Journal of the Nigerian Association of Mathematical Physics
Volume 7(2003)

Analysis of the nonlinear Diffusion Equation Associated with the Nonlinear Schr ö dinger Equation

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Abstract

We analyse numerically the Nonlinear Diffusion Equation (NLDE)

$$\frac{\partial\rho(x,\tau)}{\partial\tau}-D\frac{\partial^2\rho(x,\tau)}{\partial x^2}+\beta|\rho(x,\tau)|^p\rho(x,\tau)=0;$$

obtained from the Nonlinear Schr ö dinger Equation (NLSE). *The diffusion process is found to be ballistic. This ballistic bsehaviour increases with increase in both time* τ *and the nonlinear parameter* β*, indicating that the* NLDE *describes a quantum diffusion process. The nonlinear term is found to increase the rate of spreading or diffusion for* $\beta \succ 0$. *The dependence of the variance of the distribution function* $\rho(x,\tau)$ *on* β, τ *and p are determined.*

pp 47 - 52

1.0 Introduction

Given the linear Schr ö dinger Equation (LSE)

$$\frac{i\hbar\partial\psi(x,t)}{\partial t}=\left\{\frac{-\hbar^2}{2m}\nabla^2+V(x,t)\right\}\psi(x,t) \tag{1.1}$$

the diffusion process which corresponds to $|\psi(x,t)|^2$ is written down as the Fokker-Planck equation of a forward time evolution

$$\frac{\partial f(x,t)}{\partial t}=-\nabla[b(x,t)f(x,t)]+\frac{\hbar}{2m}\nabla^2 f(x,t) \tag{1.2}$$

where the drift term $b(x,t)$ is given through the solution of the LSE.

$$b(x,t)=\frac{\hbar}{m}\nabla(\mathrm{Im}+\mathrm{Re})In\psi(x,t) \tag{1.3}$$

and the diffusion constant is $\frac{h}{2m}$, Imafuku, Kentaro, et al (1995). The probability amplitude $|\psi(x,t)|^2$ corresponds to the propagator, Feynman et al (1965). In quantum statistical mechanics the partition function corresponds to the trace of the propagator. In Nelson's stochastic mechanic, each solution of the NLSE is associated to a diffusion process with the forward differential equation.

$$dq(t)=V_+(q(t),t)dt+dw(t) \tag{1.4}$$

or the equivalent backward equation

$$dq(t)=V_-(q(t),t)dt+dw^*(t) \tag{1.5}$$

where $w(t)$ and $w^*(t)$ are respectively forward and backward standard Wiener processes, while V_+ and V_- are drift fields that must be determined from the wave-function through the equality

$$V\pm(x,t)=\left(\nabla S\pm\nabla^2 R\right)(x,t) \qquad (1.6)$$

that is, as an extension of stochastic mechanics for a free particle, we can associate a diffusion process to any solution of the NLSE, so that the random variable "position" and "momentum" can be consistently defined. Tullio et al (1988). A. R. Kolovsky (1997) analysed the diffusion process in a physical system with Hamiltonian

$$H=\frac{p^2}{2m}-\frac{\hbar}{8}\Omega_{eff}\,f(t)\cos^2(kx) \qquad (1.7)$$

where, Ω_{eff} is a constant, and found to be ballistic. (Ballistic regime is the regime that the region of support of the wave function increases linearly with time.)

In this paper we associate a NLDE with the NLSE, in the same way that the linear diffusion equation (LDE) is associated with the LSE, and we analyse the diffusion process and the effect of the nonlinear term. (Njah, 2000). The rest of the paper is organized as follows: section 2 deals with the derivation of the NLDE (theoretical analysis). Section 3 is concerned with the methods of solution and results, section 4 discusses the results and section 5 concludes the paper.

2.0 Theoretical Analysis

The Linear Diffusion Equation (LDE) is written as

$$\frac{\partial\rho(x,t)}{\partial t}=D\frac{\partial^2\rho(x,t)}{\partial x^2}, \qquad (2.1)$$

where D = *diffusion constant* and the solution

$$\rho(x,t)=\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{x^2}{4Dt}} \qquad (2.2)$$

is called the distribution function, which gives a normal distribution if $2Dt=\sigma^2$ where the variance $\sigma^2=\langle x^2\rangle-\langle x\rangle^2$. The standard deviation $s=\sigma\sim\sqrt{t}$. The Linear Schrödinger Equation (LSE) for a free particle is

$$\frac{i\hbar\partial\psi}{\partial t}=\frac{-\hbar^2\partial^2\psi}{2m\partial x^2} \qquad (2.3)$$

If $t\to i\tau$ and $D=\frac{\hbar}{2m}$, equation (2.3) becomes

$$\frac{\partial\psi}{\partial\tau}=D\frac{\partial^2\psi}{\partial x^2} \qquad (2.4)$$

Equation (2.4) is the LDE obtained from the LSE for a free particle.

For a non free particle in a potential $V(x,t)$ the LSE is equation (1.1). By replacing the potential $V(x,t)$ om the LSE, equation (1.1) with $|\psi(x,t)|^P$ we have the NLSE

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2\partial\psi}{2m\partial x^2}+\beta|\psi|^P\psi \qquad (2.5)$$

If $t\to i\tau$ and if we replace $\psi(x,t)$ by the classical probability density $\rho(x,\tau)$ equation (2.5) gives the NLDE

$$\frac{\partial\rho}{\partial\tau}-D\frac{\partial^2\rho}{\partial x^2}+\beta|\rho|^P\rho=0 \qquad (2.6)$$

where $D=\frac{\hbar^2}{2m}$ is a diffusion parameter.

3.0 Methods of solution and results

By choosing $D = 1$, $\beta = 2$, $p = 2$ (to correspond to the standard NLSE) and using equation (2.2) (at $\tau = 0 \cdot 1$) as the initial distribution equation (2.6) was solved numerically using the finite difference method. The result obtained are shown in Figure 1 for different values of β. The expectation of x^n is given by

$$\left\langle x^n \right\rangle = \int_{-\infty}^{\infty} x^n \rho(x) dx \tag{3.1}$$

By plotting $x^n \rho(x)$ against x for the case $n = 1, 2$, we obtained the curves in Figure 2, as the final distributions at $\tau = 0 \cdot 4$. Using numerical integration (the trapezoidal rule) we integrated over each curve, that is, we integrated equation (3.1) for $n = 1, 2$, at $\tau = 0 \cdot 4$. This procedure was repeated for various values of τ and the corresponding values of the variance σ^2 were calculated from $\sigma^2 = \left\langle x^2 \right\rangle - \left\langle x \right\rangle^2$. The dependence σ^2 of τ for fixed D, fixed β is shown in Figure 3. Also the dependence of σ^2 on β for fixed τ, fixed D is shown in Figure 4, while Figure 5 shows the dependence of σ^2 on p.

4.0 Discussion of Results

The variance σ^2 of the distribution function $\rho(x,\tau)$ increases with τ as in the case of the LDE (that is, case $\beta = 0$). σ^2 decreases as p increases and tend to σ^2 for the LDE, Figure 5. This implies that the effect of the nonlinear term decreases as p increases, meaning that $\rho(x,\tau)$ is small so that $|\rho|^p \prec\prec 1$ for $p \succ\succ 1$. On the other hand, σ^2 increases as β increases Figure 4. This implies that β increases the rate of spreading. The maximum of $\rho(x)$ is found to depend on β as follows: the more negative [positive] the value of β the greater [smaller] the maximum, Figure 1. As τ increases further the diffusion process becomes ballistic. This ballistic behaviour becomes more [less] pronounced as β becomes more positive [negative]. This implies that the NLDE describes a quantum diffusion process. For quantum dynamics the distribution function does not keep its initial Gaussian shape during time evolution but develops an interference pattern (Kolovsky, et al 1994, 1997). The interference pattern is also observed for the NLDE as time increases, Figure 6. It is also found that if time is fixed this interference pattern can also develop as β increases. Hence β enhances quantum behaviour. The above observations are similar to those discussed by Kolovsky (1997) using equation (1.7).

5.0 Conclusion

The following conclusions can be drawn from the analysis:

- the NLDE associated with the NLSE describes a quantum mechanical diffusion process.
- the NLDE associated with the NLSE describes the diffusion of systems with small distribution functions.
- the nonlinear term accelerates the rate of spreading or diffusion and enhances quantum behaviour.

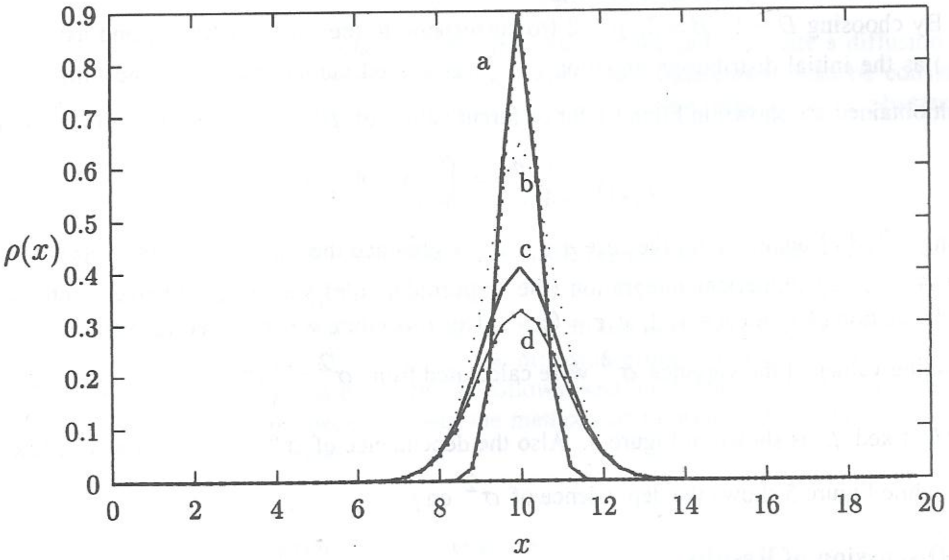

Figure 1: (a) Initial distribution (equation 2.2 with D = 1, t = 0.1); (b), (c), (d), final distribution (equation 2.6) at $\tau = 0 \cdot 4$ for $\beta = -3, 0, 3$ respectively.

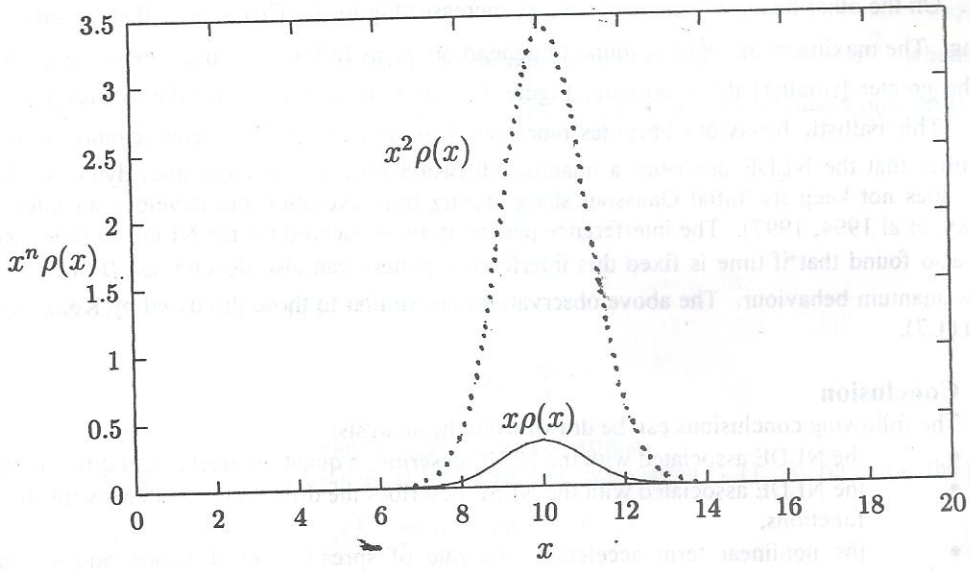

Figure 2: The final distribution of $x^n \rho(x)$, $n = 1, 2$ at $\tau = 0 \cdot 4$

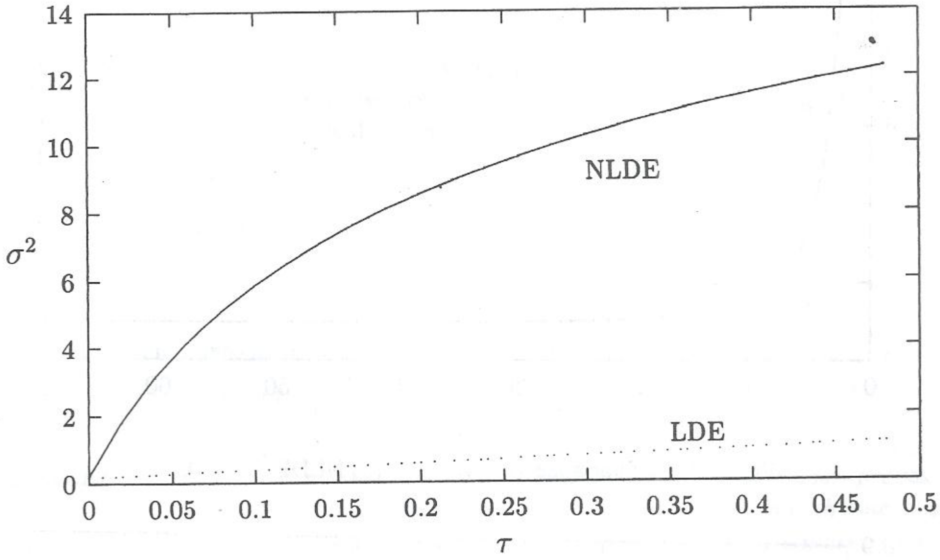

Figure 3: The variance σ^2 versus τ for the LDE and NLDE

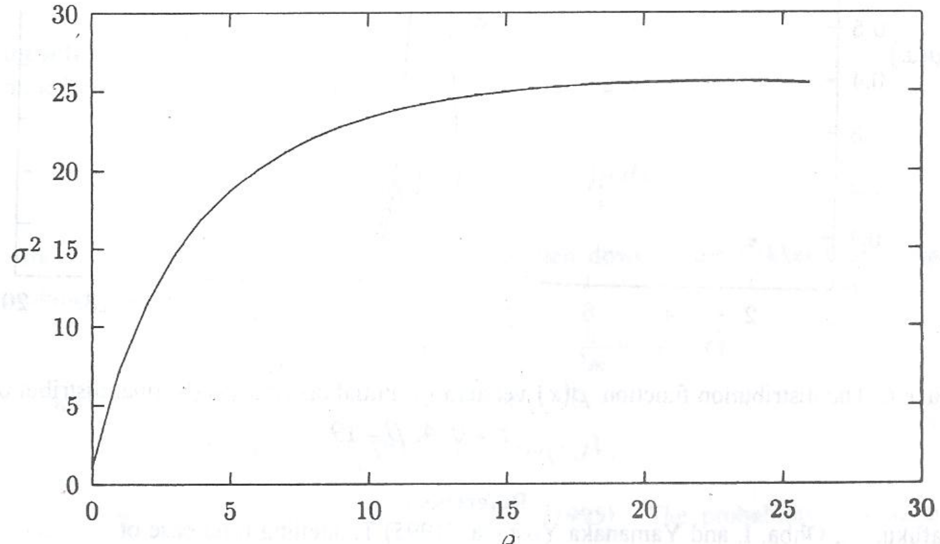

Figure 4: The variance σ^2 versus β for the NLDE

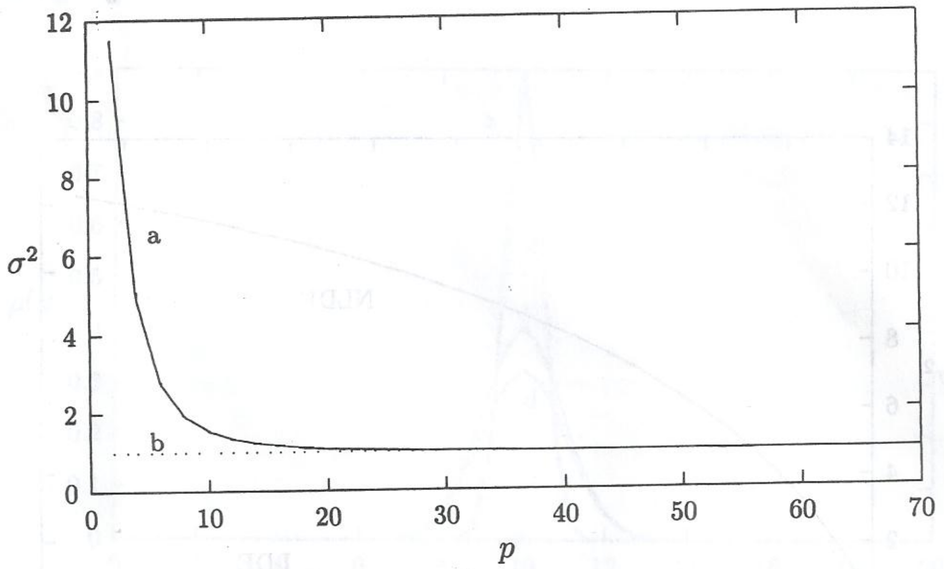

Figure 5: The variance σ^2 versus p (a) NLDE (b) LDE

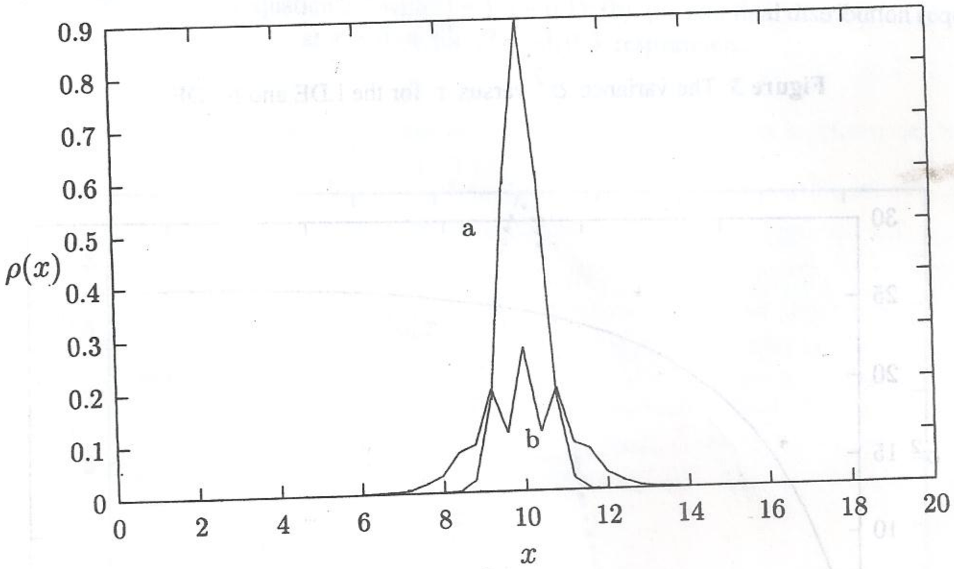

Figure 6: The distribution function $\rho(x)$ versus x (a) initial distribution (b) final distribution at $\tau = 0\cdot 4,\ \beta = 19$

References

[1] Imafuku, K., Ohba, I. and Yamanaka Yoshiya, (1995) Tunnelling time base on the quantum diffusion process approach. Physics letters A, 204(5, 6), 329-335

[2] Feynman, R. P. and Hibbs, A. R. (1965) Quantum Mechanics and Path Integrals. London: McGraw Hill Inc.

[3] Minelli, Tullio A. and Morato Laura, M. (1988) Position-momentum join probability densities for the nonlinear Schrödinger Equation and generating functions of the conserved quantities. Physics letters A, 130(4, 5), 283-8.

[4] Kolovsky, A. R. (1997) Chaotic diffusion and ballistic transport of ultra-cooled atoms in a time-modulated standing laser wave. Physics letters A, 231, 144-148.

[5] Njah, A. N. (2000) The nonlinear Schrödinger equation in bounded space and its applications to telecommunication and statistical mechanics. PhD Thesis, Department of Physics, University of Ibadan.