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Criteria for starlikeness in the unit disk

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Abstract

Let $f(z) = z + a_2 z^2 + \cdots \in T_n^{\alpha}(\beta)$ *be analytic in the unit disk* U. ***By the method of*** *differential subordination for functions f(z) to be in the class* **S*[a, b] we give a** *subordination relation for* $\frac{f(z)^{\alpha}}{z^{\alpha}}$.

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1.0 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the unit disk U = $\{z : |z| \prec 1\}$. For a function $f(z) \in T_n^{\alpha}(\beta) C S^* C A$ **we say that it is** starlike of order β if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \succ \beta \tag{1.2}$$

for some $\beta (0 \leq \beta \prec 1)$, and for all $z \in$ U, we denote by S* (β) the class of such functions. **We shall also** introduce the following notations:

(i) $S_0 = \left\{ f(z) \in A : \mathrm{Re}\left\{\frac{f(z)}{z}\right\} \succ 0,\ z \in U \right\}$

(ii) $B(\beta) = \left\{ f(z) \in A : \mathrm{Re}\left\{\frac{f(z)}{z}\right\} \succ \beta, z \in U, 0 \leq \beta \prec 1 \right\}$

(iii) $\delta(0) = \{ f(z) \in A : \mathrm{Re}\, f(z) \succ 0, \in U \}$

(iv) $B_n(\alpha) = \left\{ f(z) \in A : \mathrm{Re}\left(\frac{D^n f(z)^{\alpha}}{z^{\alpha}} \right) \succ 0, z \in U, \alpha \succ 0,\ n = 0,1,2,\cdots \right\}$

These classes of functions are well known and have been investigated **repeatedly cf [1, 2], 3, 4 and 5].** Furthermore, we denote by $T_n^{\alpha}(\beta)$ a subclass of A **consisting of functions satisfying the following conditions.**

$$\operatorname{Re}\left\{\frac{D^n\left(f(z)^\alpha\right)}{z^\alpha}\right\} \succ \beta,\ z\in U,\ n=0,1,2,\ \alpha\succ 0,\ \beta\le 1 \qquad (1.3)$$

and the operator D is the same as in $B_n(\alpha)$ namely the Salagean differential operator which is define as:

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= Df(z) = zf'(z) \\ D^n f(z) &= D\left(D^{n-1} f = z\left(D^{n-1} f(z)\right)'\right) \end{aligned} \qquad (1.4)$$

Let $f(z)$ and $g(z)$ be analytic in the unit disk U, then we say that the function $g(z)$ is subordinate to $f(z)$ in U if there exists an analytic function $w(z)$ in the unit disk U such that $|w(z)| \prec 1$ $(z \in U)$ and $g(z) = f(w(z))$. For this relation the symbol $g(x) \prec f(z)$ is used. In case $f(z)$ is univalent in U we have that the subordination $g \prec f$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$. For a function $f(z)$ belonging to A we say that it belongs to the class $S^*[a,b]$, $-1 \le b \prec a \le 1$ if and only if

$$\frac{D^1 f(z)}{D^0 f(z)} = \frac{zf'(z)}{f(z)} \prec \frac{1+az}{1+bz}, \text{ that is by (1.4)} \qquad (1.5)$$

This simply means that geometrically the image of U by $\frac{zf'(z)}{f(z)}$ is inside the open disk centred on the real axis with diameter and points $\frac{(1-a)}{(1-b)} = \frac{(1+a)}{(1-b)}$. From these and from (1.2) we conclude that $S^*[a,b] \subset S^*\left(\frac{(1-a)}{(1-b)}\right)$.
Some special selections of a and b leads to the following classes:

$$S^*[1,-1] \equiv S^*, S^*[1-2\beta,-1] \equiv S^*(\beta), 0 \le \beta \prec 1, S^*[\beta,0]$$

is the class defined by $\left|\frac{zf'(z)}{f(z)} - 1\right| \prec \beta$, $(0 \le \beta \prec 1)$ and S* is the class of starlike functions.

The general theory of differential subordinations introduced by Miller and Molanu is given in [6]. The first order differential subordinations with many interesting applications are considered by the same author in [7]. Namely if $\psi : \varphi^2 \to \varphi$ is analytic in a domain D, if $h(z)$ is univalent in U, and if $p(z)$ is analytic in U with $(p(z), zp'(z)) \in D$, $z \in U$, then $p(z)$ is said to satisfy the first order differential subordination if

$$\Psi(p(z), zp'(z)) \prec h(z) \qquad (1.6)$$

In this paper, we give a result for a function $f(z)$ of the form (1.1) for the image domain for functions $\frac{f(z)^\alpha}{z^\alpha}$, where $f(z) \in S^*[a,b]$. For the proof of this result we need the following Lemma.

Lemma 1

Let p(z) be analytic in U with p(0) = 1 and p(z) $\neq 0$ of $0 \prec |z| \prec 1$, and let $-1 \leq b \leq a \leq 1$.

(i) *Let $b \neq 0$ and μ be a complex number with $\mu \neq 0$; let a, b and $b \neq 0$ and μ satisfies:*

$$\left|\mu \frac{a-b}{b} - 1\right| \leq 1 \text{ or } \left|\mu \frac{a-b}{b} + 1\right| \leq 1 \tag{1.7}$$

if p(z) satisfies

$$1 + \frac{zp'(z)}{\mu p(z)} \prec \frac{1+az}{1+bz} \tag{1.8}$$

then $p(z) \prec (1+bz)^{\mu(a-b)/b}$

(ii) *Let $b = 0$, μ be a complex number with $\mu \neq 0$ and $|\mu a| \prec \pi$ satisfies*

$$1 + \frac{zp'(z)}{\mu p(z)} \prec 1 + az \tag{1.9}$$

then $p(z) \prec e^{\mu az}$

Proof (i)

The proof can be sourced from another Lemma which states that

Lemma 2

Let q(z) be univalent in U and let $\theta(\omega)$ and $\phi(\varpi)$ be analytic in a domain D containing q(z) with $\phi(w) \neq 0$ when $\omega \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that

(i) *Q(z) is starlike (Univalent in U with $Q(0) = 0$, $Q'(0) \neq 0$; and*

(ii) $\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} \succ 0, \ z \in U$

If p(z) is analytic in U, with p(0) = q(0), p(U) $\subset$ D and

$$\theta(P(z)) + zp'\phi(P(z)) \prec Q(q(z)) + zq'(z)\phi(q(z)) = h(z) \tag{1.10}$$

then $p(z) \prec q(z)$ *and q(z) is the best dormant* of (1.10).

From Lemma 2, we choose $\theta(\varpi) = 1$, $\phi(\varpi) = \frac{1}{\mu\varpi}$, and $q(z) = (1+bz)^{\mu\frac{(a-b)}{b}}$. The function $q(z)$ with (1.7) is univalent in U. We easily obtain that $Q(Z) = zq'(z)\phi(q(z)) = \frac{(a-b)z}{1+bz}$ is starlike (but not normalized) in U and that $h(z) = \theta q(z) + Q(z) = \frac{1+az}{1+bz}$ the conditions in Lemma 2 are also satisfied, and dthe conclusion follows from Lemma 2.

Proof (ii)

The proof is similar to that of (i).

2. Some Properties for S*[a, b] and $T_n^{\alpha}(\beta)$

Let us give the following lemma first.

Lemma 3

Let p(z) be analytic in U with $p(0) = 1$ *and*

$$P(z) \neq 0, \text{ for } 0 \prec |z| \prec 1, \text{ and let } -1 \leq b \prec a \leq 0.$$

If p(z) satisfies.

$$1 - \frac{1}{p(z)} + \frac{zp'(z)}{P(z)^2} \prec (a-b)\frac{z(2+az)}{(1+az)^2} = h_1(z) \tag{2.1}$$

then $p(z) \prec \frac{(1+az)}{(1+bz)}$

Proof

If we take $q(z) = \frac{1+az}{(1+bz)}$, $-1 \leq b \prec a \leq 0$, $\theta(\omega) = \frac{1}{\omega}$, and $\phi(\varpi) = \frac{1}{\varpi^2}$ in Lemma 2, then it is easy to show that $q(z)$, $\theta(\varpi)$ and $\phi(\varpi)$ satisfy the conditions of Lemma 2, since

$$Q(z) = zq'(z)\phi(q(z)) = \frac{(a-b)z}{(1+az)^2}$$

is a starlike in U and $h(z) = \theta(q(z)) + Q(z) = (a-b)\frac{z(2+az)}{(1+az)^2} = h_1(z)$ as defined before, which shows that the conditions (i) and (ii) of Lemma 2 will be satisfied.

We denote by $T_n^{\alpha}(\beta)$ a subclass of A consisting of functions satisfying the following conditions

$$\operatorname{Re}\left[\frac{D^n\left(f(z)^{\alpha}\right)}{z^{\alpha}}\right] \succ \beta,\ z \in U,\ n = 0,1,2\cdots,\ \alpha \succ 0,\ 0 \leq \beta \prec 1. \tag{2.2}$$

We give the following remarks and some theorem by Opoola [10].

(*i*) *for* $n = 0,\ \alpha = 1,\ \beta = 0$ we obtain the class S* (0)

(*ii*) *for* $n = 0,\ \alpha = 0,$ we obtain the class $B(\beta)$

(*iii*) *for* $n = 1,\ \alpha = 1,$ we obtain the class $\delta(\beta)$

(*iv*) *for* $\beta = 0,$ we obtain the class $B_n(\alpha)$

We also give the following theorem as stated in [10].

Theorem 2.1

$T_{n-1}^{\alpha}(\beta) \subset T_n^{\alpha}(\beta)$, *for* $n \succ 1$.

Theorem 2.2

$T_{n-1}^{\alpha}(\beta) \subset S$ *for* $n \geq 1$ *where* S* *is the subclass of* A *consisting of univalent function in* E.

Theorem 2.3

If $f \in T_n^{\alpha}(\beta)$ then $\operatorname{Re}\left[\frac{f(z)^{\alpha}}{z^{\alpha}}\right] \succ \beta,\quad z \in E$

The main aim of this paper is to establish more properties for the subclass $T_n^{\alpha}(\beta)$ using subordination method.

3. Result and Consequences

With the aid of Lemma 1, we show:

Theorem 3.1

Let $f(z)\in T_n^{\alpha}(\beta)\subset S^*[a,b]$ *where* $-1\le b\prec a\prec 1$ *and* $b=0$. *Then*

$$\frac{f(z)^{\alpha}}{z^{\alpha}}\prec(1-bz)^{\frac{\alpha(a-b)}{b}} \tag{3.1}$$

where $\alpha\neq 0$ *satisfying* (1.7). *In case* $b=0$, *that is for* $f(z)\in S^*[a,b]$, $0\prec a\le 1$, we have

$$\frac{f(z)^{\alpha}}{z^{\alpha}}\prec e^{\alpha az},\quad \text{where } \alpha\neq 0 \tag{3.2}$$

Proof

Let

$$P(z)=\frac{f(z)^{\alpha}}{z^{\alpha}} \tag{3.3}$$

then P(z) is analytic in U. For such $p(z)$ from (i) and (ii) of Lemma 1, we get relation (3.1) and (3.2) of Theorem 3.1

From Theorem 3.1, in some special cases, we have the following corollary.

Corollary (i)

For $a=1-2\beta,\ 0\le\beta\prec 1$, and $b=-1$ we have

$$f(z)\in S^*[\beta]\Rightarrow\frac{f(z)^{\alpha}}{z^{\alpha}}\prec\frac{1}{(1-z)^{2\alpha(1-\beta)}} \tag{3.4}$$

where $\alpha\neq 0$. satisfies 0, $|2\alpha(1-\beta)-1|$. For example from (3.4) for $\alpha=\frac{1}{2}$ we get

$$f(z)\in S^*[\beta]\Rightarrow\sqrt{\frac{f(z)}{z}}\prec\frac{1}{(1-z)^{(1-\beta)}},\quad 0\le\beta\prec 1,\ \text{for } \beta=0 \tag{3.5}$$

we obtain the corresponding relation S*, that is

$$f(z)\in S^*\Rightarrow\frac{f(z)^{\alpha}}{z^{\alpha}}\prec\frac{1}{(1-z)^{2\alpha}} \tag{3.6}$$

where $\alpha\neq 0$, $|2\alpha-1|\le 1$. Especially for $\alpha=1$, we have

$$f(z)\in S^*\Rightarrow\frac{f(z)}{z}\prec\frac{1}{(1-z)^{2}} \tag{3.7}$$

Corollary (ii)

For $b=0$ and $a=\beta,\ 0\le\beta\le 1$, that is for the class $S^*[\beta,0]$ defined by $\left|\frac{zf'(z)}{f(z)}-1\right|\prec\beta$ we have

$$f(z)\in S^*[\beta,0]\Rightarrow\frac{f(z)^{\alpha}}{z^{\alpha}}\prec e^{\alpha\beta z}\ \text{where } \alpha\neq 0.$$

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