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AN APPLICATION OF HOMOGENEOUS AND NON-HOMOGENEOUS MARKOV CHAINS

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ABSTRACT

This paper presents an application of homogeneous and non-homogeneous Markov Chain models to projecting school enrolment structures. These models are similar to Markov chain models used in manpower planning. The methods used in estimating the parameters of manpower planing models are adapted to the data available on the flow of pupils into, through and out of the system. The work is illustrated with a case study.

1. INTRODUCTION

In this paper, we present an application of homogeneous and non-homogeneous Markov chain models to school enrolment projection. Previous projection model applies mainly homogeneous Markov chain model (Uche, 2000). The projection derived from homogeneous and non- homogeneous Markov chain models provide insight into future structure of the school system for prompt and adequate planning for the right number of teachers, school building and classrooms, and other physical facilities. The choice to apply the two models emanated from the fact that the enrolment figure in the primary school system is determined by a number of factors - parents/guardians willingness to register their children/wards, birth/death rate, environmental as well as cultural background-which are beyond the control of the school management board. The problem of projecting school enrolment is similar to that of projecting the structure of a manpower system.

The parameters of a school enrolment system may vary from time to time, hence, non- homogeneous Markov chain is introduced for the prediction of enrolment structure.

2. RELEVANT THEORY OF MARKOV CHAINS

A Markov chain is a sequence of discrete random variables $Y_0, Y_1, \ldots, Y_{t-1}, Y_t, Y_{t+1}$ having the property that given the value of Y_t for any time instant t, the probability distribution of Y_{t+1} is completely determined by Y_t and the values Y_{t-1},

$Y_{t-2}, \ldots$ at times earlier than t are irrelevant to its determination (Thierauf, 1985). In otherwords, if the present state of a Markov chain is known, we can determine the probability of any future state without reference to the previous states.

Mathematically, a stochastic process

$$Y_t,\ t = 0, 1, 2, \ldots$$

is a Markov chain if

$$P\{Y_{t+1} = j \mid Y_0 = k_0, y_1 = k_1, \ldots, Y_t = i\}$$
$$= P\{Y_{t+1} = j \mid Y_t = i\}$$

This conditional probability $P\{Y_{t+1} = j | y_t = i\}$ is called transition probability.

If for each i and j,

$P\{Y_{t+1} = j \mid Y_t = i\} = P_{ij}$ for all t = 0, 1, …, then one-step transition probabilities are called stationary transition probabilities. A Markov chain with a stationary transition probability matrix is called a homogeneous Markov chain. In a homogeneous Markov chain, the one-step transition probabilities are independent of time. When the transition probabilities depend on time t, we have a non- homogeneous Markov chain.

In a homogeneous Markov chain, $P(y_{t+1} = j | y_t = j)$ is independent of time t for all i and j. If we regard y_t $t \in N$ as the number of pupils in a class in a school system in session t, one can see that the inflow of pupils can be described by a markov model. Fredriksen (1976), Uche (2000) use flow model similar to homogeneous Markov chain to predict school enrolment structure.

The homogeneous Markov chain model for predicting the flow of pupils in a school system is given by

$$E_j\,(t+1) = \sum_{i=1}^{k} P_{ij} \;\; E_j(t) \;+ A_j\;(t+1) \qquad (1)$$

where k is the highest class in the system,

$E_j(t)$ the total enrolment figures in class j at the beginning of sessions t,
P_{ij} the probability that a pupil is promoted from class i to class j at the end of session t, $A_j(t+1)$ the number of pupils admitted into class j at the beginning of session t+1.

Also, authors like McClean et al (1998), Ugwuowo and McClean (2000), Vassiliou (1998), Papadopoulou (1998) apply non- homogeneous Markov chain to manpower system. Tsantas et al (1998) apply it to transition probabilities of an expected population structure.

In this study, the non- homogeneous Markov chain for one-step transition probabilities is given by

$$E_j(t+1) = P_{j-1,j}(t)\; E_{j-1}(t) + P_{jj}(t)\; E_j(t) + A_j(t+1) \qquad (2)$$

Here, classes in the school system are regarded as states of a Markov chain

3.0 APPLICATION

In this section we apply the two models specified in equations (1) and (2) to the enrolment data. First we state the assumptions of the models and then estimate their parameters. Thereafter, the models are used to predict future enrolment structure based on the data obtained from public primary school in a local government area in Nigeria.

3.1 ASSUMPTIONS OF MODELS

(i) In each model, there is no demotion and no double promotion. This implies $P_{ij} = 0$ for all $i > j$ and $i < j-1$

(ii) We assume that there are k classes in the school system and that promotion is not automatic.

(iii) Pupils do not repeat the highest class. Hence $P_{kk} = 0$.

(iv) In each model, wastage is negligible. This is due to the fact that there is no withdrawal of pupils by school authority as a result of academic incompetence.

3.2 MODELS

As a result of assumption (i), the homogeneous Markov chain model for predicting the numberof pupils becomes

$$E_j(t+1) = P_{j-1,j}\ E_j(t) + P_{j,j}\ E_j(t) + A_j(t+1) \quad (3)$$

$$j = 2, 3, \ldots, k$$

Total enrolment in class 1 is given by

$$E_1(t+1) = P_{11}E_1(t) + P_1E_1(t) \quad (4)$$

where P_1 is the probability of new intake into class 1, and P_{11} is the probability that a pupil repeats class 1.

The non- homogeneous Markov chain model is as given in equation (2) as

$$E_j\ (t+1) = P_{j-1,j}\ (t)E_{j-1}(t)\ + P_{j,j}(t)\ E_j(t) + A_j(t+1)$$

$$j = 2, 3, \ldots, k.$$

And the enrolment in class 1 is

$$E_1(t+1) = P_{11}(t)\ E_1(t) + P_1(t)\ E_1(t) \quad (5)$$

3.30 ESTIMATION OF MODEL PARAMETERS

Parameters of the model are estimated in this section. These include promotion probabilities $P_{j,j+1}$, probability that pupil repeats a class ($P_{j,j}$) and transfer probability $P[A_j(t+1)]$.

3.3.1 HOMOGENEOUS MARKOV CHAIN

In the homogeneous Markov chain, the $P_{j,j}$'s are estimated using the standard progression rate as suggested by Adeyemi (1998). This is given by

$$\hat{P}_{j,j+1} = \frac{1}{T}\sum_{t=1}^{T} \frac{\hat{E}_{j+1}(t+1)}{E_j(t)} \tag{6}$$

where T is the number of academic session data that are available, and

$$\frac{\hat{E}_{j+1}(t+1)}{E_j(t)} = \begin{cases} \dfrac{E_{j+1}(t+1)}{E_j(t)} & if \ E_{j+1}(t+1) < E_j(t) \\ 1 & if \ E_{j+1}(t+1) \geq E_{j+1}(t) \end{cases}$$

Since wastage is assumed zero,

$$\hat{P}_{jj} = 1 - \hat{P}_{j,j+1} \tag{7}$$

From equation (2), we have

$$A_j(t+1) = E_j(t+1) - \left[\hat{P}_{j-1,j}E_{j-1}(t) + \hat{P}_{jj}\ E_j(t)\right] for\ j = 2,3,\ldots,k\ ]$$

Let

$$V_j(t) = \hat{P}_{j-1,j}\ E_{j-1}(t) + \hat{P}_{jj}\ E_j(t)$$

be the expected number of pupils promoted to class j from class j-1 and to repeat class j at the end of session t.

Then

$$\begin{aligned} A_j\ (t+1) &= E_j(t+1) - V_j(t) \\ &= E_j{}'(t+1) \end{aligned} \tag{8}$$

where $E_j{}'(t+1)$ is the difference between the observed number of pupils in class j at time t +1 and the expected number of pupils in class j at the end of session t.

It follows from (8) that:

$$P\left[A_j(t+1)\right] = \frac{E_j{}'(t+1)}{V_j(t)} \tag{9}$$

is the probability that a pupil either transfer in or out of class j at the beginning of time t+1.

If $P[A_j(t+1)] < 0$, pupils transfer out of the public school. Otherwise, the pupil transfers into the public school.

Hence

$$\hat{P}[A_j(t+1)] = \frac{1}{T}\left|\sum_{t=1}^{T}\frac{E_j'(t+1)}{V_j(t)}\right| \qquad (10)$$

gives the estimate of the probability that a pupil transfers into class j
In this model, the probability of new intake is given by

$$P_j = \frac{1}{T}\sum_{t=1}^{T}\frac{E_j(t+1)}{E_j(t)} \qquad (11)$$

$P[A_j(t+1)] < 0$ is in line with Raghavendra (1991) that the probability estimates arising from equation (8) cannot be guaranteed to be in the range (0,1). This anomaly is corrected in this study using equation (10).

3.3.2 NON - HOMOGENEOUS MARKOV CHAIN

For the non-homogeneous Markov chain, we proceed as follows
Let

$$P_{j,j+1}(t) = \begin{cases} \dfrac{E_{j+1}(t+1)}{E_j(t)} & if\ E_{j+1}(t+1) > E_j(t) \\ 1 & if\ E_{j+1}(t+1) \geq E_j(t) \end{cases}$$

The estimator is given by

$$\hat{P}_{j-1j}(t) = \alpha + \beta/t. \text{ Now let } z = \frac{1}{t} \text{ for all } t > 0$$

Therefore, $\hat{P}_{j-1j}(t) = \alpha + \beta Z$ (12)

Where α and β are constants

The constants α and β are obtained from the data. First we set $\hat{\alpha} = \frac{1}{T}\sum_{t=1}^{T} P_{j-1,j}(t)$. Next we estimate β from $\hat{P}_{j-1,j}(t) = \hat{\alpha} + \beta/t$ using the method of least squares. This approach ensures that $0 \le \hat{P}_{j-1j}(t) \le 1$. Thus the $P_{j-1,j}(t)$ has the desirable properties of a probability.

The transformation $z = \frac{1}{t}$ tends to 0 as $t \to \infty$.

If in the computation of α using the least squares method α is found to be outside the range (0, 1) then fix α by using equation (6). With this, equation (12) becomes

$$\left.\begin{aligned} \hat{P}_{12}(t) &= 0.94 - 0.12z \quad -(a) \\ \hat{P}_{23}(t) &= 0.95 - 0.21z \quad -(b) \\ \hat{P}_{34}(t) &= 0.95 - 0.16z \quad -(c) \\ \hat{P}_{45}(t) &= 0.95 - 0.06z \quad -(d) \\ \hat{P}_{56}(t) &= 0.91 - 0.14z \quad -(e \end{aligned}\right\} \quad (14)$$

From equation (7), we have

$$\left.\begin{aligned} \hat{P}_{11}(t) &= 1 - \hat{P}_{12}(t) \quad -(a) \\ \hat{P}_{22}(t) &= 1 - \hat{P}_{23}(t) \quad -(b) \\ \hat{P}_{33}(t) &= 1 - \hat{P}_{34}(t) \quad -(c) \\ \hat{P}_{44}(t) &= 1 - \hat{P}_{45}(t) \quad -(d) \\ \hat{P}_{55}(t) &= 1 - \hat{P}_{56}(t) \quad -(e) \end{aligned}\right\} \quad (15)$$

The probability of new intake is given by

$$P_1(t) = 0.99 - 0.04z \quad (16)$$

The table below shows the enrolment structure of public primary school in a L.G.A. of Edo State, Nigeria.

Table 1

Public Primary School Enrolment Data in Ovia North-East L.G.A. (1997/98 to 1999/2000 sessions)

	Class						
Session	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
1997/98	3614	3238	3415	3087	3009	2939	19302
1998/99	3537	2955	2903	3038	2762	2584	17779
1999/2000	3553	3120	2972	2876	2810	2634	17965

Source: Local Government Education Authority (L.G.E.A). Ovia North-East: Statistics of Pupils Enrolment Population in Public Primary School in Edo State by School, Class and Sex.

The estimators in section 3.3.1 and 3.3.2 are applied to the data in table 1 and the results are presented in tables 2 and 3.

Note that for t = 0, $E_j(t)$ assumes the values for the base year. In this case of parametric estimation the base year is 1997/98.

Table 2

The estimates of parameters of Homogeneous Markov Chain.

Class j	$\hat{P}_{ij}$	$\hat{P}_{ii}$	$\hat{p}[A_j(t+1)]$	P_i
1	0.85	0.15	0.05	0.99
2	0.95	0.05	0.05	-
3	0.94	0.06	0.06	-
4	0.91	0.09	0.09	-
5	0.91	0.09	0.09	-
6	0.00	0.00	0.01	-

The Estimates of parameters of Non-Homogeneous Markov Chain are presented in the following tables:

Table (3a)

T	$\hat{P}_{12}$ (t)	$\hat{P}_{23}(t)$	$\hat{P}_{34}(t)$	$\hat{P}_{45}(t)$	$\hat{P}_{56}(t)$	$\hat{P}_1(t)$
1	0.82	0.74	0.78	0.89	0.77	0.95
2	0.88	0.85	0.86	0.92	0.84	0.97
3	0.90	0.88	0.89	0.93	0.86	0.98
4	0.91	0.90	0.90	0.94	0.88	0.98
5	0.92	0.91	0.91	0.94	0.88	0.98

Table (3b)

T	$\hat{P}_{11}(t)$	$\hat{P}_{22}(t)$	$\hat{P}_{33}(t)$	$\hat{P}_{44}(t)$	$\hat{P}_{55}(t)$
1	0.18	0.26	0.22	0.11	0.23
2	0.12	0.15	0.14	0.08	0.16
3	0.10	0.12	0.11	0.07	0.14
4	0.09	0.10	0.10	0.06	0.12
5	0.08	0.09	0.09	0.06	0.12

4. RESULTS [EXAMPLES OF PROJECTED FIGURE BASED ON NON- HOMOGENEOUS AND HOMOGENEOUS MARKOV CHAIN

To implement the homogeneous Markov Chain, we use the equation

$$E_j(t+1) = \hat{P}_{j-j,j}\ E_i(t) + \hat{P}_{jj}\ E_j(t) + P[A_j(t+1)]\,E_j\ (t)$$

With 1999/2000 session as the base year, we project the future structure of the school system for the next five years. The results are presented in table 4. Note that $E_j(t)$ equals the base year enrolment figure for t = 0 and for all j = 1,2, .., 6.

Table 4

Enrolment Prediction Based on Homogeneous Markov Chain.

Session	Time	Class j						
	t	1	2	3	4	5	6	Total
1999/2000	0	3553	3120	2972	2876	2810	2634	17965
2000/2001	1	4051	3335	3332	3328	3129	2584	19759
2001/2002	2	4619	3792	3571	3741	3608	2877	22208
2002/2003	3	5266	4322	4046	4027	4065	3317	25043
2003/2004	4	6004	4928	4610	4541	4394	3737	28214
2004/2005	5	6845	5618	5257	5169	4904	4039	31832

We use the system in equation (2) together with the estimate of parameters in tables 3(a) and 3(b) to implement the non- Homogeneous Markov Chain. Also, with 1999/2000 session as base year, we predict the enrolment structure of the primary school system for the next five years. The results are presented in table 5.

Table 5

Enrolment Prediction Based Non Homogeneous Markov Chain.

Session	Time	Class j						
	t	1	2	3	4	5	6	Total
1999/2000	0	3553	3120	2972	2876	2810	2634	17965
2000/2001	1	4051	3335	3332	3328	3129	2584	19759
2001/2002	2	4578	4399	3393	3232	4014	2434	22050
2002/2003	3	4990	4924	4468	3463	3948	3406	25199
2003/2004	4	5390	5336	5115	4599	4113	3430	27983
2004/2005	5	5768	5711	5633	5320	5251	3657	31340

5. DISCUSSION OF THE IMPLICATION OF THE RESULTS

The results in table (4) shows that 4039 pupils completed primary six in the 2004/2005 sessions. But the total enrolment in primary 1 in the base year (1999/2000) is 3553. This implies that the total number of pupils in primary 1 in 2004/2005 session increased by 13.7% compared to the base year figure for primary one. In this homogeneous markov chain (with stationary probabilities) it is only 2234 pupils representing 62% of primary 1 base year enrolment figure that successfully complete primary six in 2004/2005 session.

Also in the non -homogeneous markov chain (with time-dependent probabilities), table (5) shows that 3657 pupils completed primary six in the 2004/2005 session. Again, only 1573 pupils representing 44.3% of the primary 1 enrolment figure in 1999/2000 session make a successful transition from primary 1to primary 6 in 2004/2005 session.

The increase in the enrolment figure observed in the tables may be as a result of pupils repetition of classes as well as transfer of pupils from private and public school outside the local Government Area.

These results in tables (4) and (5) reveal a general upward trend in enrolment figure over the prediction period. The Universal Basic Education (UBE) in Nigeria might make this prediction a reality.

The implication of this continuous increase in enrolment revealed by this model is a continuous pressure on the limited school facilities like chairs, desks classroom, library, laboratory and workshop. This is a signal for the school heads, authorities and all other bodies concerned with educational planning (specifically in the primary level) to note and plan ahead of time.

6. CONCLUSION

This paper attempts the application of homogeneous and non-homogeneous Markov chain to school enrolment projection. We make certain assumptions and also provide estimators for the parameters of the model subject to the dictates of the available data. However, some of these assumptions, likes any other assumption, can introduce some errors into model prediction. Consequently, we suggest the type of records to be kept in the school system to include:

(i) Pupils repetition figure in each class and each session.
(ii) Pupils promotion figure from each class and each session.
(iii) Drop-outs voluntary withdrawals and death in each class and each session; and,
(iv) Transfer into and outside the school system in each class and each session.

Furthermore, the prediction figures in this study would however serve as basis for quantifying the requirement for teachers, school buildings and other educational facilities by the school heads and educational authorities.

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REFERENCES

Adeyemi, J. K. (1998). Analysis of Wastage Rate in Public School in former Oredo Local Government Area of Edo State. Benin Journal of Education Studies. Vol. 11, Nos 1 and 2, pp 92 - 100.

Fredriksen B. (1976). The use of flow models for estimating the future school enrolment in developing countries. In Methods of Projecting School Enrolment in Developing Countries. UNESCO Educational Statistics.

McClean S. I., Montgomery E and Ugwuowo F. (1998). Non-Homogeneous Continuous-Time Models and Semi - Markov Manpower models. Applied Stochastic Models and Data Analysis 13, pp 191-198.

Papadopoulou A. A. (1998). Counting Transition-Entrance Probabilities in Non-Homogeneous Semi- Markov Systems. Applied Stochastic Models and Data analysis 13, pp 199-206.

Raghavendra B. J. (1991). A Bivariate Model for Markov Manpower Planning Systems. J. Opl Res. Soc. Vol 42, No 7 pp 565-570.

Thierauf R. J. (1985). Management Science: A model Formulation Approach with Computer Applications. Charles E. Merill, New York.

Tsantas N. and Georgiou A. C. (1998). Partial Maintainability of a Population Model in a Stochastic Environment. Applied Stochastic Models and Data Analysis 13, pp 183-189.

Uche P. I. (2000). The Markovian Model of Educational Process. In Contributions to the Development of Mathematics in Nigeria (A. O. E Animalu, S. O. Iyahen and H. O. Tejumola, Eds) pp 235 - 245. National Mathematics Center, Abuja.

Ugwuowo F. and McClean S. I. (2000). Modelling Heterogeneity in a Manpower System. A review:- Applied Stochastic models in Business and Industry 16, pp 99-110.

Vassiliou P.-C.G. (1998). The Evolution of the Theory of Non- Homogeneous Markov System. Applied Stochastic Models and Data Analysis 13, pp 159-176.