J. Nig. Ass. Math. Phys
Vol. 6 (2002)

INTERVAL METHODS FOR SIMULTANEOUS INCLUSION OF ALL ZEROS OF A POLYNOMIAL

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ABSTRACT
In this paper, we present some interval methods for simultaneous determination of polynomial zeros in the spirit of Petkovic [1979]. The convergence of the methods is illustrated with numerical examples. It is our desire also to identify several new directions of modifications from which significant improvements in terms of convergence, accuracy, efficiency can be made possible, although with little over head computational complexity.

keywords: *polynomials, zeros, disks, circular arithmetic, convergence.*
CR. Categories : G 1.5

1. Introduction

Quite some useful methods for the simultaneous numerical inclusion of all polynomial zeros have been proposed. These methods include those of **Aberth** [1973], Alefeld and Herzberger [1974], Gargantini and Henrici [1972], **Petkovic** and Carstensen [1993], etc. However, a comprehensive reference list exist in Petkovic [1989]. In this consideration some more interval methods are presented for polynomial inclusion of zeros. Our analysis shall be realised in complex circular arithmetic, extension to rectangular arithmetic follows analogous rules. The complex circular interval $Z=\left\{z:\left|z-a\right| \leq r; a \in \mathbb{C}, r \in R, r \geq 0\right\}$ denoted as $Z=\left\{z;r\right\}$ is a matter of convention and when there is need we shall write further that $a=mid(Z); r=rad(Z)$. The rules on arithmetic operations {+, -, x, /} on complex circular intervals are found in Gargantini [1976]. In this regard, $Z_2 \subset Z_1$ if indeed $\left|z_1 - z_2\right| \leq r_1 - r_2$. The disks have nothing in common ($Z_2 \cap Z_1 = \phi(empty)$) on the contrary that $\left|z_1 - z_2\right| > r_1 + r_2$. Furthermore, the absolute value of a disk is $\left|Z\right| = \left\{\left|z^*\right|; z^* \in Z = \left\{z;r\right\}\right\}$. The following theorems have been found useful,

Theorem 1.1

Let $Z_j = \{z_j; r_j\}; j = 1(1)m$ *be circular intervals. Then*

$$\sum_{j=1}^{m} Z_j = \left\{ \sum_{j=1}^{m} z_j ; \sum_{j=1}^{m} r_j \right\} ; m \geq 2$$

What follows is one of the most fundamental statements of interval arithmetic.

Theorem 1.2 [Inclusion Monotonicity]

Let it be that $Z_j^{(1)} = \{z_j^{(1)}; r_j^{(1)}\}; Z_j^{(2)} = \{z_j^{(2)}; r_j^{(2)}\}; j = 1(1)m$ *are circular* ***intervals such that***

$Z_j^{(1)} \subset Z_j^{(2)}; j = 1(1)m$. *Then the inclusion monotonicity*

$$Z_1^{(1)} * Z_2^{(1)} * Z_3^{(1)} * \ldots * Z_m^{(1)} \subset Z_1^{(2)} * Z_2^{(2)} * Z_3^{(2)} * \ldots * Z_m^{(2)}$$

relation holds for *an arbitrary operation* $* \in \{+, -, x, /\}$

Deduce then the following

Theorem 1.3

Let it be that the circular intervals $Z_j^{(1)}, Z_j^{(2)}$ ***are*** *such* ***that*** $Z_j^{(1)} \subset Z_j^{(2)}; j = 1(1)m$. ***Then***

(a) $$\left\{ \sum_{j=1}^{m} Z_j^{(1)} \right\} \subset \left\{ \sum_{j=1}^{m} Z_j^{(2)} \right\}$$

(b) $$\left\{ \prod_{j=1}^{m} Z_j^{(1)} \right\} \subset \left\{ \prod_{j=1}^{m} Z_j^{(2)} \right\}$$

These theorems are found useful in the implementation of the methods to be considered.

2. Improved Modifications of Existing Methods

Useful methods are obtained from the modifications of existing interval methods by adopting a correction factor in an efficient manner, such methods are found worthy of practical interest. In this regard therefore, we present the following modifications of some existing methods considered in Petkovic [1989] by incorporation of a correction process. The basic methods along with Newton's, Halley's and Weierstrass's correction as the case may be have been analysed in Petkovic [1989]. The resultant methods are obtained from correction of the centre of the disks. We present here, methods that corrects the radius and centre of the disks simultaneously. For this purpose define that

$$Z_I = \left(Z_{I,1}, Z_{I,2}, \ldots, Z_{I,n}\right); Z_{I,i} = C_{I,m,i}\,; m = 1,2,3,4\,;$$

$$N(z) = \frac{P_n(z)}{P_n^{/}(z)} \tag{2.1}$$

then we present the following modifications:

Modified Borsch-Supan's Method:

$$Z_i^{(s+1)} = z_i^{(s)} - \frac{F(z_i^{(s)})}{1 - \sum_{\substack{i=1 \\ j\neq i}}^{n} \frac{F(z_j^{(s)}).}{z_j^{(s)} - Z_i^{(s)} + C_{I,m,i}^{(s)}}}; F(z_i^{(s)}) = \frac{P_n(z_i^{(s)})}{\prod_{\substack{i=1 \\ j\neq i}}^{n}\left(z_j^{(s)} - Z_i^{(s)}\right)}, m = 1,4 \tag{2.2}$$

with any of the choice correction

$$C_{I,1,i}^{(s)} = \frac{F(z_i^{(s)})}{1 - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{F(z_j^{(s)})}{z_j^{(s)} - Z_i^{(s)}}} \qquad C_{I,4,i}^{(s)} = \frac{P_n(z_i^{(s)})}{\prod_{\substack{j=1 \\ j\neq i}}^{n}\left(z_i^{(s)} - Z_j^{(s)}\right)}$$

Modified Maehly's Method:

$$Z_i^{(s+1)} = z_i^{(s)} - \frac{1}{\frac{1}{N\left(z_i^{(s)}\right)} - S_{1,i}\left(Z_I^{(s)}, Z_I^{(s)}\right)}; Z_{I,i}^{(s)} = Z_i^{(s)} - C_{I,2,i}^{(s)}; C_{I,2,i}^{(s)} = \frac{1}{\frac{1}{N\left(z_i^{(s)}\right)} - S_{1,i}\left(Z_I^{(s)}, Z_I^{(s)}\right)} \tag{2.3}$$

with

$$a = \left(a_1, a_2, \ldots, a_n\right), b = \left(b_1, b_2, \ldots, b_n\right)$$

$$S_{k,i}(a,b) = \sum_{j=1}^{i-1} \frac{1}{\left(z_i - a_j\right)^k} + \sum_{j=i+1}^{n} \frac{1}{\left(z_i - b_j\right)^k}; k \geq 1,$$

Modified Halley-like Method:

$$Z_i^{(s+1)} = z_i^{(s)} - \frac{1}{\frac{1}{H\left(z_i^{(s)}\right)} - \frac{N\left(z_i^{(s)}\right)}{2}\left[S_{1,i}^2\left(Z_I^{(s)}, Z_I^{(s)}\right) + S_{2,i}\left(Z_I^{(s)}, Z_I^{(s)}\right)\right]}; \tag{2.4}$$

$$Z_{I,i}^{(s)} = \begin{cases} Z_i^{(s)} - C_{I,3,i}^{(s)}; C_{I,3,i}^{(s)} = \dfrac{1}{\frac{1}{H\left(z_i^{(s)}\right)} - \frac{N\left(z_i^{(s)}\right)}{2}\left[S_{1,i}^2\left(Z_I^{(s)}, Z_I^{(s)}\right) + S_{2,i}\left(Z_I^{(s)}, Z_I^{(s)}\right)\right]} \text{ or} \\ Z_i^{(s)} - C_{I,2,i}^{(s)} \end{cases}$$

where

$$H(z) = \left(\frac{P_n'(z)}{P_n(z)} - \frac{P_n''(z)}{2P_n'(z)}\right)^{-1}$$

Modified Root-Iteration Method:

$$F_3\left(z_j^{(s)}\right) = \frac{P_n\left(z_j^{(s)}\right)}{\prod_{\substack{q=1\\q\neq j}}^{j-1}\left(z_j^{(s)} - z_q^{(s)}\right) \cdot \prod_{q=j+1}^{n}\left(z_j^{(s)} - z_q^{(s)}\right)}; j = i + 1(1)n$$

The over-head cost in the computational complexity of the modifications lies in the evaluation of the disk corrections,

$C_{l,m,i}$; $m = 1,2,3,4$. It is to be noted that no extra information is needed in their computations which makes the methods efficient. These modifications points to significant improvements in terms of convergence, accuracy, efficiency though with some over head in computational cost. Acceleration of convergence of these methods is possible by adopting any of the processes

Gauss-Seidel sense of updating
intersection of generated disk
correction of generated disk
combination of methods
***root refinement** means*

or its combination on the generated disk in each case of the algorithms. Further, practical methods are obtained from a clever process of combination of methods. These methods and more require investigation and analysis we hope to provide in future.

3. Hybrid Combination of Methods

Let $\left\{Z_j^{(0)}\right\}_{j=1}^{n}$ be initial circular disks isolating the simple zeros $\left\{\lambda_j\right\}_{j=1}^{n}$ respectively of $P_n(z)$. Let the approximations $\left\{z_j^{(m)}\right\}_{j=1}^{n}$ of the zeros be obtained by any known point method after m number of iterations. A combination of methods we present is

$$z_i^{(s+1)} = z_i^{(s)} - \frac{1}{\dfrac{1}{N\left(z_i^{(s)}\right)} - \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_i^{(s)} - z_j^{(s)} + C_{I,2,j}^{*(s)}}}; \quad i = 1(1)n; s = 0(1)m-1 \tag{3.1}$$

$$Z_i^{(m,1)} = z_i^{(m)} - \frac{1}{\dfrac{1}{H(z_i^{(m)})} - \dfrac{N(z_i^{(m)})}{2}\left[\left(\displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_j^{(m)} - Z_i^{(0)}}\right)^2 + \left(\displaystyle\sum_{\substack{i=1 \\ j\neq i}}^{n} \frac{1}{z_j^{(m)} - Z_i^{(0)}}\right)^2\right]}$$

where

$$C_{I,2,j}^{*(s)} = \frac{1}{\dfrac{1}{N\left(z_i^{(s)}\right)} - \displaystyle\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_i^{(s)} - z_j^{(s)}}}$$

By this process, interval iteration is applied only once and at the final iteration step. This minimises cost of interval iterations. Other useful combination methods can be derived similarly, but in a way that makes the computational process efficient. Let $\left\{Z_j^{(m,1)}\right\}_{j=1}^{n}$ be the generated inclusion disks from the interval algorithms (3.1) and $\left\{r_j^{(m,1)}\right\}_{j=1}^{n}$ the corresponding radii and let $r^{(m,1)} = \underset{j=1(1)n}{Max}\ r_j^{(m,1)}$. It can be estimated that $r^{(m,1)} = O\left(\left(r^{(0)}\right)^{(K_I-1)K_P^m+1}\right)$ with $K_I = 4$ as the order of the interval method and $K_P = 3$ is the same for the point method in (3.1). We hope to report the success or failure of our investigations of these methods and more and their extension to point arithmetic and cases of multiple/cluster roots in future. The result of (3.1) on the problem: Henrici (1974): P[1]

$$P_3(z) = z^3 - z^2 - 2z + 2 = (z-2)(z-1)(z+1);$$

$$Z_1^{(0)} = \{2.2;0.3\}, Z_1^{(0)} = \{0.9;0.2\}, Z_1^{(0)} = \{-0.9;0.3\}$$

Table (3.1): Numerical Results on P[1].

m	$Z_1^{(m,1)}$	$Z_2^{(m,1)}$	$Z_3^{(m,1)}$
1	{2.000000001; 4.592(-14)}	{0.999999999; 1.397(-15)}	{-0.999999482;4.413(-19)}
2	{2; 3.475(-28)}	{1; 3.078(-28)}	{-1;1.561(-22)}

is such that the maximum radius of the disks is inferior to 4.59236(-14) on first iteration, see table (3.1).

4. Numerical Experiments

To justify consideration of the algorithms we present some numerical results on the polynomial; Petkovic [1989,p.66], Petkovic and Carstensen [1993]

$$P[2]:\quad P_{n=9}(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300$$

$$Z_1^{(0)} = \{-3.2 + 0.2i; r\}; Z_2^{(0)} = \{-1.1 - 0.2i; r\}; Z_3^{(0)} = \{0.1 + 1.7i; r\}; Z_4^{(0)} = \{-1.9 + 1.3i; r\};$$

$$Z_5^{(0)} = \{-1.8 - 0.8i; r\}; Z_6^{(0)} = \{2.3 + 1.1i; r\}; Z_7^{(0)} = \{1.9 - 0.7i; r\}; Z_8^{(0)} = \{1.2 + 0.2i; r\};$$

$$Z_9^{(0)} = \{0.2 - 2.2i; r\};\quad r = 0.35;\ \lambda_j = -3, \pm 1, \pm 2i, \pm 2 \pm i;\ j = 1(1)9$$

Table (4.1): Errors from Numerical Results Using the Point Equivalent Methods of Section (2) on P[1].

$$E = \underset{j=1(1)n}{Max}\left\{\left|\lambda_j - z_j^{(s)}\right|\right\}$$

n	Method	s=1	s=2	s=3	Abbreviations
n=3	MM	2.13(-3)	7.51(-9)		MM:point Maehly's Method of (2.3)
	(2.3),m=2	4.58(-5)	1.11(-16)		
n=3	BSM	2.13(-3)	7.51(-9)		BSM: point Borsh-Supan's Method of (2.2)
	(2.2),m=1	2.96(-5)			-
n=3	RITM,k=2	3.842(-4)	5.773(-15)		RITM: point Root-Iteration Method of (2.5)
	(2.5),m=2	7.665(-6)	1.110(-16)		

Table (4.1): Errors from Numerical Results Using the Point Equivalent Methods of Section (2) on P[2].

$$E = \underset{j=1(1)n}{Max}\left\{\left|\lambda_j - z_j^{(s)}\right|\right\}$$

		Problem: P[2]		
n=9	MM	2.559(-2)	8.43(-6)	4.44(-16)
	(2.3),m=2	1.28(-3)	1.07(-15)	
n=9	BSM	2.55(-2)	8.43(-6)	
	(2.2),m=1	2.16(-3)	9.69(-15)	
n=9	RITM,k=2	7.600(-10)	7.960(-10)	
	(2.2),m=2	5.479(-4)	2.220(-16)	

We have implemented the point equivalent of some of the modified algorithms in section (3), see results in table (4.1). The basic methods are obtained from setting the corrections $C_{l,m,i}$; $m = 1,2,3,4$ to zero. The presentation adopts the convention that $A(-B)=A^{-B}$. The results obtained from the modifications can be observed to compare with thinat by Petkovic [1989] and improves the accuracy of the basic methods remarkably. This is evident from tables (4.1). Noticed that the zeros have been reasonably approximated after only two iterations by the new algorithms, this is expected because of their superior convergence arising from the refinement.

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