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A Cauchy Problem for a Quasi-Linear Hyperbolic First Order Partial Differential Equation

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Abstract

A quasi-linear hyperbolic partial differential equation in a Banach space is investigated using the technique of converting such problems to those that look like the abstract Cauchy problem. It is then shown that this Cauchy problem is m-accretive and thus admits solutions.

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1.0 Introduction

A general abstract method for studying the Cauchy problem was developed by Segal (1963), which consists of recasting the problem in the form of an integral equation and solving that equation by a fixed point problem method. A less general method is proposed by Browder (1976). In any result for the abstract Cauchy problem general enough to cover examples from partial differential equations.one can worry a little about the appropriate notion of solution. Ginibre and Velo (1986) considered the Cauchy problem for non-linear Schrodinger equation and the non-linear Klein-Gordon equation and proved the existence of global weak solutions by compactness method using the energy estimates. Ben-Artz and Devinatz (1998), studied the long time behavior and regularity of solutions for the stark equation and showed that for a class of short range potentials, the gain of smoothness and the decay as time approaches infinity are close to those of the corresponding Schrodinger equation. Avgerinos (1996) considered the Cauchy problem for a parabolic distributed control system with feedback control constraint and proved the existence of a periodic trajectory. In this paper we shall consider a quasi-linear hyperbolic partial differential equation defined in a Banach space and show that it has a solution.

2.0 Some Preliminaries

Let X be a real Banach space with norm $\|.\|$ and dual X^*. An operator with domain D(A) and range R(A) in X is said to be accretive(Browder, 1967) if for all $x_1, x_2 \in D(A)$ and r > 0 there holds the inequality

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + r(Ax_1 - Ax_2)\| \qquad (2.1)$$

Also from Kato (1976), A is accretive if and only if $x_1, x_2 \in D(A)$ there is $j \in J(x_1 - x_2)$ such that $\langle Ax_1 - Ax_2, j\rangle \geq 0$ where $J(x) = \left\{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\}, x \in X$ (2.2) is the normalized duality mapping of X and $\langle .,. \rangle$ denotes the duality pairing between X and X^*. If X^* is uniformly convex, the duality mapping is uniformly continuous on bounded subsets of X^*.(Barbu;1976,Browder;1967,Kato;1976). An accretive operator A is said to be m-accretive if R(I + rA) = X for all r > 0, where I is the identity operator of X.

In terms of the concept of contractions, an operator A is said to be accretive if $(I+rA)^{-1}$ is a contraction for $r \geq 0$, that is if

$$\|(x_1+rAx_1)-(x_2+rAx_2)\| \geq \|x_1-x_2\| \tag{2.3}$$

If X is a Hilbert space, the accretive condition (2.1) reduces to

$$\mathrm{Re}\langle Ax_1 - Ax_2, x_1 - x_2 \rangle \geq 0 \; for\ all\ x_1, x_2 \in X \tag{2.4}$$

2.1 **Lemma**

Let X be a real Banach space. Let $A:D(A) \subset X \to X$ be an m-accretive operator such that $u=(A+n^{-1}I)x_1$, $v=(A+n^{-1}I)x_2$ where $u,v \in X; x_1,x_2 \in D(A); n \in R^+$. Then the operator $(A+n^{-1}I): X \to D(A)$ is continuous and bounded.

Proof

Given $u,v \in X; x_1,x_2 \in D(A); n \in R^+$ as defined above we have

$$\left\langle (A+n^{-1}I)x_1 - (A+n^{-1}I)x_2, j \right\rangle = \frac{1}{n}\|x_1 - x_2\|^2 \text{ for some } j \in J(x_1 - x_2).$$

This implies that

$\left\|(A+n^{-1}I)^{-1}u - (A+n^{-1}I)^{-1}v\right\| \leq n\|u-v\|$, which shows the continuity of $(A+n^{-1}I)^{-1}$. Now let $v=0, x_0 \in D(A)$ with $0=(A+n^{-1}I)x_0$ in the above equation, and then we have

$\left\|(A+n^{-1}I)^{-1}u\right\| \leq n(\|u\|+\|x_0\|)$. This proves the boundedness of the operator $(A+n^{-1}I)^{-1}$.

3.0 A Particular Example

We shall now consider a quasi-linear hyperbolic partial differential equation given by

$$\begin{aligned} &u_t + (f(u))_x = 0, \; 0<x<1 \\ &u(0,x) = u_0(x), 0<x<1 \\ &u(t,0) = 0, t>0 \end{aligned} \tag{3.1}$$

which we shall transform into an initial value problem

$$\frac{du}{dt} + Au = 0, u(0) = u_0 \tag{3.2}$$

defined in a Banach space $X = L^1[0,1]$, assuming *f* is continuous, strictly increasing, *f*(0) = 0 and *f*(**R**) = **R** and show that A is m-accretive.

A fundamental result, due to Browder (1976) in the theory of accretive operators states that the initial value problem as defined by (6) is solvable if A is a locally Lipschitzian and accretive operator on X. The reader is referred to Browder (1976) and Barbu (1976) for more details of the theory of accretive operators. Browder (1967) also proved that if $A: X \to X$ is locally Lipschitzian and accretive then A is m-accretive. This result of Browder (1967) was subsequently generalized by Martins (1970) to continuous accretive operators.

3.1 **Theorem**

Let $X = L^1([0,1]), D(A) = \{u \in C[0,1]: u(0) = 0, f(u) \text{ is Lipschitz continuous on } [0,1]\}$ *and* $Au = (f(u))', u \in D(A)$ *; then A is m-accretive.*

3.2 **Remarks**

The operator A acts in accordance with the partial differential equation and D(A) consists of functions satisfying the boundary conditions.

3.3 **Proof of Theorem (3.2)**

We shall first show that the operator A is accretive. Since *f* is Lipschitz on [0,1] then *f* is absolutely continuous on [0,1]. Let $g: R \to R$ such that g is Lipschitz continuous, non-decreasing with zero kernel and

$$|g| \le 1 \text{ and } \varphi(s) = \int_0^s g(t)dt$$. Then if $u, v \in D(A) \text{ and } r > 0$

we have

$$\int_0^1 (Au - Av)g(f(u) - f(v))dx = \int_0^1 (f(u) - f(v))' g(f(u) - f(v))dx \tag{3.3}$$

$$= \int_0^1 \varphi(f(u) - f(v))' dx = \varphi(f(u(1) - f(v(1)) \ge 0$$

We also observe that

$$(u - v)g(f(u) - f(v)) = |u - v||g(f(u) - f(v))| \tag{3.4}$$

so that by the assumption on g and equation (3.3) above we obtain

$$\int_0^1 |(u - v) + r(Au - Av)|dx \ge \int_0^1 |((u - v)g(f(u) - f(v)) + r(Au - Av)g(f(u) - f(v))|dx \tag{3.5}$$

$$\ge \int_0^1 |u - v||g(f(u) - f(v))|dx$$

If we set $g = g_n$ in equation (3.5) above where

$$g_n(s) = \begin{cases} \frac{s}{n}, |s| \le \frac{1}{n} \\ sign\, s, |s| > \frac{1}{n} \end{cases} \tag{3.6}$$

and let $n \to \infty$, $(u - v)g_n(f(u) - f(v)) \to |u - v|$, then from equation (3.5) we obtain

$$\|u - v + r(Au - Av)\|_1 \ge \|u - v\|_1 \tag{3.7}$$

which shows that A is accretive. It now remains to show that A is m-accretive. We observe that since *f* is Lipschitz and *f*(0) = 0 then

$$\int_0^1 |Au - Av| dx = \int_0^1 |(f(u) - f(v))'| dx = |f(u(1)) - f(v(1))| \qquad (3.8)$$
$$\le k|u(1) - v(1)|$$

Also since $u, v \in D(A)$ and from the theory of integrals of functions defined on C[0,1] we observe that

$$|u(1) - v(1)| \le \int_0^1 |u - v| dx \qquad (3.9)$$

Therefore from (3.8) and (3.9) we obtain

$$\|Au - Av\|_1 \le k\|u - v\|_1 \qquad (3.10)$$

Hence A is Lipschitz on [0,1] and since A has already been shown to be accretive; the operator A is m-accretive by the fundamental result of Browder (1967). And by lemma (2.1) the operator $(A + n^{-1}I)^{-1}$ is continuous and bounded. Similarly by another result due Browder (1976) the problem defined by (3.3) above does admit a solution.

Appendix

From (2) and for q =3, the coefficient matrix gives the form:

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