*Journal of the Nigerian Association of Mathematical Physics*
*Volume 7(2003)*

# On the Lie Algebra of an Algebraic Group

**Henry O. Omokaro**
**Department of Mathematics, University of Benin, Benin City, Nigeria.**

**Abstract**

*Given an algebraic group* G *one can associate with it a Lie Algebra* g. *In this paper we discuss how much the structure of* g *reflects the structure of* G. *Other issues concerning* g *are also looked at.*

**pp 23 - 26**

## 1.0 Introduction

### 1.1 Definition

A set G which satisfies the following conditions is called an *Algebraic group.*

(i) G is a group
(ii) G is a topological space with respect to the Zariski topology
(iii) The function $G \times G \to G$ of $(x, y) \to xy$ is a morphism of varieties.
(iv) The function $G \to G$ of $x \to x^{-1}$ is a morphism of varieties.

Examples of algebraic groups can be found in [5]. It is important to note that an algebraic group is not a topological group except in dimension zero. In a topological group G, $G \times G$ has the product topology whereas in an algebraic group G, $G \times G$ has the Zariski topology.

Studies have shown that whenever we have an algebraic group G we can identify with G a Lie algebra g [3]. The manner in which g is constructed from G is a functorial one. In the next section we discuss some of the effects of G on the structure of its Lie algebra g.

## 2 0 The Structure of Lie Algebra

Given an algebraic group G one can show that the set of all derivations Der A of A is an algebra where A = K[G]. One can also show that the bracket of two derivations of A is also a derivation of A. So, it follows that the algebra of derivations of A is in fact a Lie algebra.

When G acts on A via left translation we obtain

$$\lambda_x : (\lambda_x f)(y) = f(x^{-1} y).$$

Similarly, when it acts on A via right translation we obtain

$$\rho_x : (\rho_x f)(y) = f(yx).$$

Now consider the following subspace of Der A..

$$\{\delta \in \text{Der A} : \delta\lambda_x = \lambda_x\delta \quad \forall\, x \in G\}.$$

This is called the space *of left invariant derivations* of *A*.

Now the bracket of two derivations of A, which commutes with $\lambda_x$, must also commute with $x$, so that the left invariant derivations of A is a Lie algebra. This Lie algebra we will write as L(G). We shall call it the *Lie Algebra* of *G*.

Several natural questions arise when we take a look at L(G) and G as a way of comparison. Some of such questions are:

(i) is L(G) finite dimensional?
(ii) if L(G) is finite dimensional what is its dimension?
(iii) if $\phi : G \to G^1$ is a morphism of algebraic groups, how does this behave as a carry over to the Lie algebras L(G). L($G^1$) of the algebraic groups G and $G^1$ respectively?

(iv) How doe the structure of L(G) reflect the group structure of G?

(v) If H is a closed subgroup of G how is L(H) related to L(G)?

Let $T(G)_e$ represent the tangent space of G at the identity. Usually $T(G)_e$ is identified with $T(G^0)_e$ where $G^0$ is the identity component of G. It has the structure of a vector space over K; its dimension is equal to the dimension of G because *e* is a simple point. We write

$$T(G)_e \text{ as } g.$$

Let $\phi : G \to G^1$ be a morphism which sends *e* to *e*. An example of this is a morphism of algebraic groups. $\phi$ induces the following linear map:

$$d\phi_e : g \to g^1.$$

This map $d\phi_e$, differential of $\phi$, has the following functorial properties.

(i) $d(I_G) = I_g$ where $I_G$ and $I_g$ are the identity elements of G and g respectively.

(ii) $d(\Psi_0\phi)_e = d\psi_{e0}d\phi_e$

We can describe T(G)e algebraically as he space of point derivations from the local ring at *e* into K. But we know that such derivations are already uniquely determined by them effect on A = K[G] as a subring. Naturally, this gives one the impression that one can link up g from L[G]. This is done by evaluating functions at *e*.

The following result establishes an isomorphism between the Lie algebra between the Lie algebra L(G) of an algebraic group G and the tangent space $T[G]_e$ = *g* of G at the identity element *e*. In addition, if $\phi : G \to G^1$ is a morphism of algebraic groups it shows how the induced space map from g the tangent space of G to $g^1$ the tangent space of $G^1$ behaves.

**3.0 Theorem [3]**

Let G be an algebraic group g = $T(G)_e$, L(G) as above. Then

$$\theta : L(G)_e \to g$$

such that

$$(\theta\delta)(f) = (\delta f)(e)(e) \text{ for all } \delta \text{ in } L(G),$$

*f* in A is a vector space isormorphism. In case $\phi : G \to G^1$ is a morphism of algebraic group $d\phi_e : g \to g^1$ is a homomorphism of Lie algebra (g, $g^1$ being given the bracket product of L(G), L($G^1$) respectively)

Since there exists an isormorphism between $T(G)_e$ and L(G) from the theorem it follows that we can as well study $T(G)_e$ in place of L(G) and obtain the desired results. In particular, since the dimension of G is equal to the dimension of T(G)e, it follows that the dimension of G is equal to the dimension of L(G). Therefore, if G is finite dimensional, then *g* is also finite dimension and the first two questions above have been answered.

It is important to know that the fact that every $\delta \in L(G)$ is invariant means that the tangent vector at *e* determined by $\delta$ is assigned through left translations to the various tangent vectors at other points *e* determined by $\delta$. As a result, $\delta$ is determined by the tangent vector $f \to (\delta f)(e)$; and also any tangent vector at *e* should also give rise to such left invariant derivation. We illustrate this by looking at the following example.

**Example**

Let *x* be fixed element of an algebraic group G. Consider the function:

$$\text{Int}\, x : G \to G \text{ such that } \text{Int}\, x = xyx^{-1} \quad \forall y \in G.$$

One can show that Int *x* is a morphism of G [5]. Its differential $d(\text{Int}\, x)$ is denoted by *Adx*.

Now *Adx* is an authomorphism of the algebra *g*. To show it is invertible, consider

$$(Adx)(Adx^{-1}) = d(\operatorname{Int} x)_0 d(\operatorname{Int} x^{-1}) = d(\operatorname{Int} xx^{-1}) = d(\operatorname{Int} e) = 1g$$

where $1g$ is the identity element of the Lie algebra g associated with G.

Consider the function

$$Ad : G \to Aut(g) \to GL(g)$$

such that $Ad(x) = Adx$. Let $x, y \in G$. Consider

$$Ad(xy) = (Adx)(Ady) = Adxy.$$

Therefore $Ad$ is a homomorphism of abstract groups. In other words, it is a representation of G.

Next, let us look at the effect of $Adx$ on $g$ through the map $\theta : L(G)_e \to g$ defined earlier. Now suppose $Adx(\delta) = \delta^1$ by definition. $(\delta^1 f)(e) = \delta(\phi^* f)(e) \ \forall \ f \in K[G]$ where $\phi = \operatorname{int} x$. This applies in particular to functions of the form $\rho_x f$. But

$$\phi^*(\rho_x f)(y) = (\rho_x f)(xyx^{-1}) = f(xy) = (\lambda_{x-1} f)y,$$

that is $\phi^*(\rho_x f) = \lambda_{x-1} f$.

Now $(\delta^1 \rho_x)(f)(e) = \delta(\lambda_x^{-1} f)(e) = \lambda_{x-1}(\delta f)(e) = (\delta f)(x)$, $\delta$ is left invariant $(\delta)(f)(e)$. We conclude that $\delta^1 = \rho_x \delta \rho_x^{-1}$. Therefore $Adx(\delta) = \rho_x \delta \rho_x^{-1}$. So the effect of $Adx$ on each derivation $\delta$ is that it conjugates if with $\rho_x$ for each $x \in G$.

4.0 **Definition**

Let G be an algebraic group and g its Lie algebra. The homomorphism $Ad : G \in GL(g)$ is called the *adjoint representation* of G.

Let us consider a closed subgroup H of an algebraic group G. The induced map $i : H \to G$ is an isomorphism of H onto a closed subgroup of G. $i$ therefore induces a map $i^* : K[G] \to K[H]$. But K[H] is isomorphic to K[G]/I, where I is the ideal vanishing on H, by the fist isomorphism theorem. It follows that $d_{ie}$ identifies $T(H)_e$ with the subspace of $T(G)_e$ consisting of those $x$ for which $x(I) = 0$. But $i$ is also a morphism of algebraic groups implies $di : h \to g$ is a Lie algebra homomorphism. This allows us to view $h$ as a Lie sub algebra of g.

In [1] there is a characterization for the closed subgroups H of an algebraic group G. From the right convolution the following is a similar characterisation of $h$ in $g$ for any closed subgroup H of G.

4.1 **Theorem [1]**

Let H be a closed subgroup of G:I the ideal of K[G] varnishing on H. Then $h = \{x \in g : I^* x \subset I\}$.

5.0 **Conclusion**

It is observed that the passage from an algebraic group G to its associated Lie algebra $g$ is a functorial one which is such that most of the results true for G also have their corresponding results in $g$. One of these is that if G is finite dimensional then its Lie algebra $g$ is also finite dimensional. If H is a closed subgroup of G, through the morphical action of G on itself we have a characterization of such H. Correspondingly, through right convolution in g, we also have a characterization of $h$.

It is also observed that if $\phi : G \to G^1$ is a morphism of algebraic groups then its differential $d\phi_e : g \to g'$ the identity element is also a Lie algebra homomorphism from $g$ to $g'$.

**References**

[1] Borel, A (1985) Linear Algebraic Groups, New York: Springer Verlag

[2] Banachoff, T., Werner J. (1980) Linear Algebra through Geometry.
[3] Humphreys, J. E. (1975) Linear Algebraic groups. New York: Springer Verlag
[4] Serre, J. P. (1975) Algebraic Groups and Class field. New York: Springer Verlag.
[5] Omokaro, H. O. (1996) Action of Algebraic Groups I.C.T.P., Diploma Progamme Thesis, Trieste, Italy.
[6] Waterhouse, W. C. (1979) Introduction to Affine Group Schemes. New York: Springer Verlag.