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A Note on the Generalized Rotations of State Vectors in Complex Hilbert Spaces

E. O Ifidon
Department Of Mathematics
University of Benin

Abstract

By considering the prolonged action of the group of symmetries G on the set of states $\hat{H}$ of a quantum system, we obtain a generalized rotation of state vectors in complex Hilbert spaces for which norms are preserved. New symmetries as well as new representations are thus obtained, which for complex quantum systems aid in the simplification of the system. The structure and properties of this new algebra are extensible studied.

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1.0 Introduction

Since the pioneering work of Sophus Lie on the application of continuous group of transformations to the study of systems of partial differential equations, much work has been undertaken in the globalization, abstraction and fashionability of the Lie groups without much reference to their usefulness in obtaining solutions to differential equations which was the original reason why the theory was developed in the first place. A major reason for this is that early researchers found Lie's method of limited application in the construction of the general solution to a large number of partial differential equations occurring then. Following Noether's theorem [1], proving the one-to-one correspondence between symmetry groups of variational problems and conservation laws of their associated Euler-Lagrange equations and the work by L.V Ovsiannikov [2], new interest in the usefulness of applying Lie groups to differential equations was rekindled, which was further spurred on by interactions with soliton equations and Hamiltonian systems. This led P. J Olver into developing an improved modern version of Lie's transformation theory [3], following the work by G. Birkhoff's on hydrodynamics [4]. Olver's method provides a ready verifiable means of obtaining the group of symmetries of any partial differential equation, linear or non-linear. This paved the way for further work in this area [5] relating to the full symmetry group of point transformations for Maxwell's equations. Following E. O Ifidon and E. O Oghre [6], we extend Olver's theory to the study of the set of states $\hat{H}$ of a quantum system. It can be shown that invariance of a quantum system under the prolonged action of G give rise to a 13 dimensional subgroup of SL(6,R).

In this paper we develop the theory, which was first introduced in [6] and study in greater details the properties of the resulting algebra.

2.0 Lie Groups, Lie Algebras and the Set of States of a Quantum System

Let S denote the set of all possible states of a system.

Definition 2.1

A group G is a symmetry group of S if for each $s \in S$ and $g \in G$, $g.s \in S$ whenever g.s is defined. In other words G transforms a given state of the system to a new state of the system.

Definition 2.2

Suppose G is such that

$$g_1.(g_2.s) = (g_1 g_2).s$$
$$e.s = s \qquad g_1, g_2 \in G$$

where e is the identity element in G. Then G is called a group of symmetries for S. In quantum mechanics the set of states $S = \hat{H}$ is the set of all rays $\hat{\phi} = \{\lambda\varphi, \lambda \in C\}$ where φ is a nonzero vector in the complex Hilbert space H and C is the field of complex numbers.

Theorem 2.1

Given a ray $\hat{\phi} \in \hat{H}$, if G is a Lie group of symmetries for H, then the probability of going from the state $g.\phi$ to the state $g.\psi$ is the same as that of going from ϕ to ψ for all $g \in G$ and $\phi, \psi \in H$.

Proof

See [6]

A consequence of theorem 2.1 is that we may view our group G as acting to change the frame of reference of our quantum system. Since acting to change the frame of reference does not change the state vectors, we have a transformation for which norms are preserved. This ascertains that any local Lie group of symmetries acting on the set of states $\hat{H}$ preserves transition probabilities in H. Groups which preserves transition probabilities in quantum mechanics helps to ascertain that changing the frame of reference of the state vectors would not affect the system we are studying. This is useful a useful concept.

Definition 2.3

Let $\hat{H}$ be the set of states of a quantum system. The transformation $\phi \rightarrow g.\phi$ where g is an operation from $H \rightarrow H$ is called a generalized rotation of the state vectors in H.

Example 2.1

Let us construct a sub-bundle $Z = \{(x, \varphi) : x \in R^n \text{ and } \varphi \in H\}$. x may be viewed as the real line probability measures ϕ assigns to the various self adjoint operators in H as it evolves in time. If G acts regularly on Z, then we have for $g \in G$ close to the identity e

$$g.\phi = g(x, \varphi) = (\Lambda_g x, \lambda_g \varphi) = (\tilde{x}, \tilde{\varphi})$$

where (Λ_g, λ_g) C^∞ composition maps of G defined by

$$\tilde{x} = \Lambda_g(x, \varphi) = \Lambda_g.(e \times \varphi)(x)$$

and

$$\tilde{\phi} = \lambda_g(x, \varphi) = \lambda_g.(e \times \varphi)(x)$$

This transformation is analogous to a rotation of the classical phase space $S = R^3 \times R^3$ with

$$g.s = (g\underline{q}, g\underline{p}) \text{ if } S = (\underline{q}, \underline{p})$$

and g is a rotation of R^3.

We note here that only those group elements in the neighbourhood of the identity can in general be guaranteed to transform functions. Thus our theory is developed for only local transformation groups. Non-local symmetries have been studied recently by Muriel and Romero [7] but their procedure works for only non-abelian Lie algebras and if considered may not be guaranteed to preserve the physical observables of our quantum system. We may imagine all sorts of groups acting on the sets of states of our quantum system, but this will only be a useful concept if the symmetries preserve the observations. We now construct an appropriate space for which our group of symmetries may act. In doing this, let us first consider the manifold $Z = R^n \times H$, which is homomorphic to H, where $x \in R^n$ are the components of ϕ_t (physically the x are real line projection valued measures associated with the self adjoint operators of the system.). Any element of Z completely determines a particular pure state of the quantum system. The p-sections of Z would be determined by the dynamics of the system. Note that most systems of interest have their dynamics described by a differential equation, which in the case of quantum mechanics is the Schrodinger equation. Thus the path is of the form $\hat{\phi}_t$ where ϕ_t satisfies

$$i\hbar \frac{\partial \phi_t}{\partial t} = H\phi_T$$

Here H is a self adjoint operator called the total energy or Hamiltonian of the system. The p- sections of Z would then be given by the set

$$\left\{(x, \phi_t) : i\hbar \frac{\partial \phi_t}{\partial t} = H\phi_T\right\}.$$

The time dependence of our system may be variously viewed as a change in state vectors whose properties are imbedded in $Z_P \subset Z$. Usually having defined our Hilbert space H, the energy operator H and identifying the various self adjoint operators on H with physical observables, the problem is to find these data for given physical systems, solve the dynamics and to identify various physical observables with operators. With the help of group theory, these properties may be studied in Z and sometimes a group can be very obviously involved, for instance a rotation group if the system has spherical symmetries. Also unitary groups acting on Z create a rotation of axes without necessarily changing the state vectors. In this paper we are interested in finding more general transformations than unitary transformations in which both a change in the axes and in the state vectors may occur. To this end we shall construct an appropriate fibre bundle $\left(H^{(k)}, \pi, Z_p\right)$ where π are the canonical bundle projections and $H^{(K)}$ is the extended space of p-sections Z_P of Z called the extended k-jet bundle and it contains all the partial derivatives of ϕ_t of order $\leq k$. Due to the coordinate structure of $H^{(K)}$ no distinctions can be made between the state vectors and their dynamical variables in $H^{(K)}$. The dynamics of the system would then be described by a closed subvariety Δ_0 of $H^{(K)}$. We anticipate that "hidden" symmetries would be obtained if one considers the prolonged action of the symmetry group G on $H^{(K)}$. This can be achieved as follows. If G is a local group of symmetries acting on Z, there is induced on $H^{(K)}$ an action of G, which is due to the action of G on p-sections of Z called the k-th prolongation of G. If G acts regularly on Z in the sense of Palais [8] so that one has a natural manifold structure on the quotient space Z/G then under certain conditions G transforms elements of Δ_0 into other elements of Δ_0 i.e Δ_0 is G-invariant. If Δ_0 is invariant under the prolonged action of G, then the problem of solving the system in Z in a particular representation is in one to one correspondence with the problem of solving the system in a different representation in $H^{(K)}$ with fewer components depending on the dimensions of the orbit of G. The important point is that the dynamical variables are reduced by an amount l where l is the dimensions of the orbit of G, making the reduced system in some sense easier to solve. New conservation laws can be obtained by checking which of the vector fields are constants of motion i.e commutes with H.

2.0 Infinitesimal Generators of the Lie Algebra.

We shall assume that our quantum system consist of a single spineless particle moving in a conservative force field potential $v = v(x,y,z)$ so that the Hilbert space is $H = L^2(R^3, \eta)$ η, Lesbeque measure A state vector ϕ would be represented by a wave function $\Psi(x,y,z;t)$ defined on $R^3 \times R$ satisfying

$$\frac{\partial \Psi}{\partial t} = -iH\Psi \tag{3.0}$$

with

$$H = -\frac{\nabla^2}{2} + v$$

where ∇^2 denotes the Laplacian.

Given a local transformation group G acting on $N \times M$ the space of the independent and the dependent variables (x,u), there is induced an action of G on the space $N \times M^{(k)}$ consisting of points $(x, u^{(k)})$ where $u^{(k)}$ represents the derivatives of the dependent variables of order $\leq$ k given by

$$\partial_J u = \frac{\partial^{|J|} u}{\partial_{x_1}^{j_1} \cdots \partial_{x_p}^{j_p}}$$

$$0 \leq |J| \leq k \qquad |J| = j_1 + j_2 + \dots + j_p$$

See [9]. This induced action of G, called the k-*th* prolongation of G, can easily be obtained from the corresponding prolonged infinitesimal generators $P_r^{(k)}\chi$ of the group which are vector fields on $N\times M^{(k)}$ and have a relatively simple expression. The generators χ of the group are vector fields on N×M given by

$$\chi = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x_i} + \sum_{l=1}^{q} \varphi_l(x,u)\frac{\partial}{\partial u^l}$$

where *p* and *q* represents the number of independent variable in the space N×M respectively. The corresponding expression for the prolonged vector field is then

$$P_r^{(k)}\chi = \chi + \sum_{l=1}^{q}\sum_{J} \varphi_l^J(x,u^{(k)})\frac{\partial}{\partial u_j^l}$$

with

$$\phi_j^J = D^J\left(\phi^l - \sum_{i=1}^{p} u_i^l \xi_i + \sum_{i=1}^{p} u_{j,i}^l \xi_i\right)$$

$$D_J = D_1^{j_1}.D_2^{j_2}...D_p^{j_p}$$

where

$$D_l^i = \frac{\partial}{\partial x^i} + \sum_{i=1}^{q}\sum u_J^l \frac{\partial}{\partial u_J^l}$$

is the total derivative operator. The infinitesimal criteria for invariance of a system of partial differential equations $\Delta(x,u^{(k)}) = 0$ under the action of G states that G is a symmetry group of the system $\Delta(x,u^{(k)}) = 0$ if and only if for every infinitesimal generators χ of G.

$$P_r^{(k)}\chi\Delta(x,u^{(k)}) = 0 \tag{3.1}$$

whenever $\Delta(x,u^{(k)}) = 0$ for the case of the Schrodinger equation

$$\Delta(x,u^{(k)}) = \Psi_t + i(-\nabla^2 + v)\Psi \tag{3.2}$$

A typical vector field on $R^4\times R^2$ with coordinates (x,y,z,t,v,Ψ) is given by

$$\chi = \xi\partial_x + \eta\partial_y + \lambda\partial_z + \gamma\partial_t + \Phi\partial_v + \varphi\partial_\Psi \tag{3.3}$$

where the coefficients $\{\xi,\eta,\lambda,\gamma,\phi,\Phi\}$ are arbitrary functions of x, y, x, v and Ψ. The corresponding second prolongation of χ is

$$p_r^{(2)}\chi = \chi + \sum_J \theta^J\partial\Psi_J + \sum_J \Lambda^J\partial v_J \tag{3.4}$$

where the J-sum is over all partitions

$$J = \begin{cases} (1,0,0,0)\,(0,1,0,0)\,(0,0,1,0)\,(0,0,0,1)\,(1,1,0,0)\,(1,0,1,0) \\ (1,0,0,1)\,(0,1,1,0),\,(0,1,0,1)\,(0,0,1,1)\,(2,0,0,0)\,(0,2,0,0) \\ (0,0,2,0)\,(0,0,0,2) \end{cases}$$

Substitution of (3.3) and (3.4) in (3.1) yields

$$i\theta^{(0,0,0,1)} + \theta^{(2,0,0,0)} + \theta^{(0,2,0,0)} + \theta^{(0,0,3,0)} - v\Phi - \Psi\varphi = 0 \tag{2.5}$$

subject to

$$i\Psi_t + \Psi_{xx} + \Psi_{yy} + \Psi_{zz} - v\Psi \tag{2.6}$$

where

$$\theta^{(0,0,0,1)} = \begin{cases} \Phi_t - \Psi_x\xi_t - \Psi_y\eta_t - \Psi_z\lambda_t + \Psi_z(\gamma_\Psi - \gamma_t) \\ - \Psi_t\Psi_x\xi_\Psi - \Psi_t\Psi_y\eta_\Psi - \Psi_t\Psi_z\lambda_\Psi - \Psi_t^2\gamma_\Psi \end{cases}$$

$$\theta^{(2,0,0,0)} = \begin{cases} \Phi_{xx} - \Psi_x(2\gamma_{x\Psi} - \xi_{xx}) - \Psi_y\eta_{xx} - \Psi_z\lambda_{xx} - \Psi_t\gamma_{xx} \\ + \Psi_{xx}(\Phi_\Psi - 2\xi_x) - 2\Psi_{xx}\eta_x - 2\Psi_{xz}\lambda_x - 2\Psi_{tx}\gamma_x \\ + \Psi_x^2(\phi_{\Psi\Psi} - 2\xi_{\Psi x}) - 2\Psi_x\Psi_y\eta_{\Psi x} + 2v_x\Phi_{vx} + 2\Psi_x v_x(\phi_{v\Psi} - \xi_{vx}) \\ 2\Psi_x v_x(\varphi_{v\Psi} - \xi_{vx}) + 2\Psi_x\Psi_z\lambda_{x\Psi} - 2\Psi_x\Psi_t\gamma_{\Psi x} \\ - 2\Psi_x\Psi_z\lambda_{x\Psi} - 2\Psi_x\Psi_t\gamma_{\Psi x} \\ - 2v_x\Psi_y\eta_{xv} - 2\Psi_z v_x\lambda_{xv} - 2v_x\Psi_t\gamma_{xv} - 3\Psi_x\Psi_{xx}\xi_\Psi \\ \Psi_{xx}\Psi_y\eta_\Psi - \Psi_{xx}\Psi_z\lambda_\Psi + \Psi_{xx}\Psi_t\gamma_\Psi - 2\Psi_{xx}v_x\xi_v \\ - 2\Psi_x\Psi_{xy}\eta_v - 2\Psi_x\Psi_{xz}\lambda_\Psi - 2v_x\Psi_{xz}\eta_v - 2\Psi_x\Psi_{xt}\gamma_\Psi \\ - \Psi_x^3\xi_{\Psi\Psi} - \Psi_x^2\Psi_y\eta_{\Psi\Psi} - \Psi_x^2\Psi_z\lambda_{\Psi\Psi} \\ - \Psi_x^2\Psi_t\gamma_{\Psi\Psi} - 2\Psi_x^2 v_x\xi_{v\Psi} - 2v_x\Psi_x\Psi_y\eta_{v\Psi} \\ - \Psi_x\Psi_z v_x\lambda_{v\Psi} - 2v_x\Psi_x\Psi_t\gamma_{v\Psi} + v_x^2\Phi_{vv} \\ - \Psi_x v_x^2\xi_{vv} - v_x^2\Psi_y\eta_{vv} - v_x^2\Psi_t\gamma_{vv} + v_{xx}\Phi_v \\ - \Psi_x v_{xx}\xi_v - v_{xx}\Psi_y\eta_v - \Psi_z v_{xx}\lambda_v - v_{xx}\Psi_t\gamma_v \end{cases}$$

$$\theta^{(0,2,0,0)} = \begin{cases} \Phi_{yy} - \Psi_x\xi_{yy} + \Psi_y(2\Phi_{\Psi y} - \eta_{yy}) - \Psi_z\lambda_{yy} - \Psi_t\gamma_{yy} \\ - 2\Psi_x\Psi_y\xi_{\Psi y} + \Psi_y^2(\Phi_{\Psi\Psi} - 2\eta_{\Psi y}) - 2\Psi_y\Psi_z\lambda_{\Psi y} \\ - 2\Psi_t\Psi_y\gamma_{\Psi y} - 2\Psi_{xy}\xi_y + \Psi_{yy}(\Phi_\Psi - 2\eta_y) - 2\Psi_{zy}\lambda_y \\ - 2\Psi_{ty}\gamma_y + 2v_y\Phi_{vy} - 2\Psi_x v_y\xi_{vy} + 2\Psi_y v_y(\Phi_{v\Psi} - \eta_{vy}) \\ - 2\Psi_z v_y\lambda_{vy} - 2\Psi_t v_y\gamma_{vy} - 2\Psi_x\Psi_y^2\xi_{\Psi\Psi} - \Psi_y^3\eta_{\Psi\Psi} \\ - \Psi_y^2\Psi_z\lambda_{\Psi\Psi} - \Psi_t\Psi_y^2\gamma_{\Psi\Psi} - 2\Psi_y\Psi_{xy}\xi_\Psi - 3\Psi_y\Psi_{yy}\eta_\Psi \\ - 2\Psi_y\Psi_{xy}\lambda_\Psi - 2\Psi_y\Psi_{ty}\gamma_\Psi - 2\Psi_x\Psi_y v_y\xi_{v\Psi} - 2v_y\Psi_y^2\Phi_{\Psi v} \\ - 2\Psi_y\Psi_z v_y\lambda_{v\Psi} - 2\Psi_y\Psi_t v_y\gamma_{\Psi v} - 2\Psi_{xy}v_y\xi_v - \Psi_x\Psi_{yy}\xi_\Psi \\ - \Psi_{yy}\Psi_z\lambda_\Psi - \Psi_{yy}\Psi_t\gamma_\Psi - 2\Psi_{yy}v_y\eta_v - 2\Psi_{xy}v_y\lambda_v - v_y\Psi_{ty}\gamma_v \\ + v_y^2\Phi_{vv} - v_y^2\Psi_x\xi_{vv} - v_y^2\Psi_y\eta_{vv} - \Psi_z v_y^2\lambda_{vv} - \Psi_t v_y^2\gamma_{vv} - \Psi_z v_{yy}\lambda_v \\ - \Psi_t v_{yy}\gamma_v \end{cases}$$

$$\theta^{(0,0,0,2)} = \begin{cases} \Phi_{zz} - \Psi_x\xi_{zz} - \Psi_y\eta_{zz} + \Psi_z(2\Phi\Psi_z - \lambda_{zz}) - \Psi_t\gamma_{zz} \\ -2\Psi_y\Psi_z\eta\Psi_z + \Psi_z^2(\theta_\psi - 2\lambda\Psi_z - 2\lambda\Psi_z) - 2\Psi_z\Psi_t\gamma\Psi_z \\ -2\Psi_{xz}\xi_z - 2\Psi_{yz}\eta_z + \Psi_{zz}(\Phi\Psi - 2\lambda_z) - 2\Psi_{zt}\gamma_z \\ +2v_z\Phi_{vz} - 2v_z\Psi_x\xi_{vz} - 2v_z\Psi_y\eta_{vz} + 2\Psi_z v_z(\Phi_{v\Psi} - \lambda_{vz}) \\ -2v_z\Psi_t\gamma_{vz} - \Psi_z^2\Psi_x\xi_{\Psi\Psi} - \Psi_z^2\Psi_y\eta_{\Psi\Psi} - \Psi_z^3\lambda_{\Psi\Psi} - \Psi_z^2\Psi_t\gamma_{\Psi\Psi} \\ -2\Psi_z\Psi_{xz}\xi_\Psi - 2\Psi_z\Psi_{yz}\eta_\Psi - 3\Psi_z\Psi_{zz}\lambda_\Psi - 2\Psi_z\Psi_{zt}\gamma_\Psi \\ -2v_z\Psi_z\Psi_x\xi_{\Psi v} - 2\Psi_z v_z\Psi_y\eta_{v\Psi} - 2v_z\Psi_{xz}\xi_v - 2\Psi Z\Psi_y\eta_{v\Psi} \\ -2v_z\Psi_{xz}\xi_v - 2v_z\Psi_{yz}\eta_v - \Psi_{zz}\Psi_x\xi_\Psi - \Psi_{zz}\Psi_y\eta_\Psi - \Psi_{zz}\Psi_t\gamma_\Psi \\ -2v_z\Psi_{zz}\lambda_v - 2v_z\Psi_{zt}\gamma_v + v_z^2\Phi_{yy} - v_z^2\Psi_x\xi_{vv} - v_z^2\Psi_y\eta_{vv} \\ -v_z^2\Psi_z\lambda_{vv} - v_z^2\Psi_t\gamma_{vv} + v_{zz}\Phi_v - v_{zz}\Psi_x\xi_v - v_{zz}\Psi_y\eta_v \\ -v_{zz}\Psi_z\lambda_v - v_{zz}\Psi_t\gamma_v \end{cases}$$

from (3.5) and (3.6) the coefficient functions $\{\xi, \eta, \lambda, \gamma, \phi, \Phi\}$ satisfy the symmetry equations

$$2\Phi_{\psi x} - \xi_{xx} - \xi_{yy} - \xi_{zz} + iv\psi v\psi(\psi_x - \xi_\Psi) - i\xi_t = 0 \quad (3.7a)$$

$$2\Phi_{y\Psi} - \eta_{xx} - \eta_{yy} - \eta_{zz} + iv\Psi v\Psi(\psi_y - \eta\Psi) - i\eta_t = 0 \quad (3.7b)$$

$$2\Phi_{z\Psi} - \lambda_{xx} - \lambda_{yy} - \lambda_{zz} + iv\Psi v\Psi(\psi_z - \lambda\Psi) - i\eta_t = 0 \quad (3.7c)$$

$$\gamma_t - 2\xi_x - i(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}) - iv\Psi v\Psi = 0 \quad (3.7d)$$

$$\gamma_t - 2\eta_y - i(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}) - iv\Psi v\Psi = 0 \quad (3.7e)$$

$$\gamma_t - 2\lambda_y - i(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}) - iv\Psi v\Psi = 0 \quad (3.7f$$

$$\eta_x - \xi_y = 0 \quad (3.7g)$$

$$\lambda_x - \xi_z = 0 \quad (3.7h)$$

$$\lambda_y - \eta_z = 0 \quad (3.7i)$$

$$\eta_\psi = \eta_v = \lambda_\psi = \lambda_v = \xi_\psi = \xi_v = \gamma_\psi = \gamma_v = \gamma_x = \gamma_y = \gamma_z = 0 \quad (3.7j)$$

$$\Phi_v + iv\Psi v_v = 0 \quad (3.7k)$$

$$\Phi_{xx} + \Phi_{yy} + \Phi_{zz} - v\Phi - \Psi\Phi + iv\Psi v(\gamma_{xx} + \gamma_{yy} + \gamma_{zz}) + i\Phi_t + v\Psi(\Phi_\psi - \gamma_t) + iv^2\Psi^2\gamma_\Psi = 0$$

the most general solutions to the set of equations (3.7) are given by

$$\begin{aligned} \xi &= \frac{1}{2}(c_1 t + c_2)x - c_{10}z - c_{11}y + c_4 t + c_5 \\ \eta &= \frac{1}{2}(c_1 t + c_2)y - c_{12}z - c_{11}y + c_6 t + c_7 \\ \lambda &= \frac{1}{2}(c_1 t + c_2)z - c_{10}z - c_{12}y + c_8 t + c_9 \\ \gamma &= c_1 t^2 + c_2 t + c_3 \\ \Phi &= \frac{1}{2}\left(\frac{1}{4}c_1(x^2 + y^2 + z^2) + c_4 x + c_6 y + c_8 z + c_0\right)\Psi \\ \phi &= -(c_1 t + c_2)v + \frac{3}{4}ic_1 \end{aligned} \quad (3.8)$$

where s are arbitrary real constants. Thus the maximal invariant infinitesimal algebra is spanned by the thirteen vector fields

$$\chi_1 = \partial_x$$
$$\chi_2 = \partial_y$$
$$\chi_3 = \partial_z$$
$$\chi_4 = \partial_t$$
$$\chi_5 = x\partial_x + y\partial_y + z\partial_z + 2t\partial_t - 2v\partial_v$$
$$\chi_6 = -z\partial_x + x\partial_z$$
$$\chi_7 = -y\partial_x + x\partial_y$$
$$\chi_8 = -z\partial_y + y\partial_z$$
$$\chi_9 = t\partial_x + \frac{i}{2}x\Psi\partial_v$$
$$\chi_{10} = t\partial_y + \frac{i}{2}y\Psi\partial_v \tag{3.9}$$
$$\chi_{11} = t\partial_z + \frac{i}{2}\Psi\partial_\Psi$$
$$\chi_{12} = tx\partial_z + ty\partial_y + tz\partial_z + t^2\partial_z + \frac{1}{4}\left(x^2 + y^2 + z^2\right)\Psi\partial_\Psi + \left(\frac{3}{2}i - 2tv\right)\partial_v$$
$$\chi_{13} = \frac{i}{2}\Psi\partial_\Psi$$

The generators (3.9) satisfy the Lie commutation relations

$$[L,L] = L \quad [L,V] = V \quad [T,V] = P \quad [L,P] = P \quad [\chi_5,V] = V$$
$$[\chi_5,P] = P \quad [\chi_{5,T}] = T \quad [\chi_5,\chi_{12}] = \chi_{12} \quad [P,V] = \chi_{13} \quad [T,\chi_{12}] = \chi_5$$
$$[P,\chi_{12}] = V$$

with other commutation relations vanishing. Here

$$L_j = (r \times v)_j = \xi_{jik}\, x^i \partial_k$$
$$V_j = t\nabla_j + \frac{i}{2}x^j\Psi\partial_\Psi$$
$$P_i = \nabla_j$$
$$T = \partial_t$$

Therefore the vector field $\chi_i \quad i = 1,\ldots 13$ are closed under commutation and form a 13 dimensional Lie algebra.

The multiplication table for this algebra is

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_1
χ_1	0	0	0	0	χ_1	χ_3	χ_2	χ_{13}	0	0	0	χ_8	0
χ_2	0	0	0	0	χ_2	0	$-\chi_1$	0	χ_3	χ_{13}	0	χ_{10}	0
χ_3	0	0	0	0	χ_3	$-\chi_1$	0	0	$-\chi_2$	0	χ_{13}	χ_{11}	0
χ_4	0	0	0	0	$2\chi_4$	0	0	χ_1	0	χ_2	χ_3	χ_5	0
χ_5	$-\chi_1$	$-\chi_2$	$-\chi_3$	$-2\chi_4$	0	0	0	χ_8	0	χ_{10}	χ_{11}	$2\chi_{12}$	0
χ_6	$-\chi_3$	0	χ_1	0	0	0	χ_9	$-\chi_{11}$	$-\chi_7$	0	χ_8	0	0
χ_7	$-\chi_2$	χ_1	0	0	0	$-\chi_9$	0	$-\chi_{10}$	χ_6	χ_8	0	0	0
χ_8	$-\chi_{13}$	0	0	$-\chi_1$	$-\chi_8$	χ_{11}	χ_{10}	0	0	0	0	0	0
χ_9	0	$-\chi_3$	χ_2	0	0	χ_7	$-\chi_6$	0	0	$-\chi_{11}$	χ_{10}	0	0
χ_{10}	0	$-\chi_{13}$	0	$-\chi_2$	$-\chi_{10}$	0	$-\chi_8$	0	χ_{11}	0	0	0	0
χ_{11}	0	0	$-\chi_{13}$	$-\chi_3$	$-\chi_{11}$	$-\chi_8$	0	0	0	$-\chi_{10}$	0	0	0
χ_{12}	$-\chi_8$	$-\chi_{10}$	$-\chi_{11}$	$-\chi_5$	$-2\chi_{12}$	0	0	0	0	0	0	0	0
χ_{13}	0	0	0	0	0	0	0	0	0	0	0	0	0

The results above show that the maximal invariant symmetry algebra is projectable, non-abelian and non - solvable. The regular representation $R(\chi)$ of an arbitrary element $\chi = \sum_{i=1} a^i \chi_i$ in the vector space of this algebra is

χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{13}	χ_{12}
α^5	α^7	α^6						α^{12}	0	0	α^8	
$-\alpha^7$	α^5	α^8						0	α^{12}	0	α^{10}	
$-\alpha^6$	α^8	α^5						0	0	α^{12}	α^{11}	
α^9	α^{10}	α^{11}	$2\alpha^5$	α^{12}								
$-\alpha^1$	$-\alpha^5$	$-\alpha^3$	$-2\alpha^4$					α^8	α^{10}	α^{11}	0	$2\alpha^{12}$
α^3	0	$-\alpha^1$			0	$-\alpha^8$	α^7	α^{11}	0	$-\alpha^9$	0	
α^2	$-\alpha^1$	0			α^8	0	$-\alpha^8$	α^{10}	$-\alpha^9$	0	0	
0	α^3	α^2			$-\alpha^7$	α^6	0	0	α^{11}	$-\alpha^{10}$	0	
$-\alpha^4$	0	0						$-\alpha^5$	α^7	α^6	0	
0	$-\alpha^4$	0						$-\alpha^7$	$-\alpha^5$	α^9	0	
0	0	$-\alpha^4$						$-\alpha^6$	$-\alpha^9$	α^5	0	
0	0	0						0	0	0	0	
				$-\alpha^4$				$-\alpha^1$	α^2	$-\alpha^3$	0	$-2\alpha^5$

3.10

Where the basis vectors are obtained from (3.10) by setting

$$\alpha^i = 1,\ i = 1,2 \ldots 12 \quad \alpha^j = 0,\ j \neq i \text{ in turn.}$$

The Cartan Killing metric tensor g_{ij} of this algebra is non degenerate and is given by

$$g_{ij} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 14 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

From which we see that the algebra is non-compact and semi-simple. Actually this algebra is simple as can be seen by checking the multiplication table.

4.0 The Maximal Invariant Group

In general corresponding to a given Lie algebra, there may be more than one Lie group, but of all such Lie groups there is one and only one that is simply connected called the universal group, which in this case is the maximal invariant group of our system. This group is obtained using Taylor's exponentiation technique. Corresponding to an arbitrary element $\chi = \alpha^{i}\chi_{i}$ of the algebra is the group element $g \in G$ defined by

$$g = \exp(\alpha^{i}\chi_{i}) = \exp(\tau s^{i}\chi_{i})$$

where the last equality holds along the straight line $\alpha^{i}(\tau) = s^{i}\tau$, through the origin of the Lie algebra defined in terms of the parameter τ. Thus the one-parameter subgroups of the maximal groups corresponding to each vector field χ_i have their group actions defined by

$$G_1 = (x, y, z, t, v, \psi) \to (x + \lambda, y, z, t, v, \psi)$$

$$G_2 = (x, y, z, t, v, \psi) \to (x, y + \lambda, z, t, v, \psi)$$

$$G_3 = (x, y, z, t, v, \psi) \to (x, y, z + \lambda, t, v, \psi)$$

$$G_4 = (x, y, z, t, v, \psi) \to (x, y, z, t + \lambda, v, \psi)$$

$$G_5 = (x, y, z, t, v, \psi) \to (e^{\lambda}x, e^{\lambda}y, e^{\lambda}z, e^{-2\lambda}t, e^{-2\lambda}v, \psi)$$

$$G_6 = (x, y, z, t, v, \psi) \to (x\cos\lambda - z\sin\lambda, y, z\cos\lambda + x\sin\lambda, t, v, \psi)$$

$$G_7 = (x, y, z, t, v, \psi) \to (x, y\cos\lambda - z\sin\lambda, z\cos\lambda + y\sin\lambda, t, v, \psi)$$

$$G_8 = (x, y, z, t, v, \psi) \to \left(x + \lambda t, y, z, t, v, \psi e^{\frac{i}{2}(x+\lambda^{2}t)}\right)$$

$$G_9 = (x, y, z, t, v, \psi) \to (x\cos\lambda - y\sin\lambda, y\cos\lambda + x\sin\lambda, z, t, v, \psi)$$

$$G_{10} = (x, y, z, t, v, \psi) \to \left(x, y + \lambda t, z, t, v, \psi e^{\frac{i}{2}(y\lambda+\lambda^{2}t)}\right)$$

$$G_{11} = (x,y,z,t,v,\psi) \to \left(x,y,z+\lambda t,t,v,\psi e^{\frac{i}{2}(z\lambda+\lambda^2 t)} \right)$$

$$G_{12} = (x,y,z,t,v,\psi) \to (e^{\lambda}tx, e^{\lambda}ty, e^{\lambda}tz, e^{2\lambda}t, e^{-2\lambda}v, e^{\lambda(x^2+y^2+z^2)}\psi)$$

$$G_{13} = (x,y,z,t,v,\psi) \to (x,y,z,t,v,e^{\frac{i}{2}\lambda}\psi)$$

5.0 Some Subgroups of the Maximal Invariant Symmetry Group

Consider the infinitesimal generators $\bar{p} = -i(\chi_1, \chi_2, \chi_3) = -i\bar{\nabla}$ with corresponding Lie group $\exp(-i\bar{\lambda}.\bar{\nabla})$. These are the generators of infinitesimal displacements in space, which forms a continuously connected three-parameter, non-compact Lie group, which is isomorphic to the unitary group. The fact that the components of $\bar{p}$ commute with each other leads directly to the conclusion that the space displacement group, is abelian. Also it can be shown that $[\bar{p}, H] = 0$. That is the generators of this group commutes with the Hamiltonian of the system and are therefore constants of the motion. This shows that the system possesses space-displacement symmetry for which the momentum and energy are conserved. Thus our system may be displaced in space and still be a possible physical system characterized by a constant and well defined value of the momentum, as well as of the energy. It is well known that a free electron possesses this symmetry, but we have been able to show that this symmetry exists for any electron under the influence of a conservative field. The generator, $T = i\chi_4$ with corresponding Lie group $\exp(i\lambda\partial_T)$ is the generator of infinitesimal time-displacement of the system. This shows that the system possesses time-displacement symmetry or invariance. Thus the system can be displaced in time and still be a physical system. The fact that T commutes with the Hamiltonian of the system is the quantum mechanical analog of the law of conservation of energy. Next consider the infinitesimal generators $\bar{L} = i(\chi_8, -\chi_6, \chi_7) = \bar{r} \times \bar{p}$ with corresponding Lie group $\exp i\bar{\lambda}.(\bar{r} \times \bar{\nabla})$. These are the generators of infinitesimal rotations. Since the generators do not commute, the group is a non abelian, continuously connected, three parameter group that is easily seen to be compact. The group is called $O(3)$, the orthogonal group in 3 dimensions. It can be shown that the rotation group is doubly connected. Since $[\bar{L}, H] = 0$, we can deduce that there is energy degeneracy in the system. Also the fact that $\bar{L}$ commutes with the Hamiltonian is the quantum mechanical analog of the law of conservation of angularmomentum. The group G_5 suggests that the system is invariant under space and time dilatations. This group is useful as it gives a special state of the system in which an exact value for all three components of the angular momentum $\bar{L}$ can be specified. Other subgroups of the universal covering group can be studied.

6.0 Concluding Remarks.

The group obtained by us is actually a subgroup of the special linear group, $SL(6,R)$ which has 35 generators. This subgroup of $SL(6,R)$ which has only 13 generators is not well known in the literature and cannot be readily named by us. However, we have shown that this group is semi-simple, non- abelian , non-solvable, simple connected , projectable , compact subgroup of the special linear group and it is a direct sum of other well-known groups of mathematical physics such as the rotation group, the unitary group and so on for which known symmetry operations can be deduced. However, there are other unknown or "hidden" symmetry operations, which are consequences of the invariance of this global group from which useful deductions can be inferred about the quantum mechanical system in general.

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