low computer time for the numerous problems we have solved. The proposed procedure will therefore be useful in the development of intelligent inventory system.

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