A Method of Domain Decomposition for Solution of the Helmholtz Equation

Felix Okoro*
Department of computer science,
Ambrose Alli University, Ekpoma, Nigeria.
and
Samuel Azi
Department of Physics
Ambrose Alli University, Ekpoma, Nigeria.

Abstract:

Domain decomposition methods for the solution of partial differential equations are attractive on parallel computers. In this work, a discretization of the Helmholtz equation is reduced to sub domains and separators sets [1]. A grid point numbering makes the coefficient matrix take the block arrow head matrix form. The Cholesky decomposition of the form $L^{-1}A\ L^{-T}$ is applied to the block matrices. A computational procedure using sequential computer structure is applied to accelerate convergence. The diagonal block system is solved via the Conjugate Gradient Method (CGM). A test problem comparing the method with analytical solution is presented

Keywords: Helmholtz equation, domain decomposition, conjugate gradient method, Cholesky decomposition.

pp 33 - 38

1.0 Introduction

Various iterative methods can be classified as domain decomposition methods. Domain decomposition methods for the solution of partial differential equation are attractive on parallel processors because each processor can work independently on a large subtask. The corresponding stiffness matrix takes a sparse block structure. A popular method for solving domain decomposition with non-overlapping domain is the Schur complement method named after a German mathematician. For overlapping sub domains, the Schwarz iteration is typically used. The choice of sub domain can be influenced by different motivations. One can decompose complex domains into disjoint rectangles. In some problems, it is possible to split the complete problem in a natural way into disjoint sub problems. An example is when the sub domains model physically different materials, like wave propagation in a domain consisting of several layers which could be different [2]. In parallel computing, users are looking for decomposition of entire problem into separate sub problems which should be equally sized because of the load balancing of processors [3]. The solution of the entire problem entails a solution of the sub problems and also a solution of the coupling of the sub problems.
This paper includes three additional sections. The next section introduces the Helmholtz equation and our method of domain decomposition. Section 3 reports numerical comparison of a test example. Concluding remarks are given in section 4.

2.0 The Problem:

We consider the Helmholtz equation of the form

$$P_{xx} + P_{yy} + \lambda P = R(x, y) \qquad (2.1)$$

on a rectangular domain Ω [0,1] x [0,1]. A discretization of (1) gives

$$-P_{i+1,j} - P_{i-1,j} - P_{i,j+1} - P_{i,j-1} + [4 + \lambda h^2]\, P_{ij} = -h^2 R(x, y) \qquad (2.2)$$

*Correspondence Author

A dirichlet boundary condition is assumed. We divide the unknown into 2q-1 sets, $D_1, D_2, \ldots D_q$ and $S_1, \ldots S_{q-1}$ such that each D_i has q unknowns and the unknowns in the S_i 'separate' the unknowns in the D_i. The unknowns are lined up and partitioned as

$$D_1, S_1, D_2, S_2, \ldots, S_{q-1}, D_q \qquad (2.3)$$

The set S_i separate the unknown in the D_i so that each equation in the system contains unknowns in only one D_i. If q = 5, our grid point numbering of (3) is of the form

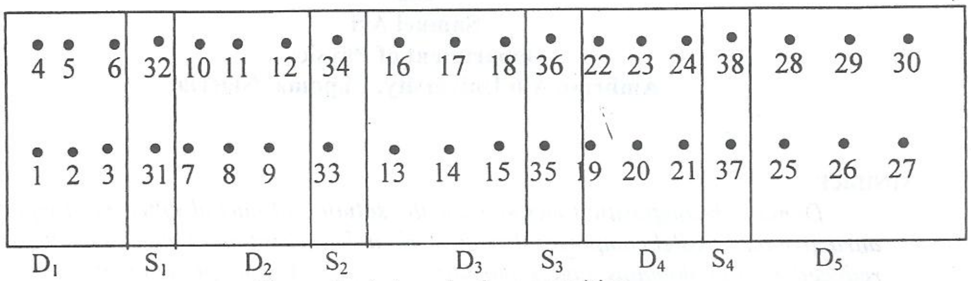

Figure 1: A domain decomposition

For this q, we have 9 sets and the system becomes a 38 x 38 matrix with the D_i = 5 blocks of 6 x 6 matrices and the S_i = 4 blocks of 2 x 2 matrices. We can write the equation in the corresponding order to get a system in the block arrow head of the form

$$\begin{bmatrix} D_1 & & & & & B_1 \\ & D_2 & & & & B_2 \\ & & D_3 & & & B_3 \\ & & & D_4 & & B_4 \\ & & & & D_5 & B_5 \\ C_1 & C_2 & C_3 & C_4 & C_5 & A_S \end{bmatrix} \qquad (2.4)$$

$$D_i = \begin{bmatrix} * & -1 & & -1 & . & . \\ -1 & * & -1 & & -1 & \\ & -1 & * & & & -1 \\ -1 & & & * & -1 & \\ & -1 & & -1 & * & -1 \\ & & -1 & & -1 & * \end{bmatrix} \qquad (2.5)$$

where $* = [4+\lambda h^2]$ and $S_i = \begin{bmatrix} * & -1 \\ -1 & * \end{bmatrix}$.

A_s is the block of separator sets,

$$C_I = B_i^T = \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$ Now, let $A_I = diag\ (D_1, \ldots\ldots, D_q)$, $B^T = (B_1^T, \ldots\ldots, B_q^T)$ and

$$C = (C_1, \ldots\ldots\ C_q). \qquad (2.6)$$

The system (2.6) can be written as

$$A_I X_I + B X_S = b_I \tag{2.7a}$$

$$C X_I + A_S X_S = b_S \tag{2.7b}$$

where $X_I^I = (X_1^T, X_2^T, \ldots\ldots, X_q^T)$ and $b_I^T = (b_1^T, \ldots\ldots b_q^T)$.

We assume that A_I is non-singular. Then if we multiply (2.7a) by $C A^{-1}$ and subtract from (2.7b), we obtain the equation

$$\bar{A} X_S = b \tag{2.8}$$

where
$$\bar{A} = A_s - C A_I^{-1} B, \tag{2.9a}$$

and
$$\bar{b} = b_s - C A_I^{-1} b_I \tag{2.9b}$$

$\bar{A}$ is the Schur complement (Reduced Matrix System). Cholesky Decomposition [4]

$$A_i = L_i^{-1} L_i^{-T}, \quad i = 1, 2, \ldots\ldots, q, \tag{2.10}$$

and since $C_i = B_i^T$ by symmetry, we have from (2.9)

$$\bar{A} = A_s - \sum_{i=1}^{q} B_i^T A_i^{-1} B \tag{2.11a}$$

$$\bar{b} = b_s - \sum_{i=1}^{q} B_i^T A_i^{-1} b \tag{2.11b}$$

If we write $B_i^T A_i^{-1} B_i = B_i^T (L_i L_i^T)^{-1} B_i = Y_i^{-1} Y_i$, where $Y_i = L_i^{-1} B_i$, and with $y_i = L_i^{-1} b_i$

$$\bar{A} = As - \sum_{i=1}^{q} Y_i^{-1} Y_i \tag{2.12}$$

$$\bar{b} = b_i - \sum_{i=1}^{q} Y_i^{-1} Y_i$$

From (2.4),

$$Y_i = L^{-1} B = \begin{bmatrix} A_1 & & & & & \\ & A_2 & & & & \\ & & A_3 & & & \\ & & & A_4 & & \\ & & & & A_5 & \\ -B_1 A_1^{-1} A_5^{-1} & -B_2 A^{-1}{}_2 A^{-1}{}_5 & -B_3 A_3^{-1} A^{-1}{}_5 & -B_4 A_4^{-1} A_5^{-1} & -B_5 A_5^{-1} A_5 & A_5 \end{bmatrix} \tag{2.13}$$

$$y_i^T Y_i = \begin{bmatrix} A_1^{-1} & & & & & \\ & A_2^{-1} & & & & \\ & & A_3^{-1} & & & \\ & & & A_4^{-1} & & \\ & & & & A_5^{-1} & \\ & & & & & A_S^{-1} \end{bmatrix} \tag{2.14}$$

Details of this method are given in the appendix.

3.0 Numerical Example

To test the performance of the method a problem has been chosen to verify it with the analytical result. We consider the Helmholtz equation with $\lambda = 11$, $P_{xx} + P_{yy} - 11P = R(x, y)$ on the unit square $[0, 1] \times [0, 1]$, $R(x, y) = 11e^{x}Sin[5]$, and the dirichlet boundary conditions on each of the sub domains:

$$P_{i,0} = P_{0j} = 0, \qquad P_{N+1,j} = P_{i,N+1} = 1.$$

$N = \frac{1}{h}$ is the number of grid point on each x and y dimension and h is the mesh size. From (2.2), $h = \frac{1}{N}$, and $(4+\lambda h^2) = \frac{155}{36}$.

Our test was conducted with q equals to the following 2, 3, 4, 5.

Table 1: solving $P_{xx} + P_{yy} - 11P = R$ on the unit square using sub domains and separators with $P = e^{x} \sin(y)$.

N	14	22	30	38
Maximal Error	4.22E-01	7.30E – 02	4.67E – 03	2.96E - 03
Relative Error	1.10E – 0.1	2.57E – 02	1.58E – 03	9.81E - 04

The true relative error is defined as Relative error = $\dfrac{\left\| P - \hat{P} \right\|_1}{\|P\|_1}$, where p and $\hat{P}$ denote the exact solution and computed solution respectively and $\|\cdot\|_1$ is the L_1 norm.

4.0 Concluding Remarks

The need to compute accurate solution to the Helmholtz equation is of interest in scientific computing. Parallel processors are attractive for solving domain decomposition problems because each processor can work independently on a sub task. Serial algorithm has been used in our approach. Using simple sub domain and separator sets, we have successfully solved the Helmholtz equation. This principle can be applied to more complex problems: The test example reflects the effectiveness of the method. However, the result depends largely on the implementation of the algorithm.

APPENDIX

From (2) and for q =3, the coefficient matrix gives the form:

$$
\left|\begin{array}{cccccccccccccccccccccc}
\ast & -1 & & -1 & & & & & & & & & & & & & & & & & & \\
-1 & \ast & -1 & & -1 & & & & & & & & & & & & & & & & & \\
 & -1 & \ast & & & -1 & & & & & & & & & & & & & & -1 & & \\
-1 & & & \ast & -1 & & & & & & & & & & & & & & & & & \\
 & -1 & & -1 & \ast & -1 & & & & & & & & & & & & & & & & \\
 & & -1 & & -1 & \ast & & & & & & & & & & & & & & & -1 & \\
 & & & & & & \ast & -1 & & -1 & & & & & & & & & & -1 & & \\
 & & & & & & -1 & \ast & -1 & & -1 & & & & & & & & & & & \\
 & & & & & & & -1 & \ast & & & -1 & & & & & & & & & & \\
 & & & & & & -1 & & & \ast & -1 & & & & & & & & & & -1 & \\
 & & & & & & & -1 & & -1 & \ast & -1 & & & & & & & & -1 & & \\
 & & & & & & & & -1 & & -1 & \ast & & & & & & & & & & \\
 & & & & & & & & & & & & \ast & -1 & & -1 & & & & & & -1 \\
 & & & & & & & & & & & & -1 & \ast & -1 & & -1 & & & & -1 & \\
 & & & & & & & & & & & & & -1 & \ast & & & -1 & & & & \\
 & & & & & & & & & & & & -1 & & & \ast & -1 & & & & & -1 \\
 & & -1 & & & & -1 & & & & & & & -1 & & -1 & \ast & -1 & & & & \\
 & & & & & -1 & & & & & & & & & -1 & & -1 & \ast & & & & \\
 & & & & & & & & & -1 & & & & & & & & & \ast & -1 & & \\
 & & & & & & & & & & & & -1 & & & & & & -1 & \ast & & \\
 & & & & & & & & & & & & & & & & & & & & \ast & -1 \\
 & & & & & & & & & & & & & & & -1 & & & & & -1 & \ast
\end{array}\right|
$$

Figure 2: A Domain Decomposition Matrix.

From (14), each of the A_i^1, i = 1, 2 ,3 gives the same entries. For the first sub domain, we have

$$
\frac{1}{*^6 - 7*^4 + 7*^2 - 1}\begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{16} \\ a_{21} & & & \\ \vdots & & & \\ \vdots & & & \\ a_{61} & \cdots\cdots & \cdots\cdots & a_{66} \end{bmatrix}.
$$

where

$$a_{11} = a_{33} = a_{44} = a_{66} = *^5 - 5*^3 + 2*$$

$$a_{22} = a_{55} = *\left(*^4 - 1\right)$$

$$a_{12} = a_{21} = a_{23} = a_{32} = a_{45} = a_{54} = a_{56} = a_{65} = *^4 - 2*^2 + 1$$

$$a_{13} = a_{31} = a_{46} = a_{64} = *(*^2 + 1)$$

$$a_{14} = a_{41} = a_{36} = a_{63} = *^2(3 - *^2)$$

$$a_{15} = a_{15} = a_{24} = a_{42} = a_{35} = a_{53} = a_{26} = a_{63} = 2*(1 - *^2)$$

$$a_{16} = a_{61} = a_{34} = a_{43} = 1 - 3*^2$$

$$a_{25} = a_{52} = 1 - *^4$$

$$* = \frac{155}{36}$$

and

$$A_S^{-1} = \frac{1}{x^2 - 1}\begin{bmatrix} * & 1 & & \\ 1 & * & & \\ & & * & 1 \\ & & 1 & * \end{bmatrix}.$$

It is easy to see that Matrices A_I^{-1} and A_S^{-1} are reducible. With a permutation Matrix, a further decomposition gives

$$A_I^{-1} = *^6 - 7*^4 \cdot 7*^2 - 1$$

and

$$A_S$$

We use these Matrices as our CGM control operat

References

[1] Gene Golub. and Ortega M. James. domain decomposition reordering Scientific computing – An introduction with parallel computing, Academic Press. Inc.. (1993). pp. 228 – 229.

[2] Elizabeth Larsson. and Sverker Holmgren. parallel solution of the Helmholtz equation in a Multi layer domain. BIT. Vol 30. No. 4. (2000). pp. 001 – 604.

[3] Larsson. E.. A domain decomposition method for the Helmholtz equation in a multiplayer domain. SIAM J. Sci. Comput. , 20. (1990). pp. 1713 – 1731.

[4] Kershaw. D.S. , The incomplete Cholesky conjugate gradient method for the iterative solution of systems of linear equations. J. Comp phys. Vol 26. (1978). pp. 46 – 65.

[5] Xian – He Sun. and Yu Zhuang. High-order direct solved for Helmholtz equation with Neumann Boundary conditions, Parallel Computing. (1995). pp. 1211 – 1267.