J. Nig. Ass. Math. Phys.
Vol. 6 (2002)

AN OPTIMAL ORDER CONTINUOUS MULTISTEP ALGORITHM FOR INITIAL VALUE PROBLEMS OF SPECIAL SECOND ORDER DIFFERNTIAL EQUATIONS

***D.O. AWOYEMI**
AND
S.J. KAYODE

Department Of Industrial Mathematics And Computer Science, Federal University Of Technology, Akure.

ABSTRACT.

An optimal symmetric continuous multi-step method is proposed in this paper for the solution of initial value problems of special second order ordinary differential equations. The method is based on collocation of the differential system at all the grid points and interpolation of the approximate solution at x_{n+j}, j = 0, 1, 2, 3. The procedure yields a consistent scheme of order eight with moderately large interval of absolute stability for non-stiff special second order odes. Also, a number of explicit schemes are developed for evaluation of y_{n+j}, j = 1, 2, 3,4, in the main method. The efficiency of the scheme is compared with the work of Awoyemi[1993].

Keywords: Collocation, differential system, grid points, interval of absolute stability, explicit schemes.

CR. Categories: G1.7

* *Author to communicate concerning this paper.*

1. INTRODUCTION

Mathematical models of mechanical systems without diss...on, satellite tracking and celestial mechanics often result into special second order initial value problems of ordinary differential equations of the form

$$y'' = f(x, y); y(x_0) = y_0, y'(x_o) = \tau \qquad (1.1)$$

The theoretical solutions of problems (1.1) are usually highly oscillatory, which put a restriction on the stepsize (h) of the conventional Linear Multistep Method (LMM)

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=o}^{k} \beta_j f_{n+j}; \qquad \textbf{(1.2)}$$

A lot of efforts by eminent numerical analysts has been expended on developing numerical methods for solving problems (1.1). Prominent among these scholars are Dahquist [1978]; Fatunla [1988]; Jeftschi [1978]; Lambert [1973]; Lambert and Watson [1976] and Jain et al [1984]. However, the different procedures adopted by them yielded methods that are not continuous and therefore made it not possible to obtain first and higher order derivatives of y with respect to x. As a result of this, the scope of this class of methods is limited in application.

Awoyemi [1993] developed a continuous method for problem (1.1). The results obtained showed a scheme of order 6 and principal error term $c_{p+q} = 0.068254$ for $k = 4$.

In this paper, we propose a collocation algorithm, which yields an order 8 scheme with continuous coefficients for solving problem (1.1).

2. DEVELOPMENT OF THE METHOD

We propose a basis function in the form

$$y(x) = \sum_{i=0}^{2k} a_i x^i \qquad (2.1)$$

to be an approximate solution to (1.1), and its first and second derivatives as

$$y^1(x) = \sum_{j=1}^{2k} j a_j x^{j-1} \qquad (2.2)$$

$$y^{11}(x) = \sum_{j=2}^{2k} j(j-1) a_j x^{j-2} \qquad (2.3)$$

Collocating the differential system (2.3) at all grid points and interpolating the approximate solution (2.1) at x_{n+i}, $i = 0(1)3$, yield

$$\sum_{j=2}^{2k} j(j-1) a_j x_{n+i}^{j-2} = f_{n+i}; \quad i = 0(1)4 \qquad (2.4)$$

$$\sum_{j=0}^{2k} a_j x_{n+i}^{j} = y_{n+i}; \quad i = 0(1)3, \qquad (2.5)$$

where $f_{n+i} = f(x_{n+i}, y_{n+i})$ and $y_{n+i} = y(x_{n+i})$, $i = 0(1)4$.

Solving (2.4) and (2.5) for a_j's and substituting their values into (2.1), we, after some algebraic evaluation, obtained a continuous scheme of the form

$$y_k(x) = \sum_{j=o}^{k-1} \alpha_j y_{n+j} + h^2 \sum_{j=o}^{k} \beta_j f_{n+j} \qquad (2.6)$$

We put

$t = (x - x_{n+k-1})/h,$

in (2.6) to obtain power series for the coefficients α_j and β_j , j = 0, 1, 2, 3, 4 to be as follows.

$$\alpha_{3}(t) = \frac{1}{1302}\{1302 + 6131t - 11508t^3 - 6559t^4 + 1995t^5 + 2506t^6 + 694t^7 + 63t^8\}$$

$$\alpha_{2}(t) = -\frac{1}{1302}\{9720t - 20412t^3 - 12033t^4 + 3339t^5 + 4578t^6 + 1326t^7 + 126t^8\}$$

$$\alpha_{1}(t) = \frac{1}{1302}\{2349t - 6300t^3 - 4389t^4 + 693t^5 + 1638t^6 + 570t^7 + 63t^8\}$$

$$\alpha_{0}(t) = \frac{1}{126}\{120t - 252t^3 - 105t^4 + 63t^5 + 42t^6 + 6t^7\}$$

$$\beta_{4}(t) = \frac{1}{312480}\{1512t - 1092t^3 + 4291t^4 + 6552t^5 + 3374t^6 + 756t^7 + 63t^8\}$$

$$\beta_{3}(t) \quad -\frac{1}{156240}\{9216t \quad 78120t^2 - 165004t^3 + 70028t^4 + 31752t^5 + 31808t^6 + 8452t^7 + 756t^8\}$$

$$\beta_{2}(t) = -\frac{1}{156240}\{441288t + 1095444t^3 - 647073t^4 + 190260t^5 + 244818t^6 + 67752t^7 + 6111t^8\}$$

$$\beta_{1}(t) = -\frac{1}{156240}\{192240t \quad 416724t^3 \quad 191548t^4 + 94248t^5 + 75208t^6 + 14652t^7 + 756t^8\}$$

$$\beta_{0}(t) \quad -\frac{1}{312480}\{16344t \quad 33628t^3 \quad 12971t^4 + 9072t^5 + 5306t^6 + 484t^7 - 63t^8\} \qquad (2.8)$$

Using (2.8) to evaluate (2.6) for t = 1 yields a symmetric scheme

$$31y_{n+4} + 128y_{n+3} - 318y_{n+2} + 128y_{n+1} + 31y_n$$

$$= \frac{h^2}{15}\{23f_{n+4} + 688f_{n+3} + 2358f_{n+2} + 688f_{n+1} + 23f_n\} \tag{2.9}$$

The scheme (2.9) has order p = 8 and the principal error constant

$$c_{p+2} = -\frac{79}{18900}.$$

3. REGION OF ABSOLUTE STABILITY OF THE METHOD

In determining the region of absolute stability of the scheme (2.9), we adopt the boundary locus method (Lambert [1973]) to have

$$h(\theta) = \frac{\rho(\exp(i\theta))}{\sigma(\exp(i\theta))}; \tag{3.1}$$

where ρ and σ are the first and second characteristic polynomials respectively. This yields

$$\bar{h}(\theta) = \frac{15\{31(\exp(4i\theta) + 128\exp(3i\theta) - 318\exp(2i\theta) + 128\exp(i\theta) + 31\}}{\{23\exp(4i\theta) + 688\exp(3i\theta) + 2358\exp(2i\theta) + 688\exp(i\theta) + 23\}} \tag{3.2}$$

$= x(\theta) + iy(\theta)$

After some algebraic manipulation, we obtain

$$x(\theta) = \frac{15\{1426\cos 4\theta + 48544\cos 3\theta + 307696\cos 2\theta + 214624\cos\theta - 572290\}}{\{1058\cos 4\theta + 63296co3\theta + 1163624\cos 2\theta + 6552512\cos\theta + 6507910\}}$$

and $y(\theta) = 0$

The result is shown in the following tabulation

θ	0°	30°	60°	90°	120°	150°	180°
$x(\theta)$	0	- 0.2742	-1.0966	-2.4654	-4.3443	-6.4157	-7.4708

This shows from the table that method (2.9) has a region of absolute stability $x(\theta) =$ (-7.47, 0) which is contained in the negative x-axis.

4. DEVELOPMENT OF PREDICTORS

Two explicit schemes are developed to calculate y_{n+4} and y_{n+3} in the main method. The schemes are

$$y_{n+4} = -16y_{n+3} + 34y_{n+2} - 16y_{n+1} - y_n + \frac{h^2}{3}\{8f_{n+3} + 44f_{n+2} + 8f_{n+1}\} \quad (4.1)$$

of order p = 6, and

$$y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n + h^2\{f_{n+2} - f_{n+1}\} \quad (4.2)$$

of order p = 4.

To evaluate y_{n+j}, j=1 and 2, in equations (2.9), (4.1) and (4.2), we adopt **the** explicit sixth-order Runge-Kutta scheme (see Lambert [1973**]),**

$$y_{n+j} = y_{n+j-1} + \frac{h}{840}\{41k_1 + 216k_3 + 27k_4 + 272k_5 + 27k_6 + 216k_7 + 41k_8\} \quad (4..3)$$

where

$$k_1 = f(x_n, y_{n+j-1});$$

$$k_2 = f(x_n + \frac{h}{9}, y_{n+j-1} + \frac{h}{9}k_1)$$

$$k_3 = f\left\{x_n + \frac{h}{6}, y_{n+j-1} + \frac{h}{24}(k_1 + 3k_2)\right\}$$

$$k_4 = f\left\{x_n + \frac{h}{3}, y_{n+j-1} + \frac{h}{6}(k_1 - 3k_2 + 4k_3)\right\}$$

$$k_5 = f\left\{x_n + \frac{h}{2}, y_{n+j-1} + \frac{h}{8}(-5k_1 + 27k_2 - 2k_3 + 6k_4)\right\}$$

$$k_6 = f\left\{x_n + \frac{2h}{3}, y_{n+j-1} + \frac{h}{9}(221k_1 + 27k_2 - 24k_3 + 6k_4)\right\}$$

$$k_7 = f\left\{x_n + \frac{5h}{6}, y_{n+j-1} + \frac{h}{48}(-183k_1 + 678k_2 - 472k_3 - 66k_4 + 80k_5 + 3k_6)\right\}$$

$$k_8 = f\left\{x_n + h, y_{n+j-1} + \frac{h}{82}(716k_1 - 2079k_2 + 1002k_3 + 834k_4 - 454k_5 - 9k_6 + 72k_7)\right\}$$

5. NUMERICAL EXAMPLES

We use the method (2.9) to solve the following examples for stepsize(h) = 0.1. The errors arising from the computed and theoretical values at selected points are compared with Awoyemi [1993] as shown in Tables 1 and 2 below.

Examples

1. $y'' = 2y^3, y(1) = 1, y'(1) = -1,$

Theoretical solution: $y(x) = \frac{1}{x}$.

2. $y'' = x\exp 3x, y(o) = -3/32, y'(0) = -\frac{5}{32},$

Theoretical solution: $y(x) = (4x-3)/32\exp 3x$.

TABLE 1:Comparison of errors arising from Awoyemi [1993] and Method (2.9)

for Example1

x	Awoyemi [1993] for k= 4	New Method (2.9) for k = 4
1.4	0.21640329D-03	0.754729612140181D-12
1.5	0.12018949D-03	0.108468789505878D-11
1.6	0.68902514D-04	0.145328193923433D-11
1.7	0.40473926D-04	0.182087678268772D-11
1.8	0.24198064D-04	0.219868567796766D-11
1.9	0.14627792D-04	0.260103050208182D-11
2.0	0.88763457D-05	0.301053626472481D-11

TABLE 2: Comparison of errors arising from Awoyemi [1993] and **Method (2.9)**

for Example2

x	Awoyemi [1993] for k = 4	New Method (2.9) for k = 4
0.4	0.33100222D-03	0.398809874013075D-10
0.5	0.50793376D-03	0.740669747756329D-10
0.6	0.76815236D-03	0.131276323145357D-09
0.7	0.11482788D-02	0.223987939307335D-09
0.8	0.17003636D-02	0.372029518302952D-09
0.9	0.24982013D-02	0.606370065270312D-09
1.0	0.36461735D-02	0.972470104443346D-09

6. CONCLUSION

An optimal four-step continuous multi-step method for general second order ordinary differential equations is developed. The method is of order eight with principal error constant $C_{p+2} = -79/18900$. Numerical results of the method are compared with the sixth-order method developed by Awoyemi [1993].

An examination of Tables 1 and 2 clearly reveals that the new method is more accurate than that of Awoyemi [1993] for k = 4. The method (2.9) is also symmetric and has a moderately large interval of absolute stability, hence it will be useful for non-stiff special second order ordinary differential equations.

REFERENCES

1. Awoyemi D.O. (1993) "On Some Continuous Linear Multistep Methods for Initial Value Problems" Ph.D Thesis (Unpublished), University of Ilorin, Ilorin, Nigeria.

2. Dalquist G. (1978) "On Accuracy and Unconditional Stability of Linear Equations".BIT 18, 133-136

3. Fatunla, S.O. (1988) Numerical Methods for IVPs in Ordinary Differential Equations. Academic Press Inc. Harcourt Brace Jovanovich Publishers, N.York

4. Jain R.K, Kambo N.S and Rakesh G. (1984) "A Sixth Order P-Stable Symmetric Multistep Method for Periodic Initial Value Problems of Second Order Differential Equations" IMA Journal of Numerical Analysis 4, 117-125.

5 Jeftsch R. (1978) "Complete Characterization of Multistep Methods with an Interval of Periodicity for Solving $y'' = f(x, y)$". Mathematics of Computation 32, 1108-1114

6. Lambert, J.D. (1973) Computational methods in ODEs. John Wiley & Sons. N York

7. Lambert, J.D and Watson A. (1976) "Symmetric Multistep Methods for Periodic Initial Value Problems". Journal of the Institute of Mathematics Applications 18, 189-202.