Journal of the Nigerian Association of Mathematical Physics
Volume 7(2003)

Conditional Expectations on Haagerup L^p-spaces

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Abstract

We consider the problem of extending conditional expectations to the Haagerup L^p-space.

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1.0 Introduction

Motivated by pure analogy the concept of (non-commutative) conditional expectation has a deep meaning in the theory of operator algebras. At the beginning of 1950s, non-commutative conditional expectation on finite von Neumann algebra appeared in the work of Umegaki [37], [38]. Since then many steps have been taken in generalizing the notion of classical conditional expectation to the von Neumann algebras context usually by using the notion of state invariant ultra weakly continuous projection of norm one (see [8-12], [28], [32-33]). In [28], Takesaki gave the necessary and sufficient conditions for the existence of such projections. However, for some problems in quantum probability (e.g., in the construction of quantum Markov chain) this notion is not suffice and so an alternative notion, the so called generalized conditional expectation, which always exists, was given by Accardi and Cecchini in [1]. This considerable achievement has been of great intrinsic interest and it now links well with several other recent developments [7], [17], [18]. In [11], Goldstein has given a construction of conditional expectation on crossed product and on Haagerup L^p-spaces which was inspired by [22]. Our approach is slightly different in that we made use of symmetric embeddings which not considered in [11].

The rest of the paper is organized as follows. In section 1 we outline some of the notations and definitions in the construction of Haagerup L^p-spaces, and in Section 2 we generalize conditional expectation to the Haagerup L^p-spaces.

2.0 Preliminaries and Notations

Let A be a von Neumann algebra with a faithful normal state φ acting on it and let B_α, $\alpha \in R^{\cdot}$, be its von Neumann sub-algebra. We denote by A_* and $(B_\alpha)_*$ the preduals of A and B_α respectively. We shall assume that A acts standardly on a Hilbert space H such that φ is implemented by a cyclic and separating vector Ω in H and that B_α acts on $\overline{B_\alpha \Omega} = H_\alpha$, the closure of $\{x\Omega : x \in B_\alpha\}$ in H, with its faithful normal state ψ. We shall not distinguish between B_α acting on H and its standard representatives acting on H_α.

We denote by $\sigma_t^\varphi, t \in R$, the modular automorphism group associated with A and φ and by $\sigma_t^\psi, t \in R$, the modular automorphism group associated with B_α, $\alpha \in R^{\cdot}$ and ψ. Let $\overline{A}$ denote the crossed product $R(A, \sigma_t^\varphi)$ of A by σ_t^φ, then $\overline{A}$ is a semi finite von Neumann algebra acting on $\overline{H} = L^2(R,H)$ [28] and is generated by the operators $\pi(x)$, $x \in A$ and $\lambda(s)$, $s \in R$, where

$$(\pi(x)\xi)(t) = \sigma_{-t}^\varphi(x)\,\xi(t), \quad \xi \in L^2(R,H), \ t \in R, \tag{2.1}$$

$$(\lambda(s)\xi)(t) = \xi(t-s), \quad \xi \in L^2(R,H), \ s, t \in R, \tag{2.2}$$

and A is identified with its image $\pi(A)$ in $\overline{A}$. We call B_α, $\alpha \in R^{\cdot}$, expected [10] if there exists a φ-invariant ultra weakly continuous projection of norm one of A onto B_α, i.e., if there exists a faithful φ-invariant σ-weakly continuous conditional expectation E_α of A onto B_α [11], [28].
Note that if B_α is expected then $\psi = \varphi|B_\alpha$, where $\varphi|B_\alpha$ means φ restricted to B_α, and by the uniqueness of modular automorphism group and [28], $\sigma_t^\psi = \sigma_t^\varphi|B_\alpha, t \in R$. Now, for expected B_α, $\overline{B_\alpha} = R(B_\alpha, \sigma_t^\psi) \subseteq R(A, \sigma_t^\varphi) = \overline{A}$

where $\overline{B}$ is the crossed product of B_α by $\sigma_t^{\psi}, \alpha \in R^+, t \in R$.

The dual action θ of R on $\overline{A}$ is the group $\{\theta_s : s \in R\}$ of automorphisms of $\overline{A}$ determined by

$$\theta_s(\pi(x)) = \pi(x), x \in A, s \in R. \tag{2.3}$$

$$\theta_s(\lambda(t)) = e^{-ist}\lambda(t), \ s,t \in R. \tag{2.4}$$

Let τ be the canonical faithful normal semi finite trace on $\overline{A}$ such that $\tau_o\theta_s = e^{-s}\tau$, $s \in R$. A closed densely defined operator h with domain D(h) is said to be affiliated with $\overline{A}$, denoted by $h\eta\overline{A}$, if $\overline{A}'D(h) \subset D(h)$ and $h'h \subset hh'$ for all h' in the commutant $\overline{A}'$ of $\overline{A}$, i.e., if $\xi \in D(h)$ and $h' \in \overline{A}'$ then $h'\xi \in D(h)$ and $h'h\xi = hh'\xi$.

An affiliated operator h is called a τ-measurable, if its domain is τ-dense, i.e., if

$$\forall \delta > 0 \ \exists p \in \overline{A}_{proj.} \quad s.t. p\overline{H} \subseteq D(h) \ and \ \tau(I-p) \le \delta, \tag{2.5}$$

where $\overline{A}_{proj.}$ is the lattice of projections in $\overline{A}$. The set of all τ-measurable operators affiliated with $\overline{A}$, denoted by $\tilde{\overline{A}}$, is *-algebra of operators on $\overline{H}$ where the sum and the product operations are in the strong sense [26]. The sets

$$N(\varepsilon,\delta) = \{h \in \tilde{\overline{A}} : \exists \ p \in \overline{A}_{proj.} : p\overline{H} \subseteq D(h), \tag{2.6}$$

$$\|hp\| \le \varepsilon, \tau(I-p) \le \delta\} \tag{2.7}$$

where ε > 0, δ > 0, form a basis for the neighbourhoods of 0 for the topology on $\tilde{\overline{A}}$ that makes $\tilde{\overline{A}}$ a topological vector space. We call this topology the topology of convergence in measure. $\tilde{\overline{A}}$ endowed with this topology is a complete Hausdorff topological *-algebra in which $\overline{A}$ is a dense subset. For details see [31].

Let $\overline{A}_+$ and $\tilde{\overline{A}}_+$ denote the set of all positive elements of $\overline{A}$ and $\tilde{\overline{A}}$ respectively. Note that if h, k are τ-measurable operators with τ-dense domains D(h), D(k) respectively and $h|E = h|E$, then h=k [31], [40], where E is a τ-dense subspace of D(h) and D(k). And that h is positive if and only if $(hx, x) \ge 0$ for x in some τ-dense subspace of D(h). So $h \ge k$ iff $\langle hx, x\rangle \ge \langle kx, x\rangle$ for x in some τ-dense subspace of $D(h) \cap D(k)$ [31], [40].

Now, if we define

$$T : \overline{A}_+ \to \hat{\overline{A}}_+ \tag{2.8}$$

by

$$T(x)(\varphi) = \int_R \varphi(\hat{\theta}_s(s))ds, \ x \in \overline{A}_+, \varphi \in (A_*)_+ \tag{2.9}$$

where $\hat{\overline{A}}_+$ is the extended positive part of $\overline{A}$ [13] and $\hat{\theta}_s$ is the extension of θ_s to $\hat{A}_+$, then T is a faithful normal semi finite operator valued weight from $\overline{A}$ to A (see [15], lemma 5.2).

If for each normal weight φ on A we put $\overline{\varphi} = \hat{\varphi} \, o \, T$, then $\overline{\varphi}$ is a normal weight on $\overline{A}$ called the dual weight of φ, where $\hat{\varphi}$ is the extension of φ to a normal weight on $\hat{\overline{A}}_+$ (see [14], proposition 1.10). Since each $h \in \hat{\overline{A}}_+$ defines a normal weight τ(h.) on $\overline{A}$ then the mapping $h \to \tau(h.)$ is a bijection of $\hat{A}_+$ onto the set of normal weights on $\overline{A}$ (see [14], theorem 1.12). Let h_φ denote the element of $\hat{\overline{A}}_+$ for $\varphi \in A_*^+$, then h_φ is a positive self adjoint operator affiliated with $\overline{A}$ and is τ-measurable, i.e., h_φ is an element of $\tilde{\overline{A}}_+$, such that

$$\bar{\varphi} = \tau(h_{\varphi}.), \quad \bar{\varphi} \in \bar{A}_*^+ \tag{2.10}$$

and the mapping $\varphi \rightarrow h_\varphi$ is a bijection of A_*^+ onto the set of elements h of $\tilde{\bar{A}}_+$ such that $\theta_s h = e^{-s}h$, $s \in R$.

The mapping $\varphi \rightarrow h_\varphi$ of A_*^+ onto $\tilde{\bar{A}}_+$ can be extended by linearity to a linear bijection mapping of A_* onto the set

$$\left\{ h \in \tilde{\bar{A}} : \forall s \in R, \theta_s h = e^{-s} h \right\}. \tag{2.11}$$

The Haagerup L^P-spaces are defined as

$$L^p(A) = \{ h \in \tilde{\bar{A}} : \forall s \in R, \quad \theta_s h = e^{-s/p} h \}, \quad p \in [1, \infty]. \tag{2.12}$$

The spaces $L^p(A)$ are subspaces of $\tilde{\bar{A}}$. For $1 \leq p < \infty$, $L^p(A)$ can be regarded as a set of closed densely defined operators h affiliated with $\bar{A}$ having polar decomposition $h = U|h|$ s.t. $|h|^p \in L^1(A)$ and $U \in A$. Let a linear functional tr on $L^1(A)$ be defined by

$$tr(h_\varphi) = \varphi(1), \ \varphi \in A_*. \tag{2.13}$$

Then the mapping $h \rightarrow tr(|h|)$ defines a norm on $L^1(A)$. The functionals $\|.\|_p$ and $\|.\|_\infty$, where

$$\|h\|_p = tr\left(|h|^p\right)^{\frac{1}{p}}, h \in L^p(A), 1 \leq p < \infty, \tag{2.14}$$

and.

$$\|h\|_\infty = \|h\|, \quad h \in L^\infty(A)(= A), \tag{2.15}$$

define norms on $L^p(A)$, $p \in [1,\infty]$. The spaces $(L^p(A), \|.\|_p)$, $p \in [1,\infty]$, are Banach spaces with A_* isometrically isomorphic to $L^1(A)$ by the isometry $\varphi \rightarrow h_\varphi$ and A isometrically isomorphic to $L^\infty(A)$. For $p \in [1,\infty)$, we have

$$L^p(A) \cap N(\varepsilon, \delta) = \{ h \in L^p(A) : \|h\|_p \, \varepsilon \delta^{1/p} \}. \tag{2.16}$$

Thus the norm topology on $L^p(A)$ is equivalent to the relative topology of the τ-measurable operators.

If $h \in L^p(A)$ and $k \in L^q(A)$ then $hk \in L^1(A)$ and $tr(hk) = tr(kh)$, where $\frac{1}{p} + \frac{1}{q} = 1$. $L^2(A)$ is a Hilbert space with the inner product.

$$\langle h, k \rangle_{L^2(A)} = tr(k^* h) = tr(hk^*); h, k \in L^2(A). \tag{2.17}$$

3.0 A Generalization of Conditional Expectation

For $\alpha \in R^+$, let E_α be the φ-invariant σ-weakly continuous conditional expectation of A onto B_α. We wish to extent E_α to $L^p(A)$.

3.1 Remark

Note that for $\varphi \in A_*$ we have $h_\varphi \in L^1(A)$ and $h_\varphi^{1/2} \in L^2(A)$. The vector $h_\varphi^{1/2} \in L^2(A)$ is cyclic and separating for A and the representation $(A, L^2(A), h_\varphi^{1/2})$ is unitarily equivalent to the representation (A,H,Ω) [4]. For the cyclic and separating vector $h_\varphi^{1/2}$, the mapping $x \rightarrow \left\| x h_\Psi^{1/2} \right\|, x \in B_\alpha, \Psi \in (B_\alpha)_*$, is a norm on B_α and the completion is $L^2(B_\alpha)$.

For each $\alpha \in R^+$, let P_α be the orthogonal projection of H onto $H_\alpha = \overline{B_\alpha \Omega}$, the closure of $\{x\,\Omega : x \in B_\alpha\}$ in H. Then $P_\alpha \in B'_\alpha$, where B'_α is the commutant of B_α, and $P_\alpha(x\Omega)=E_\alpha(x)\Omega$, $x \in A$ [5]. By identifying Ω with $h_\varphi^{1/2}$, H with $L^2(A)$ and H_α with $L^2(B_\alpha)$ we see that the conditional expectation E_α of A onto B_α extends to an expectation P_α of $L^2(A)$ onto $L^2(B_\alpha)$ and P_α is identified with E_2^α where

$$E_2^\alpha : L^2(A) \to L^2(B_\alpha) \tag{3.1}$$

is given by

$$E_2^\alpha(xh_\varphi^{1/2}) = xh_\psi^{1/2}, \quad x \in B_\alpha,\ \psi \in (B_\alpha)_*, \quad \varphi \in A_*. \tag{3.2}$$

Let the mapping

$$E_*^\alpha : A_* \to A_* \text{ be defined by } E_*^\alpha \varphi = \varphi o E_\alpha,\ \varphi \in A_*. \tag{3.3}$$

Then E_*^α is a well defined contractive mapping. Indeed, we have

$$\left\|E_*^\alpha \varphi(I)\right\| = \left\|\varphi(E_\alpha(I))\right\| \le \left\|\varphi(I)\right\|. \tag{3.4}$$

In fact, E_*^α is a projection since E_α is. Thus E_*^α is a projection of norm one. Let the mapping

$$E^\alpha : L^1(A) \to L^1(A)$$

be defined by

$$E^\alpha h_\varphi = h_{E_*^\alpha \varphi}, \quad \varphi \in A_*.$$

Then E^α is a well defined contractive mapping of $L^1(A)$ to itself. Indeed,

$$\begin{aligned}\left\|E^\alpha h_\varphi\right\|_1 &= \sup\{\left|tr(E^\alpha h_\varphi x)\right| : x \in A,\ \|x\| \le 1\} \le \sup\{tr(\left|E^\alpha h_\varphi x\right|) : x \in A,\ \|x\| \le 1\} \\ &= \sup\{tr(\left|h_{E_*^\alpha \varphi}\, x\right|) : x \in A,\ \|x\| \le 1\} = \sup\{\left|(E_*^\alpha \varphi)(x)\right| : x \in A,\ \|x\| \le 1\} \\ &= \sup\{\left|\varphi(E_\alpha(x))\right| : x \in A,\ \|x\| \le 1\} \le \sup\{\left|\varphi(x)\right| : x \in A,\ \|x\| \le 1\} = \|\varphi\| = tr(\left|h_\varphi\right|) = \left\|h_\varphi\right\|_1.\end{aligned} \tag{3.4}$$

3.2 Theorem

With the above notations, we have

(i) $E^\alpha h_\varphi = h_\varphi$ *for* $h_\varphi \in L^1(B_\alpha)$, *where* $\varphi \in (B_\alpha)_*$;

(ii) $E^\alpha(L^1(A)) = L^1(B_\alpha)$.

Proof (i) Note that $L^1(B_\alpha) \subseteq L^1(A)$ and for $\varphi \in A_*^+$, we have $h_\varphi \in L_+^1(A)$. Also for $\varphi \in A_*$ and $x \in A$, we have $tr(h_\varphi x) = \varphi(x)$ and $\varphi_o E_\alpha \in (B_\alpha)_*$, since E_α is σ-weakly continuous. Let $x \in A$ and $\varphi \in A_*$. Then $tr(E^\alpha h_\varphi x) = tr(h_{E_*^\alpha \varphi}\, x) = (E_*^\alpha \varphi)(x) = \varphi(E_\alpha(x)) = tr(h_\varphi E_\alpha(x))$ Using this, for $\varphi \in (B_\alpha)_*$ and $x \in B_\alpha$, we have $tr(E^\alpha h_\varphi x) = tr(h_\varphi E_\alpha(x)) = tr(h_\varphi x)$ and since $E^\alpha h_\varphi \in L^1(B_\alpha)$ this shows that $E^\alpha h_\varphi = h_\varphi$.

(ii) Let $h_\varphi \in L^1(A)$ and $x \in B_\alpha$, where $\varphi \in A_*$. Then $tr(E^\alpha h_\varphi x) = tr(h_{E_*^\alpha \varphi}\, x) = (E_*^\alpha \varphi)(x) = \varphi(E_\alpha(x)) = \psi(x) = tr(h_\psi x)$. Showing that $E^\alpha h_\varphi \in L^1(B_\alpha)$. Thus by (i), E^α maps $L^1(A)$ onto $L^1(B_\alpha)$.

□

3.3 **Remark**

Note that E^α is invariant under t_r. To see this let $\varphi \in A_*$ and $x \in B_\alpha$. Then $h_\varphi \in L^1(A)$ and $t_r(E^\alpha(h_\varphi)x)=tr(h_\varphi E_\alpha(x))=tr(h_\varphi x)$.

3.4 **Theorem**

Let $0 \le \alpha \le t$ with $\alpha, t \in R^+$. Then for any $Y \in L^1(A)$, we have $tr(E^t(Y)) \to tr(E^\alpha(Y))$ as $t \downarrow \alpha$.

Proof

Let $x \in A$, $\varphi \in A_*$ and define

$$L^1(A) X A \to L^1(A) \text{ by}$$

$$(h_\varphi, x) \to (E^t(h_\varphi)x) = (h_{\varphi o E_t} x).$$

Since $E_t(x) \to E_\alpha(x)$ ultra weakly as $t \downarrow \alpha$ (see [36]) and $tr(h_\varphi)$ is ultra weakly continuous we have that

$$tr(E^t(h_\varphi)x) = tr(h_\varphi E_t(x)) \xrightarrow[t\downarrow\alpha]{} tr(h_\varphi E_\alpha(x)) = tr(E^\alpha(h_\varphi)x).$$

□

3.5 **Lemma**

For $q \ge 1$ and $\varphi \in A_*^+$, $\theta_s(h_\varphi^{1/2q}) = (\theta_s h_\varphi)^{1/2q}$.

Proof

Since h_φ is in particular positive and self adjoint we can write its spectral resolution as

$$h_\varphi = \int_0^\infty \lambda de_\lambda .$$

Therefore,

$$\theta_s h_\varphi = \int_0^\infty \lambda d\theta_s e_\lambda \text{ and } (\theta_s h_\varphi)^{1/2q} = \int_0^\infty \lambda^{1/2q} d\theta_s e_\lambda$$

$$\theta_s(h_\varphi^{1/2q}).$$

which proves the result.

□

Using the above result and the fact that θ_s, $s \in R$, are automorphisms of A we have the following lemmas.

3.6 **Lemma**

Let $p,q \ge 1$ and $k \in L^p(A)$ with $\frac{1}{p}+\frac{1}{q}=1$. Then $h_\varphi^{1/2q} k\, h_\varphi^{1/2q} \in L^1(A)$, $\varphi \in A_*^+$.

Proof

By the definition of $L^1(A)$ we have to show that $\theta_s(h_\varphi^{1/2q} k\, h_\varphi^{1/2q}) = e^{-s} h_\varphi^{1/2q} k\, h_\varphi^{1/2q}$.

Now consider

$$\theta_s(h_\varphi^{1/2q} k\, h_\varphi^{1/2q}) = \theta_s(h_\varphi^{1/2q})\theta_s(k)\theta_s(h_\varphi^{1/2q}) = (\theta_s h_\varphi)^{1/2q}\theta_s(k)(\theta_s h_\varphi)^{1/2q}$$

$$= e^{-s} h_\varphi^{1/2q} k\, h_\varphi^{1/2q}$$

Showing that $h_\varphi^{1/2q} k\, h_\varphi^{1/2q} \in L^1(A)$.

□

3.7 **Lemma**

Let $p,q \ge 1$ be such that $\frac{1}{p}+\frac{1}{q}=1$. For $k \in L^1(A)$, $h_\varphi^{-1/2q} k h_\varphi^{-1/2q} \in L^p(A)$, $\varphi \in A_*^+$.

Proof

To show that $h_\varphi^{-1/2q} k h_\varphi^{-1/2q} \in L^p(A),\ \varphi \in A_*^+$ we have to show that

$$\theta_s(h_\varphi^{-1/2q} k\ h_\varphi^{-1/2q}) = e^{-s/p} h_\varphi^{-1/2q} k\ h_\varphi^{-1/2q}.$$

$$\therefore \theta_s(h_\varphi^{-1/2q} k\ h_\varphi^{-1/2q}) = \theta_s(h_\varphi^{-1/2q})\theta_s(k)\theta_s(\ h_\varphi^{-1/2q}) = (\theta_s h_\varphi)^{-1/2q}\theta_s(k)(\theta_s h_\varphi)^{-1/2q}$$

$$= e^{-s/p} h_\varphi^{-1/2q} k\ h_\varphi^{-1/2q}.$$

Showing that $h_\varphi^{-1/2q} k\ h_\varphi^{-1/2q} \in L^p(A)$. □

3.9 **Remark**

From the above lemmas, we see that for x ∈ A and $\varphi \in A_*^+$,

$$\theta_s(h_\varphi^{1/4} x\ h_\varphi^{1/4}) = e^{-s/2} h_\varphi^{1/4} x\ h_\varphi^{1/4}, \quad \theta_s(h_\varphi^{1/2} x\ h_\varphi^{1/2}) = e^{-s} h_\varphi^{1/2} x\ h_\varphi^{1/2}.$$

So for x ∈ $L^\infty(A)$=A and $\varphi \in A_*^+$, we can embed A into $L^2(A)$ by

$$w_2^\infty : A \to L^2(A), \qquad x \mapsto h_\varphi^{1/4} x\ h_\varphi^{1/4},$$

and A onto $L^1(A)$ by

$$w_1^\infty : A \to L^1(A), \qquad x \mapsto h_\varphi^{1/2} x\ h_\varphi^{1/2}.$$

Let k ∈ $L^2(A)$. Then $h_\varphi^{1/4}\ k h_\varphi^{1/4} \in L^1(A),\ \varphi \in A_*^+$. This will enable us to embed $L^2(A)$ into $L^1(A)$ by

$$w_1^2 : L^2 A \to L^1(A), \qquad k \mapsto h_\varphi^{1/4} k\ h_\varphi^{1/4}.$$

Our embeddings are consistent since

$$A \to L^2(A) \to L^1(A),\ x \mapsto \left(h_\varphi^{1/4} x h_\varphi^{1/4}\right) \mapsto h_\varphi^{1/4}\left(h_\varphi^{1/4} x h_\varphi^{1/4}\right) h_\varphi^{1/4}$$

$$h_\varphi^{1/2} x h_\varphi^{1/2}.$$

Thus we have the following diagram

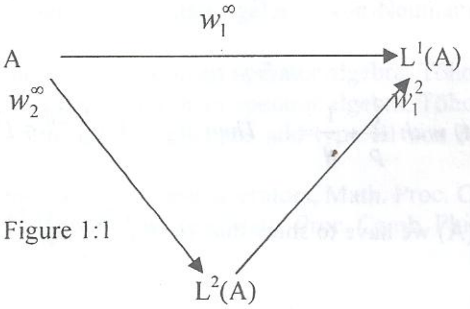

Figure 1:1

where $w_1^\infty = w_1^\infty\ o\ w_2^\infty$. By [10] w_1^∞ (A) is dense in $L^1(A)$ and w_2^∞ (A) is dense in $L^2(A)$.

3.10 **Theorem (also see Goldstein [11])**

The mapping $E_p^\alpha : L^p(A) \to L^p(A)$ *defined by*

$$E_p^\alpha k = h_\varphi^{-1/2q}\left(E^\alpha\left(h_\varphi^{1/2q} k h_\varphi^{1/2q}\right)\right) h_\varphi^{-1/2q},\ p, q \in [1, \infty],$$

$$\frac{1}{p}+\frac{1}{q}=1,\ \varphi\in A_*^+,$$

is a well defined contractive mapping of $L^p(A)$ into itself, where E^α is as in 2.2. Moreover, if k ∈ $L^p(B_\alpha)$, then

$$E_p^\alpha k = k.$$

Proof

Let k ∈ $L^P(A)$, then $h_\varphi^{1/2q}k\ h_\varphi^{1/2q} \in L^1(A)$ by 2.6. And from the definition of E^α we get

$$\left\|E^\alpha\left(h_\varphi^{1/2q}kh_\varphi^{1/2q}\right)\right\|_1 \le \left\|h_\varphi^{1/2q}kh_\varphi^{1/2q}\right\|_1.$$

Therefore

$$\left\|h_\varphi^{1/2q}(E_p^\alpha k)h_\varphi^{1/2q}\right\|_1 = \left\|E^\alpha(h_\varphi^{1/2q}\ kh_\varphi^{1/2q})\right\|_1 \le \left\|h_\varphi^{1/2q}\ kh_\varphi^{1/2q})\right\|_1$$

Hence

$$\left\|E_p^\alpha k\right\|_p \le \|k\|_p.$$

Let k ∈ $L^P(B_\alpha)$. Then, $h_\varphi^{1/2q}kh_\varphi^{1/2q} \in L^1(B_\alpha)$, $E^\alpha\left(h_\varphi^{1/2q}kh_\varphi^{1/2q}\right) = h_\varphi^{1/2q}kh_\varphi^{1/2q}$ and

$$E_p^\alpha k = h_\varphi^{-1/2q}\left(E^\alpha\left(h_\varphi^{1/2q}kh_\varphi^{1/2q}\right)\right)h_\varphi^{-1/2q} = k.$$

□

Acknowledgment

The author wishes to thank Professor C. Barnett of Imperial College London for his encouragement. He also wishes to thank Professors L. Accardi of Universitat di Roma II, Italy and S. Goldstein of Lodz University, Poland, for their faithful discussions at the University of Nottingham, England.

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