Conditional Expectations on Crossed Product of Von Neunmann Algebras

A.A Tijjani
Department of Mathematics,
Ahmadu Bello University, Zaria
Federal Republic of Nigeria

Abstract
Conditional expectations on crossed product of σ-finite von Neumann algebras are studied by showing that the crossed product of expected filtrations is expected.

pp 19 – 22

1.0 Introduction

In noncommutative probability theory the concept of conditional expectation is very important for many stochastic theories depend on is existence. Conditional expectation on a finite von Neumann algebra was first studied by Umegaki [19-20] who demonstrated that the obtained conditional expectation is a noncommutative extension of the classical conditional expectation in probability theory. Since then many steps have been taken in generalizing the notion of classical conditional expectation to the von Neumann algebras context usually by using the notion of state invariant ultraweakly continuous projection of norm one. In [12] Takesaki has given the necessary and sufficient conditions for the existence of such projections.

The motivation for this paper is contained in [7], where Lance has shown the possibility of extending conditional expectations to the crossed products. In this paper we consider the filtrations of expected increasing full right continuous σ-finite von Neumann subalgebras of a σ-finite von Neumann algebra. The main technical means used here is the tool of quantum stochastic calculus developed by Goldstein in [4].

2.0 Preliminaries and Notations

Let A be a von Neumann algebra with a faithful normal state φ acting on it and let B_α, $\alpha \in R^{\cdot}$, be its von Neumann subalgebra. We denote by A_* and $(B_\alpha)_*$ the preduals of A and B_α respectively. We shall assume that A acts standardly on a Hilbert space H such that φ is implemented by a cyclic and separating vector Ω in H and that B_α acts on $\overline{B_\alpha \Omega} = H_\alpha$, the closure of $\{x\Omega : x \in B_\alpha\}$ in H with its faithful normal state ψ. We shall not distinguish between B_α acting on H and its standard representatives acting on H_α.

We denote by $\sigma_t^{\varphi}, t \in R$, the modular automorphism group associated with A and φ and by $\sigma_t^{\psi}, t \in R$, the modular automolrphism group associated with Bα, $\alpha \in R^{\cdot}$ and ψ. Let $\overline{A}$ denote the crossed product $R(A, \sigma_t^{\varphi})$ of A by σ_t^{φ}, i.e., $\overline{A}$ is a semifinite von Neumann algebra acting on $\overline{H} = L^2(R, H)$ [13] and is generated by the operators $\pi(x)$, $x \in A$ and $\lambda(s)$, $s \in R$, defined by

$$(\pi(x)\xi)(t) = \sigma_{-t}^{\varphi}(x)\, \xi(t), \quad \xi \in L^2(R, H), \ t \in R, \tag{2.1}$$

$$(\lambda(s)\xi)(t) = \xi(t - s), \quad \xi \in L^2(R, H), \ s, t \in R, \tag{2.2}$$

where π is a faithful normal * - representation of A in L^2 (R, H) and λ is a strongly continuous unitary representation of R in L^2 (R, H), and A is identified with its image π(A) in ($\overline{A}$). We call B_α, $\alpha \in R^{\cdot}$, expected [4] if there exists a φ-invariant ultraweakly continuous projection of norm one of A onto B_α, i.e., if there exists a faithful φ-invariant σ-weakly continuous conditional expectation E_α of A onto B_α [4], [13]. Note that if B_α is expected then $\psi = \varphi| B_\alpha$, where $\varphi| B_\alpha$ means φ restricted to B_α, and by the uniqueness of modular automorphism group and [13], $\sigma_t^{\psi} = \sigma_t^{\varphi}|B_\alpha, t \in R$.

Now, for expected B_α, $\overline{B_\alpha} = R(B_\alpha, \sigma_t^{\psi}) \subseteq R(A, \sigma_t^{\varphi}) = \overline{A}$ where $\overline{B}$ is the crossed product of B_α by $\sigma_t^{\psi}, \alpha \in R^+, t \in R$. The dual action θ of R on $\overline{A}$ is the group $\{\theta_s : s \in R\}$ of automorphisms of $\overline{A}$ determined by

$$\theta_s(\pi(x)) = \pi(x), x \in A, s \in R. \tag{2.3}$$

$$\theta_s(\lambda(t)) = e^{-ist}\lambda(t), \; s,t \in R. \tag{2.4}$$

Let τ be the canonical faithful normal semifinite trace on $\overline{A}$ such that $\tau_0\theta_s = e^{-s}\tau$, $s \in R$. A closed densely defined operator h with domain D(h) is said to be affiliated with $\overline{A}$, denoted by $h\eta\overline{A}$, if $\overline{A}'D(h) \subset D(h)$ and $h'h \subset hh'$ for all h' in the commutant $\overline{A}'$ of $\overline{A}$, i.e., if $\xi \in D(h)$ and $h' \in \overline{A}'$ then $h'\xi \in D(h)$ and $h'h\xi = hh'\xi$.

An affiliated operator h is called a τ-measurable, if its domain is τ-dense, i.e., if

$$\forall \delta > 0 \exists p \in \overline{A}proj. \quad s.t. p\overline{H} \subseteq D(h) \; and \; \tau(I-p) \leq \delta, \tag{2.5}$$

where $\overline{A}_{proj.}$ is the lattice of projections in $\overline{A}$. The set of all τ-measurable operators affiliated with $\overline{A}$, denoted by $\overline{\overline{A}}$, is *-algebra of operators on $\overline{H}$ where the sum and the product operations are in the strong sense [10]. The sets $N(\varepsilon,\delta) = \{h \in \overline{\overline{A}} : \exists\, p \in \overline{A}_{proj.} : p\overline{H} \subseteq D(h),\ \|hp\| \leq \varepsilon,\ \tau(I-p) \leq \delta\}$ where ε > 0, δ > 0, form a basis for the neighbourhoods of 0 for the topology on $\overline{\overline{A}}$ that makes $\overline{\overline{A}}$ a topological vector space. We call this topology the topology of convergence in measure. $\overline{\overline{A}}$ endowed with this topology is a complete Hausdorff topological *-algebra in which $\overline{A}$ is a dense subset. For details see [15].

Let $\overline{A}_+$ and $\overline{\overline{A}}_+$ denote the set of all positive elements of $\overline{A}$ and $\overline{\overline{A}}$ respectively. Note that if h, k are τ-measurable operators with τ-dense domains D(h), D(k) respectively and h|E = h|E, then h=k [15], [22], where E is a τ-dense subspace of D(h) and D(k). And that h is positive if and only if (hx, x) ≥ 0 for x in some τ-dense subspace of D(h). So h ≥ k iff <hx, x> ≥ <kx, x> for x in some τ-dense subspace of D(h)∩D(k) [15], [22].

3.0 Conditional Expectations on Crossed Product

3.1 Remark

Since for expected B_α we can identify $(B_\alpha)_*^+$ with a subspace of A_*^+ then for expected $B_\alpha, \alpha \in R^+$, we will have $\overline{\overline{B}}\alpha \subseteq \overline{\overline{A}}$ and the measure topology of $\overline{\overline{B}}_\alpha$ is the relative measure topology of $\overline{\overline{A}}$. Our main result is the following:

3.2 Theorem

Let $\{B_\alpha : \alpha \in R^+\}$ be a family of von Neumann subalgebras of A satisfying

(i) $B_\alpha \subseteq B_t, \quad 0 \leq \alpha \leq t,$

(ii) $\left(\bigcup_{\alpha \in R^+} B_\alpha\right)'' = A,$

(iii) $\bigcap_{t > \alpha} B_t = B_\alpha.$

and $\{E_\alpha : \alpha \in R^+\}$ a family of φ–invariant σ-weakly continuous conditional expectations of A onto the family $\{B_\alpha : \alpha \in R^+\}$. Then there exists a family of ultraweakly continuous τ-invariant conditional expectations $\{\overline{E}_\alpha : \alpha \in R^+\}$ of $\overline{A}$ onto $\overline{B}_\alpha, \alpha \in R^+$ such that

(iv) $\overline{B}_\alpha \subseteq \overline{B}_t, \quad 0 \le \alpha \le t,$

(v) $\left(\bigcup_{\alpha \in R^+} \overline{B}_\alpha \right)'' = \overline{A},$

(vi) $\bigcap_{t>\alpha} \overline{B}_t = \overline{B}_\alpha.$

Before we give the prove of the above theorem, we need the following technical lemmas. Let $\overline{A}_{\lambda\pi}$ and $\left(\overline{B}_\alpha\right)_{\lambda\pi}$ be the *-algebras of linear combinations of the form $\lambda(s)\pi(x), x \in A, \ s \in R$ and $\lambda(s)\pi(x), x \in B_\alpha, \ s \in R$, respectively. The $\overline{A}$ and $\overline{B}_\alpha$ are σ-weak closures $\overline{A}_{\lambda\pi}$ and $\left(\overline{B}_\alpha\right)_{\lambda\pi}$ respectively [4], [21], and every element x of $\overline{A}_{\lambda\pi}$ may be represented as $x = \sum_k \lambda(s_k)\pi(x_k)$, where $s_1, s_2, \ldots, s_k \in R, \ x_1, x_2, \ldots, x_k \in A$.

Goldstein has proved the following

3.3 **Lemma ([4], Lemmas 4.2 and 4.3)**

For $s_1, s_2, \ldots, s_k \in R$, *and* $x_1, x_2, \ldots, x_k \in A$, *the mapping*

$$\overline{E}^{\alpha}_{\lambda\pi} : \overline{A}_{\lambda\pi} \ni \sum_k \lambda(s_k)\pi(x_k) \to \sum \lambda(s_k)\pi(E_\alpha(x_k)) \in (\overline{B}_\alpha)_{\lambda\pi}$$

is a projection of norm one and it extends to a σ-weakly continuous τ-invariant projection $\overline{E}_\alpha$ of norm one of $\overline{A}$ onto $\overline{B}_\alpha$.

3.4 **Lemma**

With the conditions given in theorem 2.2 we have $s - \lim_{t \downarrow \alpha} E_t(x) = E_\alpha(x), \quad x \in A.$

Proof: (See [3]).

3.5 **Lemma** [19]

Let $x \in \overline{A}$ *be such that* $x = \sum_k \lambda(s_k)\pi(x_k)$. *Then* $s - \lim_{t \downarrow \alpha} \overline{E}_t(x) = \overline{E}_\alpha(x)$

Proof

Since $E_t(x) \to E_\alpha(x)$ σ-strongly (strongly) as t ↓α for every x ∈ A [18] and the mapping $x \mapsto \pi(x)$ is σ-weakly continuous [14] and hence σ-strongly (strongly) continuous, we have $\pi(E_t(x)) \xrightarrow[t \downarrow \alpha]{} \pi(E_\alpha(x)), \ x \in A$, σ-strongly (strongly).

Therefore

$$\overline{E}_t(x) = \overline{E}_t \sum_k \lambda(s_k)\pi(x_k) = \sum_k \lambda(s_k)\pi(E_t(x_k)) \xrightarrow[t \downarrow \alpha]{} \sum_k \lambda(s_k)\pi(E_\alpha(x_k)) = \overline{E}_\alpha(x).$$

□

Proof of Theorem 3.2

By Lemma 3.3 the family $\{B_\alpha : \alpha \in R^+\}$ is an expected family of von Neumann subalgebras of $\overline{A}$.

(iv) follows from the fact that $B_\alpha \subseteq B_t, \ 0 \le \alpha \le t$. To show

(v) it suffices to show that

$$\overline{A} \subseteq \left(\bigcup_{\alpha \in R^+} \overline{B}_\alpha \right)''$$ since the other inclusion is obvious.

We know that $\overline{B}_\alpha = (\lambda(R) \cup \pi(B_\alpha))'' \supseteq \lambda(R) \cup \pi(B_\alpha)''$.

Therefore, $\left(\bigcup_{\alpha \in R^+} \overline{B}_\alpha \right)'' \supseteq \left(\bigcup_{\alpha \in R^+} (\lambda(R) \cup \pi(B_\alpha)'') \right)''$ Hence $\overline{A} = (\lambda(R) \cup \pi(A))''$

$$= \left(\lambda(R) \cup \pi \left(\bigcup_{\alpha \in R^+} \overline{B}_\alpha \right)'' \right)'' = \left(\bigcup_{\alpha \in R^+} (\lambda(R) \cup \pi(B_\alpha)'') \right)'' \subseteq \left(\bigcup_{\alpha \in R^+} \overline{B}_\alpha \right)'',$$

which proves (v).

(vi) It is clear that $\overline{B}_\alpha \subseteq \bigcap_{t>\alpha} \overline{B}_t$. To show the other inclusion, let $\overline{Y} \in \bigcap_{t>\alpha} \overline{B}_t$. Since for any $\overline{x} \in \overline{A}$, $\overline{E}_t(\overline{x}) \xrightarrow[t\downarrow\alpha]{} \overline{E}_\alpha(\overline{x})$ strongly, by lemma 2.5, then for this $\overline{Y}$, we have $\overline{E}_t(\overline{Y}) = \overline{Y}$, $\forall t > \alpha$. So $\overline{E}(\overline{Y})$ is a constant equal to $\overline{Y}$. So $\overline{Y} = \overline{E_\alpha}(\overline{Y})$. That is $\overline{Y} \in \overline{B}_\alpha$. It follows that $\bigcap_{t>\alpha} \overline{B}_t = \overline{B}_\alpha$.

This completes the proof.

□

References

[1] Barnett, C., Goldstein S,.and. Wilde, I.F (1994) Quantum stochastic integration and quantum stochastic differential equations, Math. Proc. Camb. Phil. Soc. 116, 535-553.

[2] Petz, D. (1988) Conditional expectation in quantum probability and applications, Lect. Notes in Math. 1303, 251-760.

[3] Dang-Ngoc, N. (1979) Pointwise convergence of martingales in von Neumann algebras, Israel J. Math. 34273-280.

[4] S. Goldstein, (1991), Conditional expectation and stochastic integrals in non-commutative L^p-spaces, Math. Proc. Camb. Phil. Soc. 110, 365-383.

[5] Jajte, R. (1985) Strong limit theorems in non-commutative probability, Lect. Notes in Math. 1110, Berlin: Springer-Verlag,

[6] Kadison R.V and Ringrose, J.R. (1986) Fundamentals of the theory of operator algebras II, , New York: Academic Press.

[7] Lance, E.C. (1978) Martingale convergence in von Neumann algebras, Math. Proc. Camb. Phil. Soc. 84, 47-56.

[8] Nelson, E. (1974) Notes on non-commutative integration, J. Funct. Anal. 15, 103-116.

[9] Pederson, G.K. and Takesaki, M. (1973) The Radon-Nikodym theorem for von Neumann algebra, Acta Math. 13053-87

[10] Segal, I.E. (1953) A non-commutative extension of abstract integration, Ann. of Math. 57, 401-457.

[11] Strătilă, S. (1981) Modular theory in operator algebras, Kent: Abacus Press.

[12] Takesaki, M. (1972) Conditional expectation in von Neumann algebras, J. Funct. Anal. 9, 306-321.

[14] Takesaki, M. (1973), Duality for crossed products and the structure of von Neumann algebras of type III, Acta Math. 131, 249-310.

[15] Takesaki, M. (1979) Theory of operator algebras I, , Heidelberg: Springer-Verlag.

[16] Terp, M. (1981) L^p-spaces associated with von Neumann algebras, Notes, Kobenhaven Universitat, Matematisk Institut, Rapport No. 3a/3b

[17] Tomiyama, J. (1957) On the projection of norm one in W^*-algebras, Proc: Japan Acad. 33, 608-612.

[18] Tomiyama, (1958) J. On the projection of norm one in W^*-algebras II, Tôhoku Math. J. 10, 204-209.

[19] Tsukada, M. (1983) Strong convergence of martingales in von Neumann algebras, Proc. Amer. Math. Soc. 8, 8537-540.

[20] Tijjani, A.A On Convergence of conditional expectations on L^p-spaces associated with crossed products, Preprint.

[20] Umegaki, H. (1954), Conditional expectation in an operator algebra, Tôhoku Math. J. 6, 177-181.

[21] Umegaki, H. (1956) Conditional expectation in an operator algebra, Tôhoku Math. J. 8, 86-100.

[22] Van Daele, A. (1978) Continuous crossed products and type III von Neumann algebras, Cambridge: Cambridge University Press

[23] Yeadon, F.J. (1973) Convergence of measurable operators, Math. Proc. Camb. Phil. Soc. 74257-268