J. Nig. Ass. Math. Phys
Vol. 6 (2002)

LONGITUDINAL SHEAR DEFORMATION OF A COMPOSITE CYLINDER SECTIONALLY LOADED ACROSS INTERFACE EDGES

JAMES N NNADI
Department of Mathematics, Abia State University
Uturu Nigeria

ABSTRACT

The study concerns the effect of subjection of a composite cylinder of radius $r = a$, to shear loads that can be adjusted, to spread or reduce, over a subsection of the lateral surface by variation of an angle 2β $(0 \le \beta \le \frac{\pi}{2})$ situated at the center of a typical cross section of the cylinder. The interface is perpendicular to the real axis and constitutes a line about which the loaded segments are symmetric. The general forms of the stress along the interface are shown to depend on material constants. The stresses

$$\sigma_{i0z}(r, \pm\frac{\pi}{2}), i = 1, 2, \quad \frac{(1-\epsilon)}{1+\epsilon} \le \frac{r}{a} \le 1, \quad \epsilon = \frac{\sin\beta}{1+\cos\beta}$$

which can induce cracks, are analysed in a graph to understand their distribution as the length of a load site is compared with the semi-interface length, a. It is deduced that results for the special case when $\epsilon = 1$ agrees with those obtained when the entire lateral surface is loaded.

1. INTRODUCTION

issimilar semi – circular solids are used in composing a long cylinder with perfectly bonded interface. Shear loads are applied across each end of the interface on minor segments situated symmetrically, relative to the interface, so that equal segments on each material is loaded. The stressed segments jointly subtend an angle, 2β $(0 \le \beta \le \frac{\pi}{2})$ at the center of the circular cross–sectional surface : $y = r\sin\theta$, $x = r\cos\theta$, $r \le a$, $-\pi < \theta \le \pi$.

The length of a loaded segment varies with β so that a load site can be made smaller (or greater) than corresponding length of the semi – interface, a, Fig I. We seek the character of the deformation fields under these conditions.

Composite materials have been studied by several authors using various techniques (see for example Cook and Erdogan [1], Zhang and Hasebe [2]

Herrmann et al [3]). Here, a typical circular cross-section is conformally mapped onto a right half – plane and Mellin transform is then applied.

2. Main Equations

Let μ_i, $i = 1, 2$ denote the shear modulus of each material which due to the loads of magnitude T experiences displacements $w_i(r, \theta)$. The fields are then governed by the following equations:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)w_i(r,\theta)=0 \quad , \quad r \le a \, , \quad -\pi \le \theta \le \pi \quad i = 1 \, ,$$

$$w_1\left(r, \pm\frac{\pi}{2}\right) = w_2\left(r, \pm\frac{\pi}{2}\right) \qquad (2a)$$

$$\mu_1 \frac{\partial w_1}{\partial \theta}\left(r, \pm\frac{\pi}{2}\right) = \mu_2 \frac{\partial w_2}{\partial \theta}\left(r, \pm\frac{\pi}{2}\right) \qquad (2b)$$

$$\frac{\partial w_1}{\partial r}(a, \theta) = \begin{cases} \dfrac{T}{\mu_1} \, , & \beta + \dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2} \qquad (2c) \\ -\dfrac{T}{\mu_1} \, , & -\beta - \dfrac{\pi}{2} \le \theta \le -\dfrac{\pi}{2} \qquad (2d) \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial w_2}{\partial r}(a, \theta) = \begin{cases} -\dfrac{T}{\mu_2} \, , & -\dfrac{\pi}{2} \le \theta \le \beta - \dfrac{\pi}{2} \qquad (2e) \\ \dfrac{T}{\mu_2} \, , & \dfrac{\pi}{2} - \beta \le \theta \le \pi \qquad (2f) \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma_{\theta z}(r,\theta) = \frac{\mu_i}{r}\frac{\partial w_i}{\partial \theta}(r,\theta) \quad , \quad \sigma_{rz}(r,\theta) = \mu_i \frac{\partial w_i}{\partial r}(r,\theta) \quad , \quad i = 1 \, , \, 2 \qquad (3)$$

3. Solution Technique

Our technique of solving (1) subject to (2) requires the conformal transformation of the original plane of analysis onto a right half plane, Fig II, through the conformal mapping function

$$\lambda(z) = \frac{a + iz}{a - iz} \, , \; z = x + iy \qquad (4)$$

Writing $\lambda(z) = u(r, \theta) + i\, v(r, \theta)$ and choosing polar coordinates (ρ, ϕ) for the λ-plane so that $\lambda(z) = \rho e^{i\phi}$ we get $u = \rho \cos\phi$, $v = \rho \sin\phi$ where

$$u(r, \theta) = \frac{a^2 - r^2}{a^2 + 2ar\sin\theta + r^2}, \; v(r,\theta) = \frac{2ar\cos\theta}{a^2 + 2ar\sin\theta + r^2}$$

Hence

$$\tan\phi(r, \theta) = \frac{2ar\cos\theta}{a^2 - r^2} \qquad (5)$$

$$\rho(r, \theta) = \{u^2(r, \theta) + v^2(r, \theta)\}^{1/2} \qquad (6)$$

Consequently

$$\frac{\partial\phi}{\partial\theta}\left(r, \pm\frac{\pi}{2}\right) = \frac{\pm 2ar}{a^2 - r^2} \, , \; r < a \qquad \frac{\partial\phi}{\partial\theta}(a \, , \theta) \quad \frac{1}{a\cos\theta}$$

$\frac{\partial\rho}{\partial\theta}(r, \pm\frac{\pi}{2}) = 0$ and $\frac{\partial\rho}{\partial\theta}(a, \theta) = 0$

Also $\rho(r, \frac{\pi}{2}) < 1$ while $\rho(r, -\frac{\pi}{2}) > 1$, Fig II. From (6) we get

$$\rho^2(a, \theta) = \left(\frac{\cos\theta}{1+\sin\theta}\right)^2 \quad \text{since } \rho \text{ must be positive.}$$

Considering $\rho(a, \theta)$ for the points $D(a, \frac{\pi}{2} - \beta)$, $F(a, \frac{\pi}{2} + \beta)$, $H(a, -\frac{\pi}{2} - \beta)$, and $B(a, -\frac{\pi}{2} + \beta)$, (Fig I) we define

$$\delta = \frac{\sin\beta}{1 - \cos\beta} \qquad \in = \frac{\sin\beta}{1 + \cos\beta} \qquad (7a, b)$$

Then

$$1 \leq \delta \leq \infty \ , \qquad 0 \leq \in \leq 1, \qquad \in\delta = 1 \ (\beta \neq 0) \qquad (8a, b, c)$$

At the boundaries we have the relations

$$\frac{\partial w_i}{\partial\theta}(a, \theta) = \frac{\partial W_i}{\partial\phi}(\rho, \pm\frac{\pi}{2}) \frac{\partial\phi}{\partial\theta}(a, \theta) \ , \quad i = 1, 2 \qquad (9)$$

where θ is specified in (2c) – (2f) and

$W_i(\rho, \phi)$, $i = 1, 2$ are λ - plane displacements corresponding to $w_i(r, \theta)$.
Along the common boundary, one set of relations is given by chain rule as

$$\frac{\partial w_i}{\partial\theta}(r, \pm\frac{\pi}{2}) = \frac{\partial W_i}{\partial\phi}(\rho, 0) \frac{\partial\phi}{\partial\theta}(r, \pm\frac{\pi}{2}) \ , r < a, \rho > 0 \qquad (10)$$

From (4)

$$\lambda(a, \theta) = \frac{1 + ie^{i\theta}}{1 - ie^{i\theta}} = \rho(a, \theta)\, e^{i\theta}, \quad -\pi < \theta \leq \pi \ , \quad \phi = , \pm\frac{\pi}{2}$$

Referring to Fig I and Fig II, the segments AB, DE, EF, HA correspond and determine coordinate relations given by

$$AB: \ ie^{i\theta} = (i\rho - 1)(i\rho + 1)^{-1} \ , \quad \phi = \frac{\pi}{2} \ , \rho > \delta$$

$$DE: \ ie^{i\theta} = (i\rho - 1)(i\rho - 1)^{-1} \ , \quad \phi = \frac{\pi}{2} \ , \rho < \in$$

$$EF: \ ie^{i\theta} = (i\rho + 1)(i\rho - 1)^{-1} \ , \quad \phi = -\frac{\pi}{2} \ , \rho < \in$$

$$HA: \ ie^{i\theta} = (i\rho + 1)(i\rho + 1)^{-1} \ , \quad \phi = -\frac{\pi}{2} \ , \rho > \delta$$

These expressions yield the form which $\frac{\partial\phi}{\partial r}(a, \theta)$ *takes, and which taken together with (2a) – (2f) and (9) give the boundary conditions when* $\phi = \pm\frac{\pi}{2}$.

The effect of the transformation is that

$W_i(\rho, \phi)$, $i = 1, 2$, are to be sought for in the problem:

In deriving (17b), the behaviour of the fields for $0 < \rho < \epsilon$ require that Res > - 1 while it is required that Res < 1 for $\rho > \delta$. We note that, for integral values of s, which are not poles, each series in (17b) is absolutely convergent.

Now, considering the solution of (16) given by

$$\overline{W}_i(s,\phi) = A_i(s)\sin s\phi + B_i(s)\cos s\phi \ , \ i=1,2 \tag{19}$$

we substitute (18a) into (19) to get (20)

$$B_1(s) = B_2(s)$$

Substituting (18b) into (19) gives (21)

$$\mu_1 A_1(s) = \mu_2 A_2(s)$$

Applying (17a) to (19) yields

$$A_1(s)\cos\frac{\pi}{2}s + B_1(s)\sin\frac{\pi}{2}s = -\frac{2aT}{\mu_1 s}h(s)$$

$$A_2(s)\cos\frac{\pi}{2}s - B_2(s)\sin\frac{\pi}{2}s = -\frac{2aT}{\mu_2 s}h(s)$$

When (20) is considered, we see that

$$A_1(s) + A_2(s) = -\frac{2aT}{\mu_1\mu_2}(\mu_1-\mu_2)\frac{h(s)}{s\cos\frac{\pi}{2}s}$$

Incorporating the other consequence of continuity given by (21), it follows that

$$A_i(s) = \frac{2aT}{\mu_i}\left(\frac{(\mu_1-\mu_2)}{(\mu_1+\mu_2)}\right)h(s) \quad , \quad i=1,2$$

Therefore

$$B_1(s) = \frac{-4aT}{(\mu_1+\mu_2)}\cdot\frac{h(s)}{s\cos\frac{\pi}{2}s}$$

Substituting the results for $A_i(s)$ and $B_i(s)$ into (19), we obtain

$$\overline{W}_i(s,\phi) = 2aT\left\{\frac{\mu_1-\mu_2}{\mu_i(\mu_1+\mu_2)}\frac{\sin s\phi}{s\cos\frac{\pi}{2}} - \frac{2\ \mathrm{con}\ s\phi}{(\mu_1+\mu_2)s\ \sin\frac{\pi}{2}s}\right\}h(s) \ , \ i=1,2 \tag{22}$$

The formula for inverting the Mellin transform is defined by

$$W(\rho,\phi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\overline{W}_i(s,\phi)\rho^{-s}\,ds \ , \ -1 < c < 1$$

Next, use is made of (22) and (17b) to deduce the displacement as

$$W_m(\rho,\phi) = \frac{2aT}{\mu_m}2\pi i\int_{c-i\infty}^{c+i\infty}\left\{\begin{array}{l}\left(\frac{\mu_1-\mu_2}{\mu_1-\mu_2}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\epsilon^{2k-1}}{s+2k-1}\frac{\sin s\phi}{s\cos\frac{\pi}{2}s} \\ -\frac{2\mu_m}{\mu_1+\mu_2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\epsilon^{2k-1}\epsilon^{2k-1}}{s+2k-1}\frac{\cos s\phi}{s\sin\frac{\pi}{2}s}\end{array}\right\}\left(\frac{\rho}{\epsilon}\right)^{-s}ds$$

$$+\frac{2aT}{\mu_m}2\pi i\int_{c-i\infty}^{c+i\infty}\left\{\begin{array}{l}\left(\frac{\mu_1-\mu_2}{\mu_1-\mu_2}\right)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\delta^{1-2k}}{s-2k-1}\frac{\sin s\phi}{s\cos\frac{\pi}{2}s} \\ -\frac{2\mu_m}{\mu_1+\mu_2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\epsilon^{2k-1}\epsilon^{2k-1}}{s-2k+1}\frac{\cos s\phi}{s\sin\frac{\pi}{2}s}\end{array}\right\}\left(\frac{\rho}{\epsilon}\right)^{-s}ds \tag{23}$$

Residue theory is used in evaluating (23) after which Jordan's Lemma is applied to write the expressions for $W_m(\rho, \phi)$, in terms of convergent series with respect to the intervals $0 < \rho < \in$, $\in < \rho < \delta$ and $\rho > \delta$. Only residues that lead to bounded solutions are considered.

When $0 < \rho < \in$, use is made of the double poles of the first term in the first integrand at $s = -(2n - 1)$, $n = 1, 2, 3, \ldots$ and two classes of simple poles of the second term at $s = -(2n - 1)$ and $s = -2n$, $n = 1, 2, 3, \ldots$ the poles of the integrand of the second integral are all simple and located at $s = -(2n - 1)$ for the first term and at $s = -2n$, $n = 1, 2, 3, \ldots$ for the second term. The Lemma then requires the closure of the contour in the left half plane $\mathrm{Re}s < 1$, for $\rho < \in$ (which is turn implies $\rho < \delta$) to get:

$$W_i(\rho,\phi) = \frac{2aT}{\mu_i}\left\{\left(\frac{\mu_1-\mu_2}{\mu_1-\mu_2}\right)\frac{2}{\pi}\left[-\ln\left(\frac{\rho}{\in}\right)\sum_{n=1}^{\infty}\frac{\sin(2n-1)\phi}{2n-1}\rho^{2n-1} + \sum_{n=1}^{\infty}\frac{\sin(2n-1)\phi}{(2n-1)^2}\rho^{2n-1}\right.\right.$$

$$\left. -\phi\sum_{n=1}^{\infty}\frac{\cos(2n-1)\phi}{2n-1}\rho^{2n-1} + \sum_{n=1}^{\infty}\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{(-1)^{k+n-2}}{2(k-n)}\frac{\in^{2k-1}}{(2n-1)}\sin(2n-1)\phi\left(\frac{\rho}{\in}\right)^{2n-1}\right]$$

$$+\frac{4\mu_i}{\pi(\mu_1+\mu_2)}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(k-n)-1}\in^{2k-1}\frac{\cos 2n\phi}{2n}\left(\frac{\rho}{\in}\right)^{2n} - \frac{2\mu_i}{(\mu_1+\mu_2)}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\phi}{2n-1}\rho^{2n-1}$$

$$+\left[\frac{2}{\pi}\left(\frac{\mu_1-\mu_2}{\mu_1-\mu_2}\right)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(k+n-1)}\delta^{1-2k}\frac{\sin(2n-1)\phi}{2n-1}\left(\frac{\rho}{\delta}\right)^{2n-1}\right.$$

$$\left.\left.+\frac{4\mu_i}{\pi(\mu_1+\mu_2)}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{1-2(k+n)}\delta^{1-2k}\frac{\cos 2n\text{-}1\phi}{2n}\left(\frac{\rho}{\delta}\right)^{2n}\right]\right\}, \quad i=1,2 \tag{24}$$

When $\in < \rho < \delta$, the contour in the first integral of (23) is closed in the right half plane $\mathrm{Re}s > 1$ for $\rho > \in$ because, there, the integrand has simple, poles at $s = 2n - 1$ and $s = 2n$, $n = 1, 2, 3, \ldots$. The contour in the second integral is closed in the plane $\mathrm{Re}s < 1$ for $\rho < \delta$ since, there, the poles are all simple and located at $s = -(2n - 1)$ and $s = -2n$, $n = 1, 2, 3, \ldots$. Therefore

$$W_i(\rho,\phi)=\frac{4aT}{\pi\mu_i}\left\{\left(\frac{\mu_1-\mu_2}{\mu_1-\mu_2}\right)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k-n-1}}{2(k+n-1)}\in^{2k-1}\frac{\sin(2n-1)\phi}{2n-1}\left(\frac{\rho}{\in}\right)\right.$$

$$\left.\frac{2\mu_i}{\mu_1+\mu_2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(n+k)-1}\in^{2k-1}\frac{\cos 2n\phi}{2n}\left(\frac{\rho}{\in}\right)^{-2n}\right\}$$

$$+\frac{4aT}{\pi\mu_i}\left\{\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(1-k-n)}\delta^{1-2k}\frac{\sin(2n-1)}{2n-1}\left(\frac{\rho}{\delta}\right)^{2n-1}\right.$$

$$\left.+\frac{2\mu_i}{(\mu_1+\mu_2)}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{1-2n-2k}\delta^{1-2k}\frac{\cos 2n\phi}{2n}\left(\frac{\rho}{\delta}\right)^{2n}\right\},\quad i=1,\ 2 \qquad (25)'$$

Consideration of $\rho > \delta$ (which implies $\rho > \in$) requires closure of the contours in (23) in the right half plane Res > 1. This should be done because relevant simple poles concerning the first integrand are located at $s = 2n - 1$ for the first term and at $s = 2n$, for the second, when $n = 1, 2, 3, \ldots$. The second integral is associated with double poles for its first term at $s = 2n - 1$, $n = 1, 2, 3, \ldots$ and two classes of simple poles of its second term at $s = 2n$ and $s = 2n - 1$, $n = 1, 2, 3, \ldots$. Hence, for this case:

$$W_i(\rho,\phi)=\frac{4aT}{\pi\mu_i}\left\{\left(\frac{\mu_1-\mu_2}{\mu_1-\mu_2}\right)\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-3}}{2(n+k-1)}\in^{2k-1}\frac{\sin(2n-1)\phi}{2n-1}\left(\frac{\rho}{\in}\right)^{-(2n-1)}\right.$$

$$\left.+\frac{2\mu_i}{\mu_1+\mu_2}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(n+k)-1}\in^{2k-1}\frac{\cos 2n\phi}{2n}\left(\frac{\rho}{\in}\right)^{-2n}\right\}$$

$$-\frac{4aT}{\pi\mu_i}\left\{\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\left[\ln\left(\frac{\rho}{\delta}\right)\sum_{n=1}^{\infty}\frac{\sin(2n-1)\phi}{2n-1}\rho^{-(2n-1)}+\sum_{n=1}^{\infty}\frac{\sin(2n-1)\phi}{(2n-1)^2}\rho^{-(2n-1)}\right.\right.$$

$$\left.-\phi\sum_{n=1}^{\infty}\frac{\cos(2n-1)\phi}{2n-1}\rho^{-(2n-1)}-\sum_{n=1}^{\infty}\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{(-1)^{k+n-2}}{2(n-k)}\delta^{1-2k}\frac{\sin(2n-1)\phi}{2n-1}\rho^{-(2n-1)}\right]$$

$$-\frac{2\mu_i}{\mu_1+\mu_2}\left[\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(n-k)+1}\delta^{1-2k}\frac{\cos 2n\phi}{2n}\left(\frac{\rho}{\delta}\right)^{2n}\right.$$

$$\left.\left.+\frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\phi}{2n-1}\rho^{-(2n-1)}\right]\right\}\qquad i=1,\ 2 \qquad (26)$$

The original displacement is obtained from (24) - (26) by the fact that $w_i(r,\theta) = W_i(\rho,\phi)$, $i = 1, 2$ with (5) and ρ given by

$$\rho(r,\theta)=\left(\frac{a^2-2ar\sin\theta+r^2}{a^2+2ar\sin\theta+r^2}\right)^{\frac{1}{2}}$$

4. Interface Stress

We seek the character of the stress along the common boundary of the materials where $\theta = \pm\frac{\pi}{2}$, $r \le a$ for which

$$\delta^{-1}\left(r\ ,\ -\frac{\pi}{2}\right)=\frac{a-r}{a+r}=\rho\left(r\ ,\ \frac{\pi}{2}\right)<1\quad ,\ \phi=0$$

From (24) - (26) we see that for $i=1,\ 2$

$$\frac{\partial W_i}{\partial \phi}(\rho\,,0)=\frac{4aT}{\pi\mu_i}\left(\frac{\mu_1-\mu_2}{\mu_1-\mu_2}\right)\left[-\ln\left(\frac{\rho}{\epsilon}\right)\sum_{n=1}^{\infty}\rho^{2n-1}+\sum_{n=1}^{\infty}\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{(-1)^{k+n-2}}{2(k-n)}\epsilon^{2k-1}\left(\frac{\rho}{\epsilon}\right)^{2n-1}\right.$$

$$\left.+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-2}}{2(1-k-n)}\delta^{1-2k}\left(\frac{\rho}{\delta}\right)^{2n-1}\right\}\ ,\ 0<\rho<\epsilon \qquad (27a)$$

$$=\frac{4aT}{\pi\mu_i}\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-2}}{2(k+n-1)}\epsilon^{2k-1}\left(\frac{\rho}{\epsilon}\right)^{(2n-1)}\right.$$

$$\left.+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-2}}{2(1-k-n)}\delta^{1-2k}\left(\frac{\rho}{\delta}\right)^{2n-1}\right\}\ ,\ \epsilon<\rho<\delta \qquad (27b)$$

$$=\frac{4aT}{\pi\mu_i}\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\left[-\ln\left(\frac{\rho}{\delta}\right)\sum_{n=1}^{\infty}\rho^{(2n-1)}-\sum_{n=1}^{\infty}\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{(-1)^{k+n-2}}{2(k-n)}\delta^{1-2k}\left(\frac{\rho}{\delta}\right)^{(2n-1)}\right.$$

$$\left.+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n-2}}{2(k-n=1)}\epsilon^{2k-1}\left(\frac{\rho}{\epsilon}\right)^{(2n-1)}\right]\ ,\ \rho>\delta \qquad (27c)$$

$$\frac{\partial W_i}{\partial \rho}(\rho\,,0)=\frac{8aT}{\pi(\mu_1+\mu_2)}\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(k-n)-1}\epsilon^{2(k-1)}\left(\frac{\rho}{\epsilon}\right)^{2n-1}-\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(k+n)-1}\delta^{-2k}\left(\frac{\rho}{\delta}\right)^{2n-1}\right\}$$

$$-\frac{4aT}{\mu_1+\mu_2}\sum_{n=1}^{\infty}\rho^{2(n-1)}\qquad 0<\rho\leq\epsilon \qquad (28a)$$

$$=\frac{8aT}{\pi(\mu_1+\mu_2)}\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{-2(n+k)+1}\delta^{-2k}\left(\frac{\rho}{\delta}\right)^{2n-1}\right.$$

$$\left.-\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(k+n)-1}\epsilon^{2(k-1)}\left(\frac{\rho}{\epsilon}\right)^{(2n-1)}\right\}\ ,\qquad \epsilon<\rho<\delta \qquad (28b)$$

$$=\frac{-8aT}{\pi(\mu_1+\mu_2)}\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{2(k+n-1)}\epsilon^{2(k-1)}\left(\frac{\rho}{\epsilon}\right)^{2n-1}+\sum_{k=1}^{\infty}\frac{(-1)^{k+n-1}}{-2(k-n)+1}\delta^{-2k}\left(\frac{\rho}{\delta}\right)^{2n-1}\right\}$$

$$\frac{-4aT}{\mu_1+\mu_2}\sum_{n=1}^{\infty}\rho^{-2n}\qquad \rho>\delta \qquad (28c)$$

The desired stresses along $\left(r, \pm\frac{\pi}{2}\right)$ may now be computed using (3), (10)

and the relations

$$\frac{\partial w_i}{\partial r}\left(r, \pm\frac{\pi}{2}\right) = \frac{\partial W_i}{\partial \rho}(\rho, 0)\frac{\partial \rho}{\partial r}\left(r, \pm\frac{\pi}{2}\right), \quad i = 1, 2 \tag{29}$$

where

$$\frac{\partial \rho}{\partial r}(r, \theta) = \frac{-2a}{(a+r)^2} = \frac{1-\rho^2}{2r}, \quad \theta = \frac{\pi}{2} \tag{30a}$$

$$= \frac{2a}{(a-r)^2} = \frac{\rho^2-1}{2r}, \quad \theta = -\frac{\pi}{2} \tag{30b}$$

From (27a,c) we get

$$\frac{\partial W_i}{\partial \phi}(\rho, 0) = \frac{4aT}{\pi\mu_i}\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\left[(\ln\in - \ln\rho)\frac{\rho}{1-\rho^2} + \sum_{k=1}^{\infty}\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{(-1)^{k+n}}{2(k-n)}\in^{2(k-n)}\rho^{2n+1}\right], \ \rho < 1 \tag{31a}$$

$$= \frac{4aT}{\pi\mu_i}\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\left[(\ln\in + \ln\rho)\frac{\rho}{\rho^2-1} + \sum_{k=1}^{\infty}\sum_{\substack{k=1\\k\neq n}}^{\infty}\frac{(-1)^{k+n}}{2(k-n)}\in^{2(k-n)}\rho^{-2n+1}\right], \ \rho > 1 \tag{31b}$$

From (28a c) we get

$$\frac{\partial W_i}{\partial \rho}(\rho, 0) = \frac{-8aT}{\pi(\mu_1+\mu_2)}\in^{-1}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n}\in^{2(k-n)}}{2(k-n)-1}\rho^{2n-1} - \frac{4aT}{(\mu_1+\mu_2)}\frac{1}{(1-\rho^2)}, \quad \rho < \in \tag{32a}$$

$$= \frac{-8aT}{\pi(\mu_1+\mu_1)}\in^{-1}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n}\in^{2(k-n)}}{2(k-n)-1}\rho^{-2n-1} - \frac{4aT}{(\mu_1+\mu_2)}\frac{1}{(\rho^2-1)}, \quad \rho > \in^{-1} \tag{32b}$$

Next, we deduce the form of the stresses for

$$a\frac{(1-\in)}{1+\in} \leq r \leq a, \quad \theta = \pm\frac{\pi}{2}, \rho = \rho\left(r, \pm\frac{\pi}{2}\right)$$

utilizing (3a) and (10). That is, for $i = 1, 2$

$$\sigma_{i\theta z}\left(r, \pm\frac{\pi}{2}\right) = \frac{\mu_i}{r}\frac{\partial W_i}{\partial \phi}(\rho, 0)\frac{\partial \phi}{\partial \theta}\left(r, \pm\frac{\pi}{2}\right)$$

together with (31) and

$$\frac{\partial \phi}{\partial \theta}(r, \theta) = \frac{-2ar}{a^2-r^2} = -\frac{(1-\rho^2)}{2\rho}, \quad \theta = \frac{\pi}{2}$$

$$\frac{\partial\phi}{\partial\theta}(r,\theta)=\frac{2ar}{a^2-r^2}=\frac{\rho^2-1}{2\rho}\;,\quad \theta=\frac{-\pi}{2}$$

yield

$$\sigma_{i\theta z}\left(r,\pm\frac{\pi}{2}\right)=\frac{-2aT}{\pi r}\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\left[\ln\in-\ln\left(\frac{a-r}{a+r}\right)\right.$$

$$\left.+\frac{4ar}{(a+r)^2}\sum_{n-1}^{\infty}\sum_{\substack{k=1\\k\neq n}}^{2n-1}\frac{(-1)^{k+n}}{2(k-n)}\in^{2(k-n)}\left(\frac{a-r}{a+r}\right)^{2(n-1)}\right]\; i=1,2 \qquad (33)$$

The other stress components derived from (3b) (29) and (32) for

$a\dfrac{(1-\in)}{1+\in}<r<a$ as

$$\sigma_{irz}\left(r,\pm\frac{\pi}{2}\right)=\pm\mu_i\left[\frac{8aT\in^{-1}}{\pi(\mu_1+\mu_2)}\frac{2a}{(a+r)^2}\sum_{n=1}^{\infty}\sum_{k=1}^{2n}\frac{(-1)^{k+n}\in^{2(k-n)}}{2(k-n)-1}\left(\frac{a-r}{a+r}\right)^{2n-1}\right.$$

$$\left.+\frac{2aT}{(\mu_1+\mu_2)r}\right] \qquad (34)$$

5 Conclusion

The displacement field is distributed within three subregions corresponding to $0<\rho<\in$, $\in\leq\rho<\delta$ and $\rho>\delta$. When $0<\rho<\in$, the corresponding subsection of the original cross-section encloses the edge $(a, \pi/2)$, and is bounded by the loaded segment, from (a, β) to $(a, \frac{\pi}{2}+\beta)$ and the curve obtained by considering $\rho=\in$; that is

$$r=a\frac{\left\{(1+\in^2)\sin\theta-\left[(1+\in^2)^2\sin^2\theta-(1-\in^2)^2\right]^{\frac{1}{2}}\right\}}{1-\in^2} \qquad (35)$$

where $\beta\leq\theta\leq\frac{\pi}{2}+\beta$, $\beta<\frac{\pi}{2}$ and $\in<1$, with a minimum at $\theta=\frac{\pi}{2}$ for which

$$\frac{r}{a}=\frac{1-\in}{1+\in}$$

The subsection of the original plane that corresponds to that of [the] described by $\rho > \delta$, encloses the edge $(a, -\frac{\pi}{2})$ and is a reflection o[f] associated with $\rho < \in$, about the real line. Because $-\theta$ replaces θ wh[en] (35) still holds. The middle subsection is therefore, the compliment of the [other] two in $/z/ \leq a$, Fig. III.

From (25) and $w_i (r, \theta) = W_i (\rho, \phi)$, $i = 1, 2$, it is clear that, since $\in = \delta^{-1}$

$$w_i\left(r,\pm\frac{\pi}{2}\right)=\frac{-8aT\in^{-1}}{\pi(\mu_1+\mu_2)}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+n}\in^{2(k+n)}}{2(k+n)-1}\frac{\left(\rho^{-2n}-\rho^{2n}\right)}{2n}$$

where

$$1<\rho\left(r,-\frac{\pi}{2}\right)=\frac{a+r}{a-r},\quad \rho\left(r,\frac{\pi}{2}\right)=\frac{a-r}{a+r}<1$$

Consequently $0<r<\frac{a(1-\in)}{1+\in}$ for $\in<\rho<\delta$. But $\rho^{-2n}\left(r,-\frac{\pi}{2}\right)=\left(\frac{a-r}{a+r}\right)^{2n}=\rho^{2n}\left(r,\frac{\pi}{2}\right)$

Hence $w_i\left(r,\pm\frac{\pi}{2}\right)=0$ for $0<r<\frac{a(1-\in)}{1+\in}$, $i=1, 2$

The form of the stresses given in (33) and (34) indicate their dependence o[n] material properties and $\sigma_{i\theta z}$ $(r, \pm\frac{\pi}{2})$ which would have been absent had **the** material been homogeneous, is singular at the edges $(a, \pm\frac{\pi}{2})$, **implying that** cracking may initiate there.

If the length of half of the loaded segment is L, then $L = a\beta$. We may use (33) for the graph in Fig IV to understand the distribution of the stresses $\sigma_{i\theta z}$ $(r, \pm\frac{\pi}{2})$. for various values of $\frac{r}{a}$, when (1) load site length, 2L is less than semi-interface length, a (II) load site length, 2L is equal to semi-interface length, a (III) load site length, 2L is greater than a.

In Fig. IV the interval $\frac{1-\in}{1+\in} \leq \frac{r}{a} < 1$ determines the angle at which the effect of the stress is felt at a point on the circle $r = \frac{m}{n}a$; m, n integers and $n > m$.

When $\beta_0 = 28.6479^0$, $2L = a$, $\in = .2533$ and $\frac{1-\in}{1+\in} = .5932$ (all approximations to four decimal places). In this case, part of the interface along which $\frac{r}{a} < 0.5932$ is not displaced. The other part for which $0.5932 \leq \frac{r}{a} < 1$ experiences a higher magnitude of the stress. Fig. III is a sketch of the curve (35), for the case, $2L = a$. The curve intersects the interface at its minimum point, $\frac{r}{a} = .5932$, $\theta = \frac{\pi}{2}$. If $\beta < \beta_0$ $(2L < a)$ the subsection that is not displaced increases, since the interval $0 < \frac{r}{a} < \frac{1-\in}{1+\in}$ expands in each case. On the other hand, if $\beta > \beta_0$ $(2L > a)$, the subsections not displaced decrease, since in each case, the interval $0 < \frac{r}{a} < \frac{1-\in}{1+\in}$ contracts until (35) degenerates into a straight line as $\in \to 1$.

It is note worthy that in the special case when $\beta = \frac{\pi}{2}$, and $\in = 1$, the loading corresponds to that on the entire lateral surface, $r = a$, $0 < \theta < \pi$. To see that the series in (33) vanishes if $\in = 1$, we observe that the finite series

$$\sum_{\substack{k=1 \\ k\neq n}}^{2n-1} (-1)^{k+n} \frac{\in^{2(k-n)}}{k-n}$$

has even number of terms, for each n, given by

$$\in^{-2} - \in^{2} \text{ for } n = 2\ ;\ -\frac{\in^{-4}}{2} + \frac{\in^{-2}}{1} - \frac{\in^{2}}{1} + \frac{\in^{4}}{2} \text{ for } n = 3$$

$$\frac{\in^{-6}}{3} - \frac{\in^{-4}}{2} + \frac{\in^{-2}}{1} - \frac{\in^{2}}{1} + \frac{\in^{4}}{2} - \frac{\in^{-6}}{3} \text{ for } n = 4\ ;\ \ldots$$

Thus for each n, there are $2(n-1)$ terms which can be paired to get terms in the form $\pm\frac{1}{m}(\in^{-2m} - \in^{2m})$ $m = 1, 2, 3, \ldots,$ $(n-1)$ which are clearly zero when $\in = 1$.

Therefore, the expression for $\sigma_{i\theta z}$ (r, $\pm\frac{\pi}{2}$), for the special case when the entire lateral surface is loaded, is obtained from the general functional form (33), by setting $\in = 1$.

6 REFERENCES

[1] T. S. Cook and F. Erdogan: Stresses In Bonded Materials With A crack Perpendicular To The Interface. Int, J. Enging Sci, 1972, Vol. 10 pp 677 – 697.

[2] Xiangzhon Zhang and Norio Hasebe: Longitudinal Shear and Torsion of Bimaterial Solids. J. of Appl Mech, 1994 Vol. 61 pp 495 – 497.

[3] T Honein, E Honein and G Herrmann: Circularly Cylindrical Plane Layered Media In Anti-plane Elastostatics: J of Appl Mech, Vol. 61, 1994, pp 243 – 250.

[4] S. R. Chor and Y Y Ermme: Analysis of A Kinked Crack In Anti – plane Shear. Mech. Of Materials, 1990, Vol. 9 pp 529 – 533.

[5] R. V. Church, II, J. W. Brown and R. F. Verhey: Complex Variables And Applications, McGraw – Hill, Tokyo, 1974 pp 315 – 324.

FIGURES

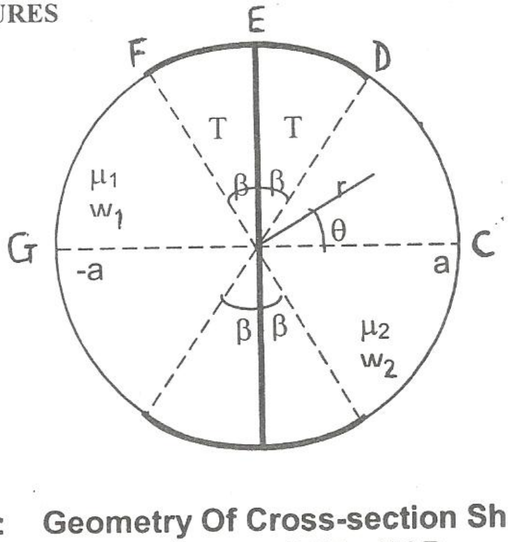

Fig. I: Geometry Of Cross-section Showing Interface AE And Load Sites DEF : HAB

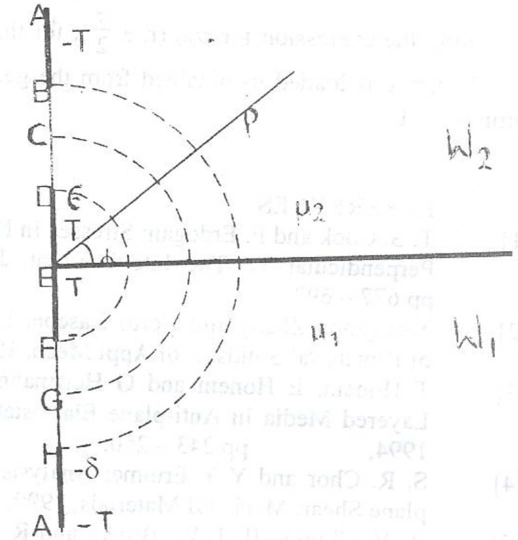

Fig. II. Significant correspondence on the Right Half plane

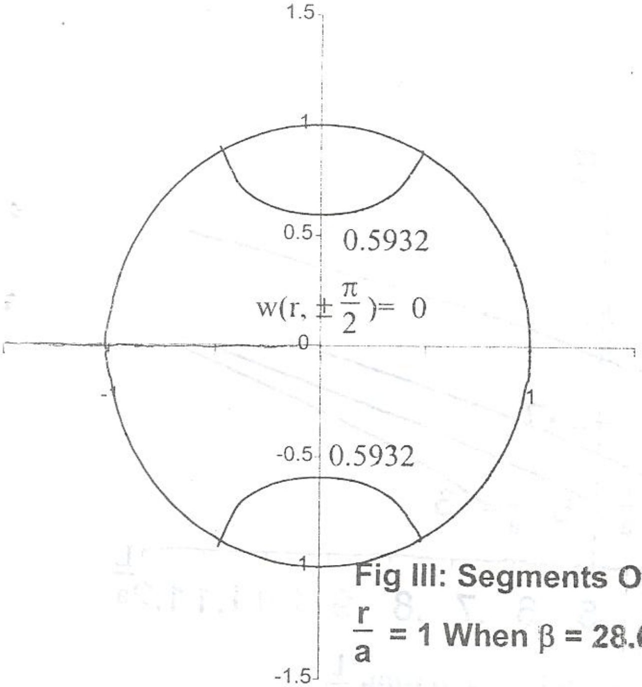

Fig III: Segments On The Cross section $\frac{r}{a} = 1$ When $\beta = 28.6477^0$ ($\frac{L}{a} = .5$)

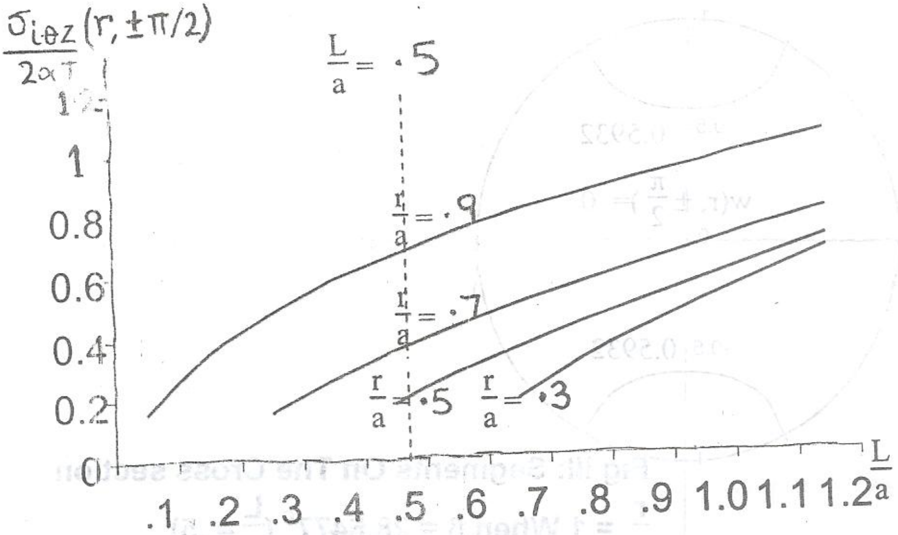

Fig IV: Variation of $\frac{\sigma_{l\theta z}}{2\alpha T}$, $\alpha > 0$ With $\frac{L}{a}$