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A Class of (2m+1) Degree Polynomials Derived Via the Generalized Optimization of the Binding Energy per Nucleon, B

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Abstract

A Hilbert spaces generalized functions, $\chi_{m,n}(r)$, are identified for the variational method determination of the binding energy per nucleon, B, for all nuclei. In their application, the optimisation of B for all m,n=1,2,3..., results in the emergence of a class of (2m+1) degree polynomials, P_n^m, which promise to have significant physical implication. These polynomials were analysed with the use of the MATLAB.

pp 63 - 70

1.0 Introduction

In an earlier work [1], the generalized wavefunctions defined by

$$\chi_{m,n}(r) = Cr^m e^{-n\alpha r} \tag{1.1}$$

where $m,n \in$ K, and α is the Ritz variational parameter,
were introduced in the variational method determination of the binding energy per nucleon, B, for all nuclei. It will be shown later that $m \neq 0$; while a necessary condition for the physical acceptability of $\chi_{m,n}(r)$ is $n > 0$ - a condition which ensures the convergence of $\chi_{m,n}(r)$ as $r \to \infty$.

Furthermore, due to an impending complexity encountered in the above mentioned work [1], only the case for which m=n=1 was considered. However, in this work, a more general consideration of the problem is carried out and the optimisation of B with respect to α is done for all $m, n = 1,2,3,\ldots$.

An interesting outcome of this general approach is the emergence of a class of (2m+1) degree polynomials, P_n^m, that promise to have significant physical application. The MATLAB is used to perform a brief analysis of these polynomials.

2.0 A Hilbert Spaces Generalized Functions

Given the functions, $\chi_{m,n}(r)$, as defined by equation (1), a set ,$G(\infty)$, say, may be identified such that

$$\chi_{m,n}(r) = Cr^m e^{-n\alpha r} \in G(\infty) \tag{2.1}$$

where a required orthonormalization condition gives the normalization constant

$$C = \sqrt{\frac{(2n\alpha)^{2m+1}}{(2m)!}} \tag{2.2}$$

Since the functions, $\chi_{m,n}(r)$, have piecewise continuous second derivatives in r, are square integrable and vanish at infinity, it is therefore possible to construct a convergent series [2]

$$\psi(r) = \sum_{k=1}^{\infty} a_k \chi_k \in G(\infty) \tag{2.3}$$

where

$$\int_{-\infty}^{+\infty} |\chi_k|^2 < +\infty \tag{2.4}$$

with the coefficients, a_k, having their usual statistical interpretation, so that

$$|a_k|^2 = 1 \tag{2.5}$$

It can easily be shown [3] that $\chi_{m,n}(r) \in G(\infty)$ is a Hilbert spaces generalized functions; where

$$\chi_{m,n}(r) \in G(\infty) \subset S(R^n) \subset H \tag{2.6}$$

H being a Hilbert space and $S(R^n)$ is dense in *H*.

3.0 A Class of (2m+1) Degree Polynomials

The functions $\chi_{m,n}(r)$ can be used to calculate for the binding energy per nucleon, B, [1] via the variational method from the expression

$$B = \langle \chi_{m,n} | \hat{H} | \chi_{m,n} \rangle \tag{3.1}$$

where $\hat{H}$ is the Hamiltonian operator, and $\chi_{m,n}$ are normalized according to equation (2.1). For this purpose, the attractive Yukawa potential [4], which is essentially attractive everywhere, is so modified as to make the nucleus attractive only beyond a "critical" distance from its center, $r' \sim 10^{-15} m$; and that this attractive part, may be described by the Yukawa-like potential:

$$V\left(V_0, \frac{r_>}{a}\right) = -V_0 \frac{e^{-\frac{r_>}{a}}}{(r_>/a)} \tag{3.1}$$

where $r_> = r > r' \sim 10^{-15} m$; V_0 is the interaction strength, a constant, which differs from one nucleus to another, and the range of the nuclear force which is within the range [5]: $1.2x10^{-15} m \le a \le 2.0x10^{-15} m$

By considering the isobaric spin state T = 0, with $l = 0$ and S = 1, equation (3.1) will then lead to

$$B = \frac{n^2}{(2m+1)} \frac{\hbar^2 \alpha^2}{2\mu} - \frac{1}{2m} \frac{(2na\alpha)^{2m+1}}{(1+2na\alpha)^{2m}} \cdot V_0 \tag{3.2}$$

where the condition $m \neq 0$, is imposed so as to prevent equation (3.2) from going to infinity. Here μ is the nucleon-nucleon reduced mass and $\hbar$, the reduced Plank's constant. Minimizing B with respect to the Ritz variational parameter, α:

$$\frac{\partial B}{\partial \alpha} = 0 \tag{3.3}$$

results in an expression which can be used to obtain a class of polynomials that may not only be peculiar to this problem:

$$m\left(n\hbar^2\right)\left(\alpha + 2na\alpha^2\right)(1+2na\alpha)^{2m} - \\ -\left(4m^2 + 4m + 8m^2 na\alpha - 4mna\alpha + 1\right)\left(a\mu V_0\right)(2na\alpha)^{2m} = 0 \tag{3.4}$$

Due to the first term on the LHS of equation (3.4), this equation can be solved for α by applying the binomial theorem for positive integrals [6]; therefore for any given value of m = 1,2,3,.... Obviously, the degree of the resulting polynomial(s) will depend only on the value of m.

By expanding equation (3.4), it will be seen that these class of polynomials can be generated from:

$$\sum_{i=0}^{2m} \binom{2m}{i} (2na)^i \alpha^{i+1} + \sum_{i=0}^{2m} \binom{2m}{i} (2na)^{i-1} \alpha^i + \\ + \frac{4(m+1)\mu a V_0 (2na)^{2m}}{n\hbar^2} \alpha^{2m} - \frac{2(m-0.25m-1)\mu V_0 (2na)^{2m}}{n^2\hbar^2} \alpha^{2m-1} = 0 \tag{3.5}$$

with *m, n* = 1,2,3,....

These polynomials, P_n^m ,can, in general, be represented by

$$P_n^m = \alpha^{2m+1} + \frac{1}{n}A(V_o)\alpha^{2m} + \frac{1}{n^2}B(V_o)\alpha^{2m-1} + \frac{1}{n^3}C\alpha^{2m-2} + \frac{1}{n^4}D\alpha^{2m-3} + \ldots \quad (3.6)$$

All the coefficients, A,B,C,... depend on the value of m, while in addition, both A and B are dependent on V_o; V_o, on the other hand, is constant only for a given nucleus i.e. it is different for different nuclei.

With the use of the MATLAB and for the purpose of illustration, the first three of these (2m+1) degree polynomials, i.e. $P_n^{m=1,2\, and\, 3}$, are plotted as shown in Figures (1), (2) and (3) for $n = 1$ and V_o=65.9MeV (the interaction strength of the element Z=37).

Each of these polynomials, which are characterized by m, is seen to have 3 real roots and 2(m-1) imaginary roots. The order of the real roots, for a given value of n, increases by 1 as the value of m is increased by1, beginning with $\alpha \sim 10^{14} m^{-1}$ for m=1. It has been shown [1] that in the optimisation of B, with $m = n = 1$, the range of the physically acceptable α for all nuclei (with each nucleus having different V_o) is

$$5.0x10^{14}m^{-1} \le \alpha \le 9.0x10^{14}m^{-1} \quad (3.7)$$

Meanwhile, unlike in figure (1), figures (2) and (3), which display only the real roots, reveal that the polynomials are not represented by smooth curves for m ≥ 2 at small values of α (see inserts).

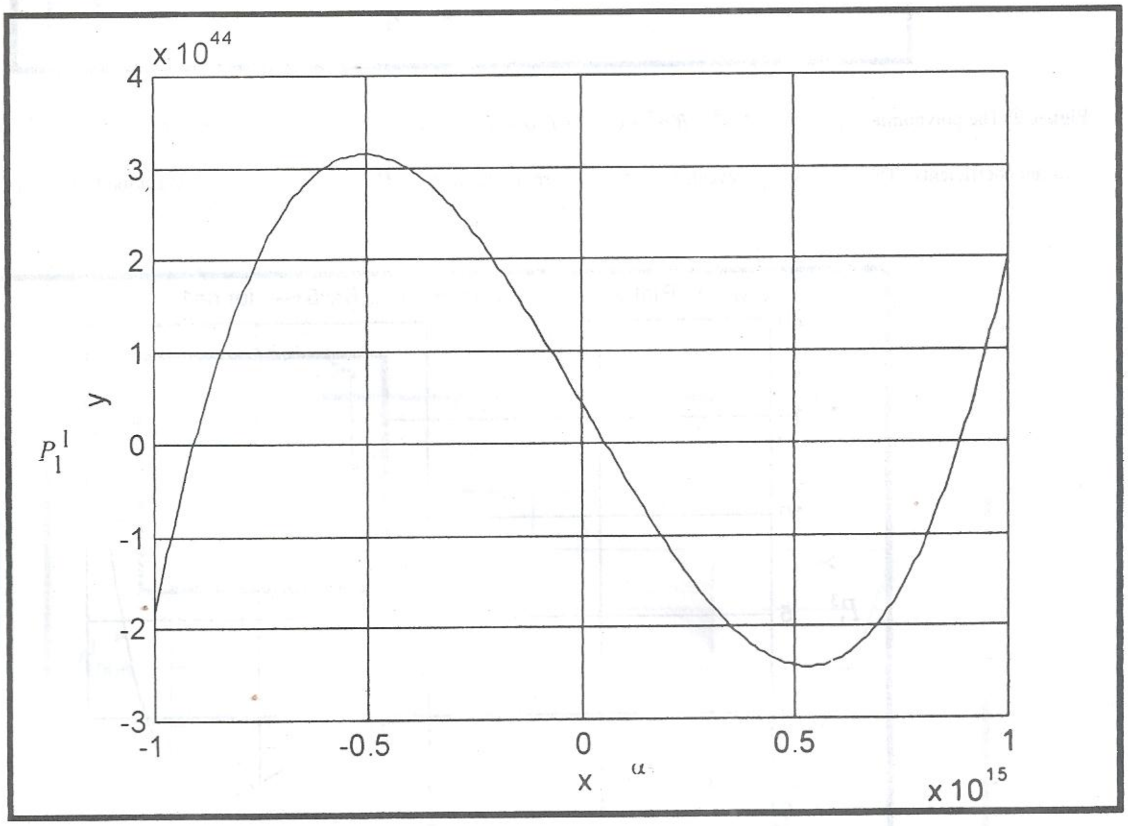

Figure 1: The polynomial $P_1^1 = \alpha^3 + A\alpha^2 + B\alpha + C$, where A B and C are constant coefficients, m=n=1 and V_o=65.9MeV

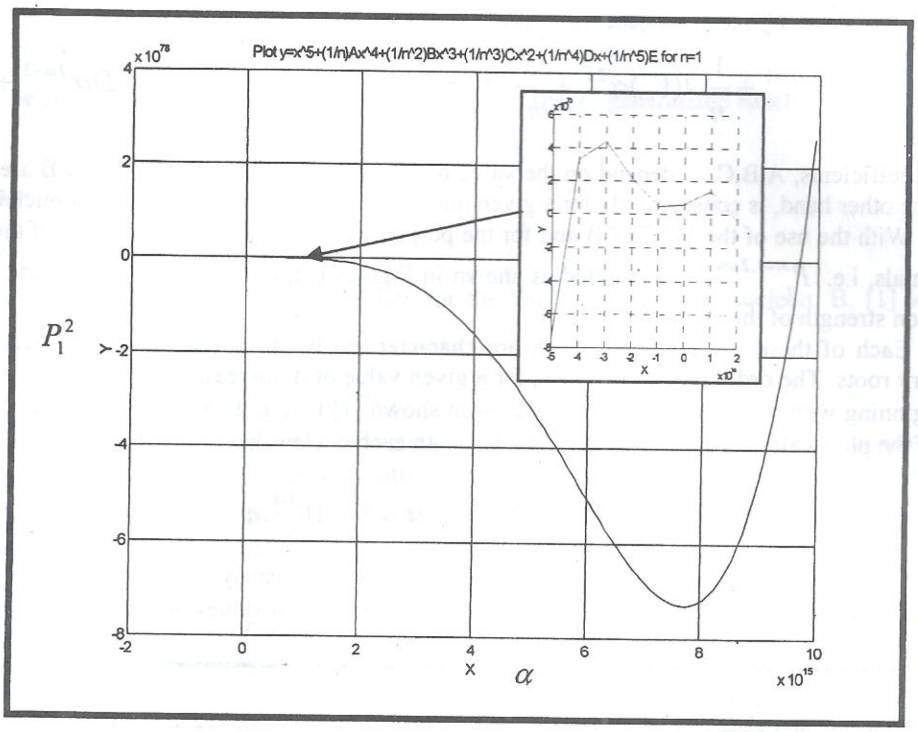

Figure 2: The polynomial $P_1^2 = \alpha^5 + A'\alpha^4 + B'\alpha^3 + C'\alpha^2 + D'\alpha + E'$, where m=2, n=1, Vo=65.9MeV and $A', B', C', D' and E'$ are constant coefficients. The insert, which reveals two of the real roots, shows that P_1^2 is not represented by a smooth curve at small values of α

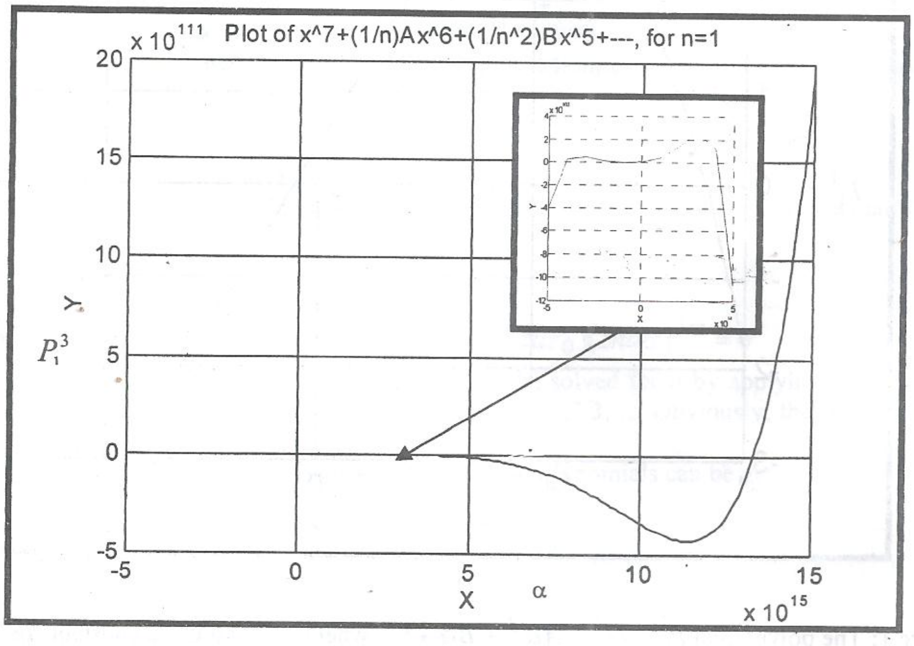

Figure 3: The polynomial $P_1^3 = \alpha^7 + A''\alpha^6 + B''\alpha^5 + C''\alpha^4 + D''\alpha^3 + E''\alpha^2 + F''\alpha + G''$ where A", B",...G" are constant coefficients, m=3, n=1 and V_o=65.9MeV. The insert, which reveals two of the real roots, shows that P_1^3 is not represented by a smooth curve at small values of α.

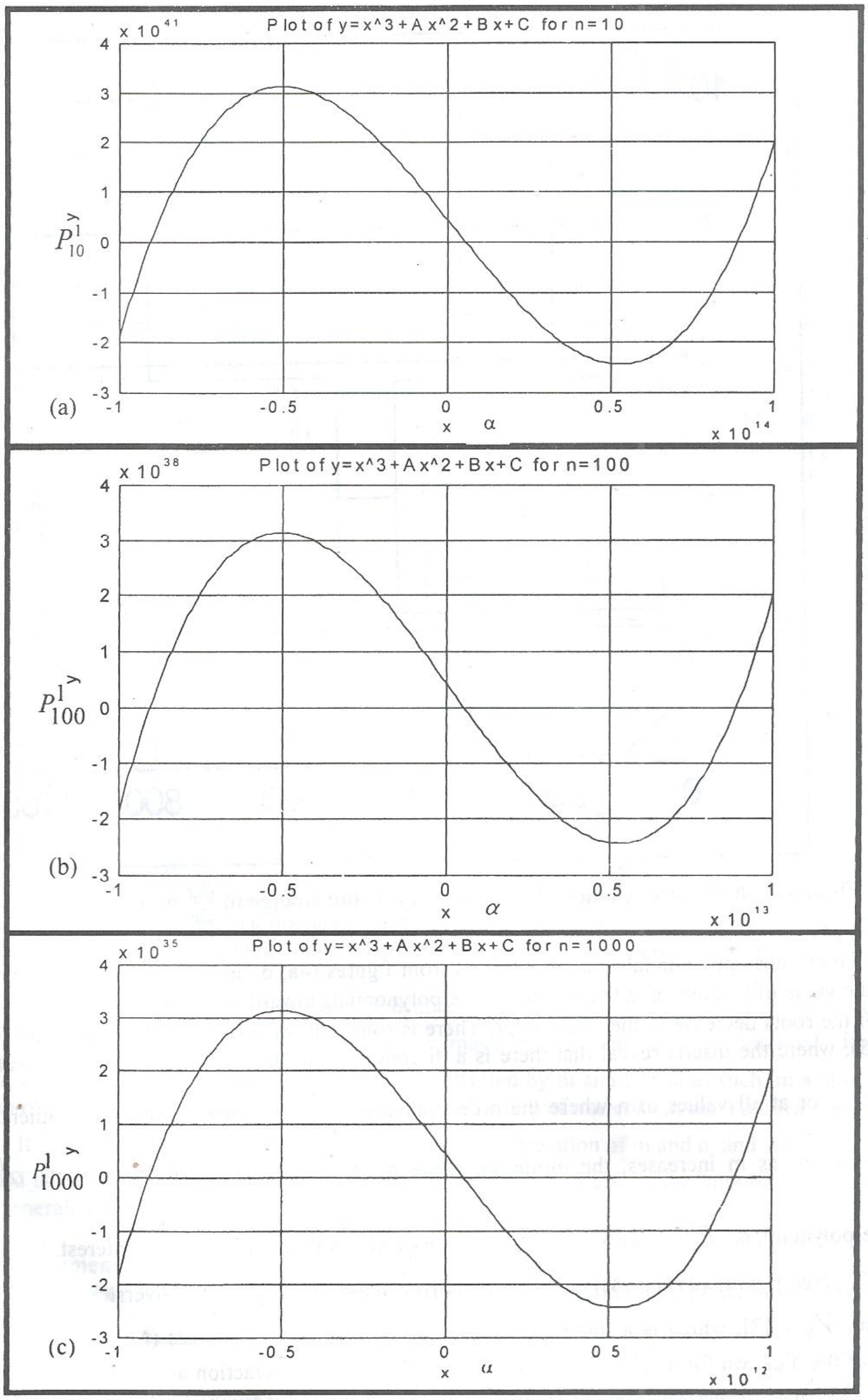

Figure 4: The shift-effect of n on P_n^m for a given m and V_o can be seen by comparing figures (a), (b) and (c).

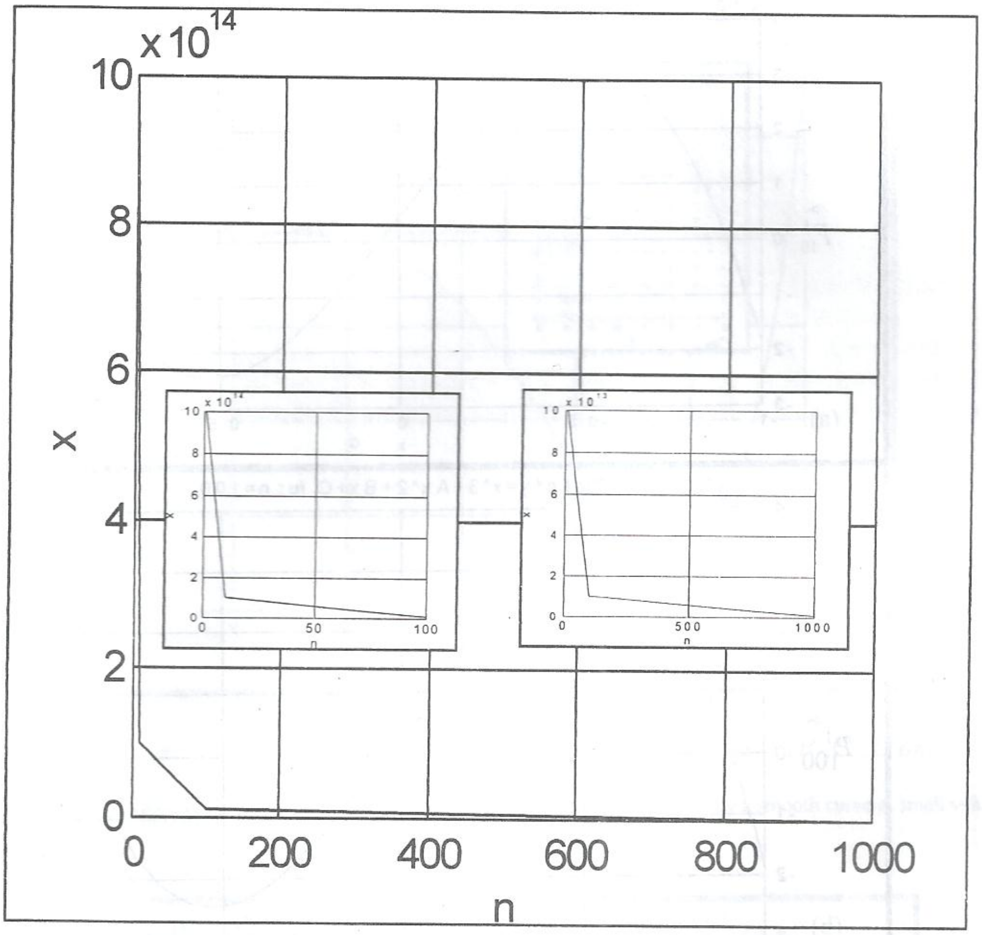

Figure 5: The inverse relaltion of α and n. Note the change in $\frac{\partial\alpha}{\partial n}$ at $n = 10,100\cdots$

The effect of n on these polynomials can be deduced from figures (4a, b, and c). For any given m and V_o, an increase in the value of n results in a lateral shift of the polynomials toward decreasing α; so that as n increases, the values of the roots decrease in the same order. There is some sort of inverse relationship between n and α, see figure (5), where the inserts reveal that there is a discontinuity in the linear variation at the values of n = 10,100,1000,..., or at all values of n where the order increases by 1. At these points, the gradients, $\frac{\partial\alpha}{\partial n}$, is reduced. Moreover, as m increases, the minimum value of P_n^m decreases while their roots $\alpha\left(P_n^m = 0\right)$, increases.

The polynomials, $P_n^{m\geq 2}$, with m≥2, can be viewed in a way that might be of interest. The large scale view of $P_n^{m\geq 2}$, (see figures (2) and (3)), suggests a "mirror image" (i.e the lateral inversions) of the Ried soft-core potential, V_{RSC} [7], which is a repulsive nuclear potential at short distances (Morse-like) with a radial dependence of the Yukawa form. This agrees with the sort of nuclear interaction assumed in the earlier part of this section. V_{RSC} has successfully applied in obtaining the energy levels of nuclear systems.

Furthermore, with the sort of inverse relationship established between n and α (Figure (5)), it should be expected, from the factor $e^{-n\alpha r}$ in $\chi_{m,n}(\vec{r})$, that r and α will also exhibit a kind of inverse law which could make $P_n^{m\geq 2}$ a "mirror image" of V_{RSC}; in which case, the minimum value of each $P_n^{m\geq 2}$ could have a one-to-one correspondence with V_o. Therefore, the strength of the nuclear force, V_o, may actually depend on m and n which determine the degree of $P_n^{m\geq 2}$ and scales their values respectively. And, as such, they could be related to the number of exchange particles responsible for the binding of nucleons in the Yukawa sense.

The roots of the polynomials p_1^1 obtained for the different values of V_o is used to evaluate B for all the elements in the periodic table, and the results obtained are in good agreement with experiment (see Table 1 below):

Table 1: Comparison of Calculated B, B_{cal}, With the Correspondig Experimental Values, B_{exp}.

Z	A	B_{exp}(MeV)	V_o(MeV)	$\alpha_{min}(x10^{14}m^{-1})$	B_{cal}(MeV)
1	2	1.1125	50.58	5.765	1.8
2	3	2.573	53.75	6.41	2.81
10	18	7.34	63.28	8.371	7.32
20	39	8.37	65.34	8.78	8.5
30	61	8.614	65.83	8.875	8.83
37	81	8.65	65.9	8.89	8.87
40	87	8.63	65.86	8.883	8.87
70	168	8.11	64.82	8.683	8.23
103	257	7.4	63.4	8.396	7.37

However, it is expected that, at the least, with an appropriate choice of n, any of these polynomials can be used to obtain B.

4.0 Summary/Conclusion

In this work, $\chi_{m,n(\vec{r})} = Cr^m e^{-n\alpha r} \in G(\infty)$, were presented as a Hilbert spaces generalized functions. These functions can be adequately applied in the determination of the binding energy per nucleon, B, of all nuclei via the variational method; where the nucleus was described by a Yukawa-like potential that is attractive only beyond a given range.

In the optimisation of B for all m,n=1,2,3,..., a class of (2m+1) degree polynomials was identified. These polynomials were then analysed with the use of the MATLAB.

While it was noticed that the degree of these polynomials depended only on m, it was concluded that n represents a scaling factor on their roots; where it had been demonstrated that these roots are essential in the calculation of B.

Though the complete physical implication of these polynomials is yet to be understood, the polynomials $P_n^{m\geq 2}$ were observed to be a mirror-image of the Reid soft-core potential, V_{RSC}, for nuclear systems. It was also argued that V_o may actually be affected by m and n; and as such, m and n could be related to the number of exchange particles responsible for the binding of nucleons in the Yukawa sense.

It would be of some interest if a clear physical interpretation of m and n, and therefore of P_n^m, as could be related to nuclear systems is arrived at, which, in turn, would illustrate the important role played by Hilbert spaces generalized functions.

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