Journal of the Nigerian Association of Mathematical Physics
Volume 7(2003)

On The Controllability of Perturbations of Non-Linear Delay Systems to Equilibrium State

V.A. Iheagwam
Department of Mathematics and Computer Science
Federal University of Technology, Owerri

Abstract

The purpose of this research is to obtain a set of criteria for the controllability of perturbations of non-linear functional differential systems to equilibrium state. In this study, a relationship is established between the controllability to the origin of the system under study and the location of the origin in the interior of the domain of null controllability; to provide a necessary and sufficient condition for the existence of equilibrium state for given non-linear dynamics.

pp. 71 - 76

1.0 Introduction

The subject of equilibrium state is very important in all real life situations. Every system, be it political system, economic system needs a guarantee of little equilibrium to survive. In Physics and Engineering, the achievement of equilibrium state for some dynamics has been the preoccupation of many researchers. Controlling a system to its equilibrium state therefore poses a challenge to scholars of mathematical control theory

The controllability to equilibrium state has however, received very little attention. Emphasis has been on "null controllability" (see [1]) or controllability to the origin (see[2]).The thrust of the efforts on both types of controllability is to have the origin at the centre of the reachable set (see[3]).The controllability to the origin of ordinary systems and their perturbations have been investigated by some researchers among whom are Onwuatu [1], Dauer [4], Olbrot [5], Hermes and Lasalle [6],Pandolfi [7].
Results obtained from these studies have been applied by, Chukwu [2], Cooke and Yorke [8], Takayama [9] and Arrow [10] in the controllability of some socio-economic dynamics.

Investigation into the controllability of Functional differential systems to the origin is attracting attention with the growing interest in disease control models (see[11]) where the number of affected individuals are desired to be controlled to zero. However, the absence of standard controllability techniques for perturbations of nonlinear delay systems has hindered many incursions into the realm. The method of linearization has proved useful in nonlinear analysis. In [12], Chukwu used the technique of linearization in establishing the conditions for the equilibrium state for nonlinear neutral system in response to the question: Controllability to equilibrium state: Will the centre hold? Recently, the linearization method was used in [13] by Iheagwam to establish the relative controllability of the nonlinear perturbation, with implicit derivative given by

$$x(t) = L(t, x_t, u_t)x_t \cdot B(t, x_t, u_t)u_t + f(t, x_t, x_t, u_t) \qquad (1.1)$$

The continuation of investigation of the above system in the direction of the new information pointed to by Chukwu [12] is the main objective of our research here.

2.0 Notation and Preliminaries

Let E be the real line and E^n the n–dimensional Euclidean space with norm denoted by $\|.\|$. The symbol $C = C([-h, 0], E^n)$ denotes the space of continuous functions defined on the interval, [-h,0] into E^n with the sup. norm $\|.\|$ defined by $\|\phi\| = \sup |\phi(\theta)|$, $\phi \in C$; $-h \leq \theta \leq 0$, $h > 0$. $C^1 = C^1([-h, 0], E^n)$ denotes the space of differentiable functions mapping the interval [-h, 0] into E^n.

Let $(X, \|.\|)$ be a Banach space and Q a bounded subset of X. The measure of non– compactness of Q is given as

$$\mu(Q) = \inf \{r > 0: \text{Q can be covered by a finite number of balls of radii less than } r\}$$

For the space of continuous functions C ($[t_0, t_1], E^n$) the measure of non compactness of a set Q is given by

$$\mu(C) = \tfrac{1}{2} W_0(C) = \tfrac{1}{2} \lim_{h \to 0+} W(c,h)$$

where W(c,h) is the common modulus of continuity of the functions which belong to the set C, that is ,

$$W(c, h) = \sup_{x \varepsilon C} \{\sup |x(t) - x(s)| : |t-s| \leq h\}$$

For the space of differentiable functions C^1 ([-h, 0], E^n), we have

$$\mu(C^1) = \tfrac{1}{2} W_0(DC)$$

where DC = $\{x : x \varepsilon C\}$.

For the function $x \varepsilon C^1$ defined by $x:[t_0-h, t_1] \to E^n$, h>0 then for t ε $[t_0, t_1]$ then x_t denotes the functions. On [-h, 0] defined by

$$x_t(s) = x(t+s)$$

for s ε [-h, 0]. Also, for functions $u:[t_o-h, t_1] \to E^n$ and t ε $[t_o, t_1]$, u_t denotes the function on [-h, 0] defined by

$$u_t(s) = u(t+s) \quad \text{for} \quad s \, \varepsilon \, [-h,0]$$

Consider the system of interest

$$x(t) = L(t, x_t, u_t)x_t + B(t,x_t,u_t)u_t + f(t,x_t,x(t),ut) \tag{2.1}$$

where $L(t, s, \Psi)x_t = \int_{-h}^{0} d\eta\,(t,s,\phi,\Psi)x\,(t+s)$

With the $n \times n$ matrix function $\eta(t,s,\phi,\Psi)$ measurable in (t , s) ε $E \times E$, normalised so that

$$\eta(t,s,\phi,\Psi) = 0 \;;\; s \geq 0 \quad \text{for all } \phi, \Psi$$

$$\eta(t,s,\phi,\Psi) = \eta(t,-h,,\phi,\Psi) \quad \text{for all } s \leq -h.$$

$\eta(t,s,\phi,\Psi)$ is continuous from the left in s on [-h,0] and has bounded variation in s on [-h,0] for each t, ϕ, Ψ. | $L(t,s,\phi,\Psi)x_t| \leq m(t) \| x_t \|$ for all t ε $(-\infty, \infty)$,Ψ ε C. m(t) is an integrable function. The n x m matrix $B(t, x_t,u_t)$ u_t given by

$$B(t,x_t,u_t)\,u_t = \int_{-h}^{0} d_sH(t,x(t+s), u(t+s))\,u\,(u+s)$$

is continuous on all variables and is of bounded variation in s on [-h,0]. Also, the function f is continuous and satifies the Lipschitz condition in all its arguments. Integration is in the Lebesgue–Steiltjes sense. Enough smoothness conditions on L and f are imposed to ensure the existence of solution of system (2.1) and the continuous dependence of same on initial data. (see ref 13)

Definition 2.1

The set y(t) = $\{x(t),x_t,u_t\}$ is the complete state of system (2.1)

Definition 2.2

System (2.1) is ***relatively controllable*** on $[t_0,t_1]$, if for every initial complete state $y(t_0)$ and every x_1 ε E^n, there exists a control u(t) defined on $[t_0,t_1]$ such that the corresponding trajectory of system (2.1) satisfies $x(t_1) = x_1$. If $x_1 = 0$, the system (2.1) is said to be ***relatively null controllable.***

Definition 2.3 (Domain of Null Controllability)

The domain of relatively null controllability of system (2.1) is the set of all initial values ϕ (to) for which the system is relatively null controllable.

Definition 2.4

System (2.1) is controllable to the equilibrium state if zero is in the interior of the domain of relatively null controllability

Throughout this work, ***int.*** stands for ***interior***, ***sup*** for **supremum**, ***Det*** for ***determinant.***

3.0 Linearization Process

Consider the nonlinear system

$$\dot{x}(t) = L(t, x_t, u_t)x_t + B(t, x_t, u_t)u_t \tag{3.1}$$

which is the unperturbed part of system (2.1). By replacing the arguments, x_t, u_t of L and B inside the brackets by specified functions $z \varepsilon C^1$ and $v \varepsilon C$. System (3.1) becomes

$$\dot{x}(t) = L(t, z, v)x_t + B(t, z, v)u_t \tag{3.2}$$

which is a linear approximation of the nonlinear system (3.2). System (2.1) can thus be approximated by the linear perturbation

$$\dot{x}(t) = L(t, z, v)x_t + B(t, z, v)u_t + f(t, x_t, \dot{x}_t, u(t)) \quad 3.3)$$

Owing to the difficulty of investigating system (2.1) directly, we usually extract information about the system by studying system (3.3).

3.1 Variation of Constraints Formula for System (3.3)

Consider the homogeneous part of system (3.2)

$$\dot{x}(t) = L(t, z, v)x_t \tag{3.4}$$

Let X(t,s) = X(t,s,z,v) be the transition matrix for system (3.4); so that

$$\partial X(t,s) / \partial t = L(t,z,v)\, X(t,s) \tag{3.5}$$

where

$$X(t,s) = \begin{cases} 0, & for\ s-h \le t \le s \\ 1, & for\ t=s\ (I, identity matrix)) \end{cases}$$

and

$$X_{t(\cdot,s)}\,\theta = X(t+\theta, s); \quad -h \le \theta \le 0$$

We can thus express the solution of system (3.3) by

$$x(t) = x(t, t_0, \phi, 0) + \int_{t_0}^{t} X(t,s)\left(\int_{-h}^{0} d_\theta H(s, z(\theta), v(0)u(s+0)) \right) ds + \int_{t_0}^{t} X(t,s) f(s, x_s, x(s) \cdot v(s)) ds \tag{3.6}$$

Using the unsymmetric Fubini theorem on the change of the order of integration (3.6) becomes

$$\begin{aligned} x(t) = x(t, t_0, \theta, 0) + X(t, t_o) \int_{0+s}^{t0} \left\{ \int_{-h}^{0} X(t_o, s-\theta) d_0 [H(s-\theta, (z)\theta), v(\theta)] u_{t0} \right\} ds\, + \\ X(t, t_o) \int_{0}^{t} \left\{ \int_{-h}^{0} X(t_o, s-\theta) d\theta [H(s-\theta, (z)\theta), v(\theta)] u(s) \right\} ds \\ + X(t, t_0) \int_{0}^{0} X(t_o, s) f(s, x_s, x(s), u(s),) ds \end{aligned} \tag{3.7}$$

where
$$\overline{H}(s,z,v) = \begin{cases} H(s,z,v), & \text{for } s \le t \\ 0, & \text{for } s \succ 0 \end{cases}$$

Let us now define the following set functions at $t = t_1$. The Reachable Set for System (3.2) at time t_1 from the variation of constraint formula (3.7); we extract the set function called the reachable set; denoted by $R(t_o, t_1)$ given as

$$R(t_1,t_0) = \left\{ \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t_1, s-\theta) d_0 \{H(s-0, z(0), v(0))\} u(s) \right] ds : u \in U \right\}$$

The reachable set is closed and bounded. See ref [3]. Define a function $Z = Z(t_o, s, z, v)$ by

$$Z(t_o, s, z, v) = \int_{-h}^{0} X(t_o, s-\theta)\, d_\theta H(s-\theta, z(\theta), v(\theta))$$

The controllability grammian of system (3.2) at time t_1 is given as $W(t_o, t_1, z, v) = \int_{t_0}^{t_1} Z \cdot Z^T ds$, T denotes transpose.

3.2 **Controllability Results.**

Consider the non-linear approximation (3.3) of system (2.1) given by

$$\dot{x}(t) = L(t, z, v)x_t + B(t, z, v)u_t + f(t, x_t, \dot{x}(t), u_t)$$

In ref [13] under smoothness and boundness conditions on L, B, f, it was established that if the Infimum det $W(t_o, t_1, z, v) > 0$, $z \varepsilon C^1$ then system (3.3) is relatively controllable on $[t_o, t_1]$. We shall state and prove a theorem that guarantees the controllability of system (3.2) to equilibrium state. Under the assumptions on system (3.2), we have

Theorem (3.1)

If 0 ε interior $R(t_1,t_o)$ then system (3.2) is relatively controllable on the finite interval $[t_o, t_1]$.

Proof: The proof is immediate following Chukwu-like argument presented in [3].

3.3 **Main Results**

We shall consider system (3.3) with all the conditions imposed on it.

Theorem 3.2

Suppose further that

(i) *System* (3.3) *is relatively controllable on the interval $[t_o, t_1]$; and*

(ii) *$f(t, t_o, 0, 0, 0) = 0$ then the domain of relative null controllability of systems* (3.3) *contains zero in its interior .*

Proof

By the relative null controllability of systems (3.3) and condition (ii) there exists a $u \in U$ such that the solution of system (3.3) satisfies

$$x(t_o, t_o, x_o, u, f) = x_o$$
$$x(t_1, t_o, x_o, u, f) = 0$$

This implies that

$$0 = X(t_1, t_o)\left[x_o + \int_{0+s}^{t_0} d_H \left[\int_{-h}^{0} X(t_o, \tau-s) H(\tau-s, s) u\, t_o\, d\tau \right] + \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t_o, \tau-s) dH(\tau-s, s) \right] u(\tau)\, d\tau \right.$$
$$\left. + \int_{t_0}^{t_1} X(t_o, \tau) f(\tau, x(\tau), x(\tau), u(\tau), u(\tau-h)) d\tau \right.$$

Simplifying, we have

$$x_o = -\int_{0+s}^{t_0} d_H \left[\int_{-h}^{0} X(t_o, \tau-s) H(\tau-s, s) u_{t0} d\tau \right] - \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t_o, \tau-s) dH(\tau-s, s) \right] u(\tau) d\tau$$

$$- \int_{t_0}^{t_1} X(t_o, \tau) f(\tau, x(\tau), x.(\tau), u(\tau), u(\tau - h)) d\tau$$

We now define the set function

$$K(t_1, t_o) = \left\{ - \int_{t_{0+s}}^{t_o} dH \left[\int_{-h}^{0} X(t_o, \tau - s) H(\tau - s, s) u t_o \, d\tau \right] - \int_{t_0}^{t_1} \left[\int_{-h}^{0} X(t_o, \tau - s) dH(\tau - s, s) \right] u(\tau) d\tau \right.$$

$$\left. - \int_{t_{0+s}}^{t_o} X(t_o, \tau) f(\tau, x(\tau), \dot{x}(\tau), u(\tau), u(\tau - h)) d\tau : u \in U \right\}$$

Since U is symmetric about 0, the second summand is the reachable set $R(t_1,t_0)$. Denote the other terms by $Q(t_1,t_0)$, then we have

$$K(t_1,t_0) = \{(R(t_1,t_0) \cup Q(t_1,t_0) : u \in U\} \tag{3.8}$$

From the definition of the domain of relative null controllability, D of system (3.3)

$$X(t_1, t_0, D, x_0, 0, f) = \{x(t_1, t_0, x_0, \dot{x}_0, 0, f) : x_0 \in D\}$$

Clearly, D is a subset of E^n and such $x(t_1, t_0, D, x_0, 0, f) \subseteq \{R(t_1, t_0) \cup Q(t_1, t_0) : u \in U\} = K(t_1,t_0)$

By condition (ii) of the theorem, we have that 0 is a trivial solution of system (3.3) and hence, $0 \in K(t_1, t_0)$ for each $t_1 > t_0$. By theorem (3.1), 0 is in the interior of $R(t_1,t_0)$. Next, we show that 0 is in the interior of $Q(t_1, t_0)$. Since $0 \in \text{int } R(t_1,t_0)$, there exists an open ball B_1 such that $0 \in B_1 \subset R(t_1, t_0)$, hence $B_1 + Q(t_1,t_0) \subseteq R(t_1, t_0) \cup Q(t_1,t_0) = K(t_1, t_0)$, but $B_1 + Q(t_1, t_0)$ is a ball around $Q(t_1, t_0)$. Therefore, $0 \in Q(t_1, t_0) \subseteq K(t_1,t_0)$ for $t_1 > t_0$. Since $K(t_1, t_0) \subseteq D$. Clearly, 0 is in D. Suppose 0 is on the boundary and not in the interior of D, then, there exists a countable sequence of initial points $\{x_0^i\} \in E^n, 1,2,3,\cdots$ such that $x_0^i \to 0$ as $i \to \infty$ $x_0^i \notin D$ for any i, (that is the system is not relatively null controllable) hence, $z_i = x(t_1, t_0, x_0^i, \dot{x}_0^i, u, f) \neq 0$ for any i and $z_i \in K(t_1, t_0)$ for any i and $t_1 > t_0$, but $z_i \to 0$ as $i \to \infty$ therefore 0 is not in the interior of $K(t_1, t_0)$ for $t_1 > t_0$. This establishes that 0 must be in the interior of the domain of relative null controllability for the system (3.3) to be relatively null controllable.

The next theorem provides necessary and sufficient conditions for controllability to equilibrium state of system (2.1) vis a vis system (3.3).

Consider system (3.3) with its assumptions.

Theorem 3.3

Assume further that

(*i*) *system (3.4) is uniformly asymptotically stable so that there exist constants* α, k *such that every solution* $x(t)$ *satisfies*

$$\| x(t, t_0, \phi, 0) \| \leq k \| \phi \| e^{-\alpha(t - t_0)} \quad \text{for} \quad t > t_0$$

(*ii*) $f(t, x(t), x(t), x(t-h), 0, 0) = g(t, x(t), x(t), x(t-h)) + h(t, x(t), x(t), x(t-h))$

where $|g(t, x(t), \dot{x}(t), x(t-h))| \leq \pi(t) [\, |x(t)| + |\dot{x}(t)| + |x(t-h)| \,]$,

and $|h(t, \dot{x}(t), x(t), x(t-h))| \leq \epsilon[\, |x(t)| + |\dot{x}(t)| + |x(t-h)| \,]$. $\pi = \int_{0}^{\infty} \pi(t) dt$; $\epsilon > 0$

(*iii*) $f(t, t_0, \theta, 0, 0) = 0$ *then system (3.3) is relatively null controllable*

Proof

By conditions (*i*) and (*ii*) every solution of the free system, (i.e. when $u = 0$):

$$\dot{x}(t) = L(t, x_t)z_t + f(t, x(t), \dot{x}(t), x(t-h), 0, 0),$$

satisfies

$$x(t, t_0, \phi, 0, f) \to 0 \text{ as } t \to \infty. \tag{3.9}$$

By condition (iii), the system (3.3) admits trivial solution so that $0 \varepsilon D$, the domain of relative null controllability. At some finite time, say $t_2 > t_0$, there exists (by (3.9)) a ball $B_r(0)$ centre at 0 and radius r, such that $x(t_2, t_0, \phi, 0, f) \varepsilon B_r(0) \le \text{Int } D$. By definition of the domain of relative null controllability of system (3.3) there exists a control $u \varepsilon U$ such that for some $t_3 > t_2$, the solution $x(t_2, t_0, \phi, 0, f)$ satisfies $x(t_3, t_2, x(t_2, t_0, \phi, 0, f), u, f) = 0$. This establishes the relative null controllability of system (3.3). By theorem (3.2), 0 is in the interior of the domain of relative null controllability of system (3.3). Hence by definition (2.4), system (3.3) is controllable to equilibrium state.

Conclusion

From the sequel, the relative null controllability of system (2.1) vis a vis system (3.3) have been established. Also established is the relationship between relative null controllability of the systems and their domains of relative null controllability. This study has been able to show that if a system is relatively null controllable then zero is in the interior of the domain of relative null controllability. From the above results, the necessary and sufficient conditions for controlling systems (2.1) vis a vis system (3.3) to equilibrium have also been proved. They are a guarantee that the free parts of the systems under investigation have to be asymptotically stable in the large.

References

[1] Onwuatu, J.U. (1993) Null controllability of nonlinear infinite neutral system KYBERNETIKA (29)X

[2] Chukwu, E.N. (1992) Control of global economic growth. Will the centre hold, Ed. J. Weiner and J.K Hale) Longman Scientific and technical, 19-23.

[3] Chukwu, E.N. (1979) Euclidean controllability of linear delay systems with limited controls. IEEE Transactions on automatic control (AC24)5

[4] & Dauer, J.P. (1976) Non Linear perturbations of quasi – linear control systems Journal of Math. Analysis Applications (54)3, 717 – 725

[5] Olbrot, A. W. (1972) On the controllability of linear systems with time delays in control. I.E.E.E Trans. Automatic control,. 664 – 666.

[6] Hermes, H. and Lasalle, J.P. (1969) Functional Analysis and Time optimal control. New York: Academic Press

[7[Olbrot, A. W. and Pandolfi, L. (1988) Null controllability of a class of functional differential systems. International Journal of control (47)1, 193 – 208.

[8] Cooke, K. L. and Yorke, J.A. (1972) Equations modelling population growth , Economic growth and gonorrhea., epidemiology in ordinary differential equations (ed. L. Weiss) New York: Academic Press

[9] Takayama, A. (1994) Mathematical Economics, 2nd Edition Cambridge: University Press.

[10] Arrow, K.J. (1985) Production and Capital: Collected papers of Kenneth J. Arrow. The Beknop Prec. Cambridge:Harvard University Press.

[11] Manitius, A. (1975). Optimal control of Hereditary systems I.A.E.A – SM 17/95

[12] Chukwu, E.N. (1994) Mathematical Controllability theory of the growth of Wealth of Nations: The Japan Journal of Industrial and Applied Mathematics (11)1

[13] Iheagwam, V.A. (2000). On the Relative Controllability of perturbations of Nonlinear Functional Differential systems with Distributed Delays in Control Journal of Neg. Assoc. Math. Physics 4 .

[15] Hale J.K. (1977). Theory of Functional Differential Equations. New York: Springer Verlag