J. Nig. Ass. Math. Phys.
Vol. 6 (2002)

ON QUASI LINEAR KINEMATICS WAVES

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ABSTRACT

We present results of a selective study of the asymptotic solution of wave equations satisfying certain initial value conditions for kinematic waves. Furthermore, the characteristics for the equation and the transition from stable to unstable solutions were discussed illustrating the results obtained with relevant initial value problems (IVP). Development of shocks governed by the same IVP and initial data was evaluated with its characteristic equations. It was shown that displacement; u (x, t) is constant along characteristics which propagates with speed c (u). The dependence of on u produces a gradual nonlinear distortion of wave profile as it propagates through the medium. It is also follows that c (u) must be constant along characteristics, thus characteristics must be straight lines in the x-t plane.

1.0 INTRODUCTION

One of the broadest scientific subjects is the study of wave motion, at any technical level. The occurrences such as the behavior of water waves and the propagation of light and sound are familiar. These have been studied extensively and there exists a whole wide range of knowledge from these studies.

In recent times, problems such as sonic booms or moving bottlenecks in traffic flow has been of considerable interest. To the non-specialist, it is appreciated in a descriptive way. On the other hand, almost all fields of Science and Engineering have something to do with wave phenomena.

These had led specialists in the field to develop mathematical concepts and techniques for understanding the phenomena of wave motion and thus solve the problems that emanate from them. These problems vary from application to application with each having its own peculiar constraints.

In the sea, waves appear for a number of reasons. Those generated by winds are called wind waves. Those generated by relatively rapid variations in atmospheric pressure over the sea are called seiches or anemobaric waves. Those due to seismic activity over the sea bottom are called tsunamis while those generated by astronomical forces are called tidal waves. The list goes on and on and this explains why waves cannot be easily classified in any precise way. However, two main classes that are not mutually exclusive can be generally distinguished:

i. Those formulated in terms of hyperbolic partial differential equations are called hyperbolic waves; and
ii. Those in which the phase speed depends on wave number are called dispersive waves.

The application of mathematical analysis to the solution of the wave problems dates back to 1687 and is credited to Isaac Newton. He was the first to apply mathematical analysis to the solution of the wave problem. Laplace also demonstrated that wave oscillations in water are formed by water particles moving along elliptical orbits which flatten out toward the bottom, while at the bottom, they degenerate into straight lines. Within this same period, Langrage conceived a different opinion from Laplace. He concluded that shallow water wave velocity depended only on depth of the water layer and on wave-length in deep water. Gertstner, in his trocoidal wave theory proposed that particles in water of infinite depth move in circular orbits and that wave speed depended only on the length of the wave. (Milton 1964)

Those theories of the early 19th century, that is, water particles moving in elliptical orbits or in circular orbits combined with Poisson and Cauchy's theory in which they concluded that waves propagate with increasing speed were eliminated by the studies of green, Stokes, Rayleigh, Reynolds and other great Mathematicians when it became possible to establish the limits on the applications of these theories and to determine their inter-relationship.

These studies constitute what is generally known today as linear wave theory, i.e. investigations into waves of stable shape and of small amplitude. As mentioned earlier, waves cannot be studied without considerations to the actions of the following effects:.

i. Forces due to gravity;
ii. Deflecting forces produced by the earth's rotation (in the study of very long waves like tidal waves);
iii Surface tension (for small amplitude waves with wavelength few centimentres long);

Waves can be two – dimensional. In this case, it is called plane waves. Further ,in three dimensions, it is called spatial waves. The former is recognized if the orbits of water particles involved in wave motion are directed parallel to the vertical planes, coinciding with the direction of wave motion but do not depend on the distance from it. Spatial waves are those in which the water particle orbits do not lie in any plane parallel to the vertical planes.

Kinematic waves are waves that depend primarily on the conservation equation

$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$ (where ρ is the density per unit length and q is flux per unit time) governed by

$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0$ with a given initial data.

2. PROBLEM FORMULATION

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \nabla^2 u \quad \text{for velocity } c_0 \tag{2.0.1}$$

is often taken as the simplest equation of hyperbolic type. Suppose that we have an infinitely long uniform string of density ρ, which is taut with tension T. the displacement u at the point of abscissa x at time t, for small vibrations, satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \text{ with } c^2 = T/\rho. \tag{2.0.2}$$

We can determine the shape at any instant if we know the initial displacement and initial velocity of each point of the string, i.e. if we are given

$u(X,0) = \phi(X),\ u_t(X,0) = \Psi(X)$

By this the data are of Cauchy type and there is a unique solution analytic near every point of the x-axis if the initial data are analytic.

Introducing characteristic variables $\xi = x + ct,\ \eta = x - ct$, equation 2.0.2 is

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

We thus have

$$u = F(\xi) + G(\eta) = F(x + ct) + G(x - ct)$$

so that the general solution represents the sum of two waves propagated with velocities $\pm c$. With the initial conditions, we obtain the complete solution

$$u = \tfrac{1}{2}\phi(x - ct) + \tfrac{1}{2}\phi(x + ct) + \frac{1}{c2}\int_{x-ct}^{x+ct}\Psi(\tau)d\tau \tag{2.0.3}$$

when $t \geq 0$ which has continuous derivatives of the second order, provided that $\Phi''(\tau)$ and $\Psi'(\tau)$ are continuous.

A simpler formulation of equation 2.0.1 is given by

$$\frac{\partial u}{\partial t} + c_0\frac{\partial u}{\partial x} = 0 \tag{2.0.4}$$

Equation 2.0.4 governs the evolution of the linear wave transformations.

A linear depressive system is one that admits solutions of the form

$$u = Ae^{i(kX - \omega t)} \tag{2.0.5}$$

where k = wave number and ω = frequency, and ω = ω(k).

The generalized form of equation (2.0.4) is

$$\frac{\partial u}{\partial t} = c(u)\frac{\partial u}{\partial x} = 0 \qquad (2.0.6)$$

where c (u) is the speed of propagation and is a function of the disturbance, u. Studies for equations of these types provides much information for the study of more general nonlinear equations and all its associated features such as wave breaking and the formation of wave shocks. etc. It will be noted that even though equation (2.0.6) has its first derivative appearing in a linear fashion, it is nonlinear in u and hence it is called quasi-linear. The general nonlinear first order equation for u(x,t) is any functional relation between u, u_x *and* u_t e.g. the equations of hydrodynamics given by

$$\frac{d}{dt}q = \frac{\partial}{\partial t}q + q.grad.q = 0 \qquad (2.0.7)$$

where $\frac{d}{dt}$ is the particle derivative, $\frac{\partial}{\partial t}$ is the local or time derivative and q.grad is the convection or space derivative or element changing position. These equations are often linearized especially when the disturbances are of small amplitude by neglecting all quadratic terms thus making the governing equations linear. We consider the equation

$$\frac{du}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad (2.0.8)$$

When the disturbances are weak about some stable equilibrium value of u, u_o *say*, $\left|\left(u - u_o\right) / U_o << 1$ and equation (2.0.8) becomes on linearization

$$\frac{\partial u}{\partial t} + u_o\frac{\partial u}{\partial x} = 0 \qquad (2.0.9)$$

with general solution $u - u_o = f\left(x - U_{ot}\right)$.

Let $\epsilon > 0$ be a measure of the maximum initial value of $\left(u - u_o\right) / u_o$, and we seek a solution of (2.0.8) in the form

$$u = u_o + \epsilon u_1(x,t) + \epsilon^2 u_2(x,t) + \ldots \qquad (2.0.10)$$

on the assumption that this expansion converges to a limit. Putting equation (2.0.10) into equation (2.0.8) and equating coefficients of ϵ^n to zero, we obtain a hierarchy of equations of the form

$$\phi_t + u_o\phi_x = \Phi(x,t) \qquad (2.0.11)$$

with Φ calculated from the previous step.

Let

$$y = x - u_o t \qquad (2.0.12)$$

be the characteristic co-ordinate. Introducing this into equation (2.0.11), we may then write with condition

$$\left\{\frac{\partial \phi}{\partial t}\right\}_{y = constant} = \Phi\,(y + u_o t, t) \qquad (2.0.13)$$

whence

$$\Phi = \int_o^t \Phi(y + u_o \tau, \tau)d\tau + \Psi(y) \qquad (2.0.14)$$

The condition on u may be written in the form $u = u_o + \in P(x)$ at $t = 0$ and is satisfied by $c_1 = P(x_1)$, $c_n = 0$ $(n > 1)$ at $t = 0$. it has been shown that successive terms in the assumed series for u are of order $\in^n t^{n-1}$, and so the series is not uniformly valid as $t \to \infty$, witham (1974). The breakdown occurs because the linearized theory approximates the characteristics as $x - u_o t$ = constant. The slight inclination of the true characteristic lines relative to each other accumulates to a large displacement as $t \to \infty$. Lesser (1970) demonstrated that this problem can be overcome for simple systems, at least partially. He developed a uniformly valid perturbation series for linear wave propagation in an in homogenous medium based on a constant formulation of the characteristic equation. This clearly demonstrates that linear theory is not uniformly valid in general for large values of the co-ordinates (space as well as time).

When the disturbances are weak and oscillate about some stable equilibrium value of $u = u_o$ say, $|(u - u_o)| / u_o << 1$ and equation (2.0.7) becomes

$$-u_o \frac{\partial}{\partial \theta} u + \frac{1}{2}\frac{\partial}{\partial \theta} u^2 + k\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (2.0.15)$$

with solution $u - u_o = f(x - u_o t)$.

2.1 THE FIRST ORDER QUASI-LINEAR EVOLUTION EQUATION

The general case of the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

is of the form

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = f(x, y, u) \qquad (2.1.1)$$

This is considered as the most fundamental, first order, quasi-linear in homogenous evolution equation. If f = 0, it reduces to

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0$$

and is called a kinematic wave equation. This is the case if the dissipative and dispressive forces are ignored.

2.2 SHOCKS DEVELOPMENT

We next review the development of shocks as governed by the initial value problem

$$\frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0, \; -\infty < x < \infty \qquad (2.2.1)$$

with initial data

$$u(x,0) = f(x). \qquad (2.2.2)$$

The equations of the characteristics associated with (2.2.1) are:

$$\frac{dt}{1} = \frac{dx}{c(u)} = \frac{du}{0} \qquad (2.2.3)$$

, These are

$$\frac{du}{dt} = 0 \qquad (2.2.4)$$

$$\frac{dx}{dt} = c(u) \qquad (2.2.5)$$

The solution (2.2.4) and (2.2.5) represent the characteristics of the equation (2.2.1). Along these characteristics lines,

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + c(u)\frac{\partial u}{\partial x} = 0 \qquad (2.2.6)$$

This proves that u(x, t) is constant along characteristics which propagates with speed c (u). The dependence of c on U produces a gradual nonlinear distortion of wave profile as it propagates through the medium. It also follows that c(u) must be constant along characteristics, thus characteristics must be straight lines in the x-t plane.

Consider a characteristic line which cuts x – axis at $x = \xi$ as shown in figure 1. Along this, u = u (ξ) is constant. It follows that $c[u(\xi)] = F(\xi)$ is also a constant. Thus,

$$\frac{dx}{dt} = F(\xi) \qquad (2.2.7)$$

which gives

$$\frac{dx}{dt} = F(\xi) + A \qquad (2.2.8)$$

Along x – axis, t = 0, thus A = ξ. Equation of the characteristic line is thus given by

$$\frac{dx}{dt} = F(\xi) + \xi \qquad (2.2.9)$$

Equation (2.2.9) is the equation of that characteristic line which cuts the x – axis at x = ξ.

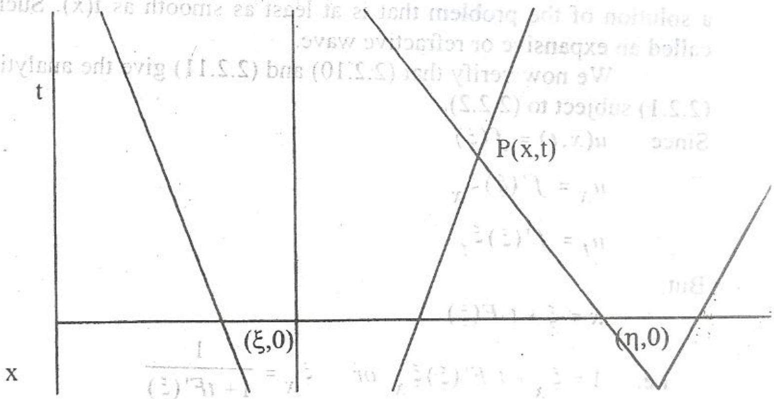


Figure 1. characteristic Lines of different slopes: - $\frac{1}{F(\xi)}$

Since u(x,t) is constant on the characteristics, it follows from (2.2.9) that

$$u(x,t) = u[\xi + t\,F(\xi), t] = u(\xi, 0) = f(\xi)$$

In general, the solution of the initial value problem (2.2.1) and (2.2.2) is of the from

$$u(x,t) = f(\xi) \qquad (2.2.10)$$

where $\xi = x - t\,F(\xi)$ (2.2.11)

Since f(x) is the given initial data, the solution reduces to

$$u(x,t) = f[x - t\,F(\xi)] \qquad (2.2.12)$$

2.3 INTERSECTION OF CHARACTERISTICS AND FORMATION OF THE SHOCKS.

If there are two points (ξ,0) and (η,0) as shown in figure 1 for which ξ < η, with

$$m_1 = \frac{1}{F(\xi)} < \frac{1}{F(\eta)} = m_2 \qquad (2.3.1)$$

then, the characteristic starting from (ξ,0) will intersect those from (η,0) at P(x,t). Thus at P(x,t), f(x) has two different values f(ξ) and f(η). Consequently, we have no unique solution at P(x,t).

If no two characteristic lines intersect in the half-plane t > 0, there exist a unique solution of the initial value problems (2.2.,) and (2.2.2) as differentiable functions for all t > 0. the necessary condition forthis is that

$$F(\xi) \le F(\eta) \quad for\ \xi < \eta.$$

In other words, the family of characteristics spread only for t > 0 and generates a solution of the problem that is at least as smooth as f(x). Such a solution is called an expansive or refractive wave.

We now verify that (2.2.10) and (2.2.11) give the analytical solution of (2.2.1) subject to (2.2.2).

Since $u(x,t) = f(\xi)$

$$u_x = f'(\xi)\xi_x$$

$$u_t = f'(\xi)\xi_t$$

But,

$$x = \xi + t\,F(\xi)$$

i.e. $1 = \xi_x + t\,F'(\xi)\xi_x \quad or \quad \xi_x = \dfrac{1}{1+tF'(\xi)}$

and $0 = \xi_t + F(\xi) + tF'(\xi)\xi_t$

$$\therefore \quad \xi_t = -\frac{F(\xi)}{1+tF'(\xi)}.$$

Consequently,

$$u_x = \frac{f'(\xi)}{1+tF'(\xi)}$$

$$u_t = -\frac{f'(\xi)F(\xi)}{1+tF'(\xi)} \text{ and}$$

$$u_t + uu_x = -\frac{f'(\xi)F(\xi)}{1+tF'(\xi)} + \frac{f'(\xi)f(\xi)}{1+tF'(\xi)} = 0$$

provided that $t \neq -\dfrac{1}{F'(\xi)}$

if $c(u) = c$ = constant, then

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \ -\infty < x < \infty \qquad (2.3.2)$$

with U(x,0) = f(x)
therefore, equation of characteristics are:

$$\frac{du}{dt} = 0, \frac{dx}{dt} = c \qquad (2.3.3)$$

from (2.3.3),

$$\frac{dx}{dt} = c \ i.e. \ x = ct + A$$

$$when \ t = 0, \ x = \xi$$

$$\therefore \ x = \xi + ct$$

$$\Rightarrow \xi = x - ct$$

$$\frac{du}{dt} = 0$$

$$\Rightarrow u = u(x) = f(x)$$

Thus, parametrically, the solution

$$u = f(\xi) = u(x,t) = u(x,0)$$

along the characteristics with

$$\xi = x - ct$$

$$\therefore u(x,t) = f(x - ct)$$

From the relation

$$x = \xi + ct$$

the characteristics are straight lines with constant gradient $-\frac{1}{c}$. No two of such characteristic lines intersect. Thus, they are parallel straight lines and no shock is formed in this case. This is shown in figure 2.

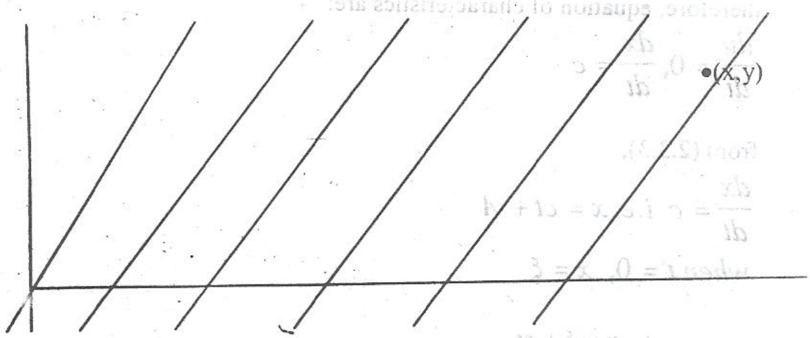


Figure 2. Lines of characteristics with constant gradient

The time for the possible formation of shocks is $t = -\dfrac{1}{F'(\xi)}$ and at this time t = t*, the solution of the initial value problem no longer exists as a continuous function of x and t.

Consider the problem given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad t > 0, x \in R$$

$$\text{with } u(x,0) = \begin{cases} \left(a^2 - x^2\right) & |x| \le a \\ 0 & |x| > a \end{cases}$$

$$x = \xi + tF(\xi) = \xi + t\left(a^2 - \xi^2\right), \quad |x| < a$$

$$= \xi \qquad |x| > a$$

$$u(x,t) = a^2 - \xi^2, \quad |\xi| < a$$

$$= a^2 - \xi^2 = \frac{1}{2t^2}\left[2xt \pm \sqrt{(1 - 4xt + 4a^2t^2)}\right], \quad t \ne 0 \qquad (2.3.4)$$

$$0 = \qquad |\xi| > a$$

and for small t, take only the positive sign in (2.3.4). figure 3 represents the propagation of the initial parabolic pulse with a = 1. As t increases, the initial pulse distorts progressively.

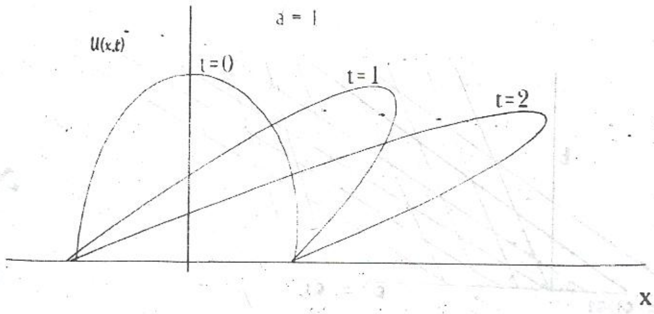


Figure 3. Propagation of the initial parabolic pulse with a = 1

This progressive change in the initial wave pulse is the result of the nonlinear term $u\dfrac{\partial u}{\partial x}$.

In the linear case,

$$\frac{\partial u}{\partial u}+c_0\frac{\partial u}{\partial x}=0,\quad t>0,\ c_0=constant\ with\ u(x,0)=\left(a^2-x^2\right),|x|<a,|x>0$$

and characteristics $x=\xi+c_0t$. It can be deduced that the initial pulse propagates without change of shape or disturbances.

2.4 CASE OF INITIAL-VALUE PROBLEM WITH DISCONTINUOUS INITIAL DATA

The interest in this regard concerns the difficulties caused by the discontinuities of initial data in the solution. Consider the Reimann problem for the scalar advection equation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0,\quad x\in R,\ t>0$$

$$with\quad u(x,0)=\begin{cases}U_1, & x<0\\ U_2, & x>0\end{cases}$$

$$C(x,0)=F(x)=\begin{cases}c_1,=c(U_1) & x<0\\ c_2,=c(U_2) & x>0\end{cases}$$

where U_1 and U_2 are constants. We consider the two cases : $U_1 > U_2$ and $U_1 < U_2$.

<u>Case 1:</u> $U_1 > U_2$ with $c_1 > c_2$.

Here, breaking will occur immediately as shown in the figure (4a) below.

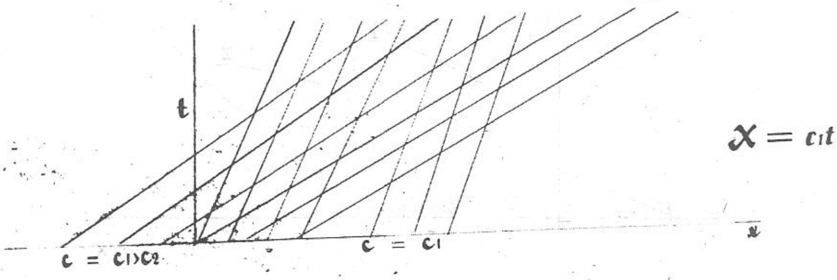
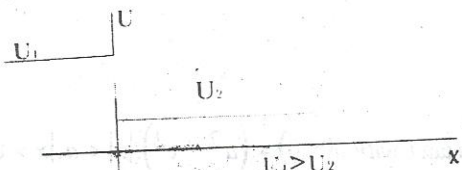


Figure 4(a). Case of IVP with discontinuous initial data



Case 2: $U_2 > U_1$ **with** $C_2 > C_1$.

Here, there is a perfectly good continuous solution as the initial condition is expensive and a fan of characteristics in the x = t plane as in the figure (4b) below is generated with each member of the fan having a different slope.

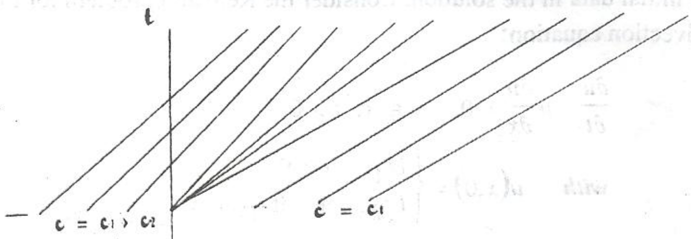
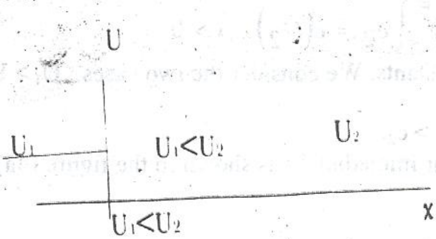


Figure 4 (b). Case of IVP with continuous initial data



2.5 Discontinuous solutions and Shock waves.

There is a continuous distribution of either material or some state of the medium in problems of wave propagation. Suppose we define a density $\rho(x,t)$ as mass per unit length and corresponding flux q(x,t) per unit time. We can then define a flow velocity v(x,t) by

$$v = \frac{q}{\rho} \tag{2.5.1}$$

For a conserved state, we can stipulate that the rate of change of the total amount of it in any section $x_1 > x > x_2$ must be balanced by the net inflow across x_1 and x_2, i.e.

$$\frac{d}{dt}\int_{x_2}^{x_1} \rho(x,t)dx + q(x_1,t) - q(X_2,t) = 0 \tag{2.5.2}$$

If $\rho(x,t)$ has continuous derivatives, we may take the limit as $x_1 \to x_2$ so that the conservation equation becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad \forall\ x \in R, t > 0 \tag{2.5.3}$$

the solution for which the following assumptions are made:

i. ρ and q are continuous $\forall\ x \in R, t > 0$

ii. $q = Q(\rho)$

In the analysis of shock formation, only (ii) will apply. We also allow discontinuity at x = s(t) and there are points $x = x_1$ and $x = x_2$ such that $x_1 < s(t) < x_2$ and $\frac{ds}{dt} = u(t)$. Thus, x = x (s) is the equation of discontinuity which moves with the speed u(t).

Without any sink or source, the integral form of conversation law (2.5.2) gives:

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho\, dx = q(x_1,t) - q(x_2,t)$$

$$\text{or} \quad \frac{d}{dt}\int_{x_1}^{x_2} \rho\, dx + q\left(x_2,t\right) - q\left(x_1,t\right) = 0$$

$$\text{or} \quad \frac{d}{dt}\left[\int_{x_1}^{s^-} \rho\, dx + \int_{s^+}^{x_2} \rho\, dx\right] + q\left(x_2,t\right) - q\left(x_1,t\right) = 0$$

$$\text{or} \quad \int_{x_1}^{s^-} \rho_t\, dx + s\rho\left(s^-,t\right) + \int_{s^+}^{x_2} \rho_t\, dx - s\rho\left(s^+,t\right) + q\left(x_2,t\right) - q\left(x_1,t\right) = 0$$

where $\rho\left(s^-,t\right)$ *and* $\rho\left(s^+,t\right)$ are the values of $\rho\left(s,t\right)$ as $x \to s$ from below and above.

Since ρ_t is bounded in $x_1 < x < s$ *and* $s^+ < x < x_2$ respectively, the integrals in the above vanish as

$x_1 \to 5$ *and* $x_2 \to s^+$. *Thus,* $q\left(5,t\right) - q\left(5^+,t\right) = U\left\{\rho\left(5,t\right) - \rho\left(s^+,t\right)\right\}$, where $U = \dot{s}$.

That is, $[q] = U[\rho]$, where $[q] = \left(\bar{q} - q^+\right)$ *and* $[\rho] = [\rho^- - \rho^+], u = [q]/[\rho]$

2.6 WEAK OR GENERALISED SOLUTION

We generalize the classical solution of our problem to include the disco... uous and non continuously differentiable functions.

Consider the initial value problem

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \qquad \forall\, x \in R, t > 0 \qquad (2.6.1)$$

$$\rho\left(x,0\right) = f\left(x\right) \qquad \forall\, x \in R \qquad (2.6.2)$$

$q\left(x\right)$ is a continuously differentiable function on R. The classical solution of (2.6.1) is the smooth function which satisfies the problem, then, $\rho\left(x\right)$ is the classical solution of (2.6.1).

2.7 WEAK SOLUTION OF THE CAUCHY INITIAL PROBLEM – SHOCK FORMATION.

At time $t = t^* = -1 / F(\xi)$, the solution U(x, t) experiences what is known as 'gradient catastrophe' or 'blow-up'. This implies that there cannot exist a smooth beyond $t = t^*$. The solution now obtained is called 'weak solution' and this allows for discontinuity in the wave profile.

To examine this, the given equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad (2.7.1)$$

has conservative form which can be written as

$$\frac{\partial}{\partial t}R(u) + \frac{\partial}{\partial x}S(u) = 0 \qquad (2.7.2)$$

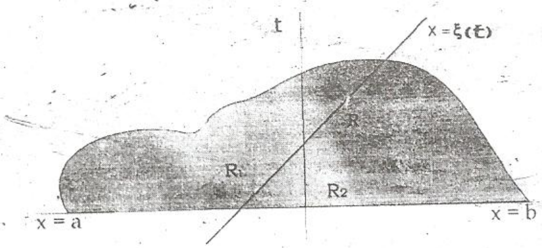


Figure 5. Weak solutions of the Cauchy IVP.

This is,

$$R'(u)\frac{\partial u}{\partial t} + S'(u)\frac{\partial u}{\partial x} = 0 \qquad (2.7.3)$$

Comparing 2.7.1 and 2.7.3,

$R'(u) = 1$ or $R(u) = u$

$S'(u) = u$ or $S(u) = u^2/2$

Also, $u\,R'(u) = S'(u)$. We have assumed that $S(0) = R(0) = 0$

Integrating 2.7.2 from $x = a$ to $x = b$, we have

$$0 = \frac{d}{dt}\int_a^b R[u(x,t)]dx + S[u(b,t)] - S[u(a,t)] \quad (2.7.4)$$

Thus, any bounded measurable u(x,t) with bounded measurable initial data which satisfies 2.7.4 is called the Weak solution of the initial value problem.

We now consider the case where u(x,t) is a C^1-solution of the initial value problem in each of the two regions of x-t plane separated by a curve $x = \xi(t)$ across which the value of u(x,t) undergoes a jump ('shock'). Denoting the limit of u(x,t) from the left and right respectively by u^- and u^+, we have

$$0 = S[u(b,t)] - S[u(a,t)] + \xi' R(u^-) - \xi' R(u^+) - \int_a^{\xi} \frac{\partial S(u)}{\partial t}dx - \int_{\xi}^{b} \frac{\partial S(u)}{\partial t}dx$$

But on the axis, S[u(x,0)] = S(x)

$$\frac{\partial S(u)}{\partial t} = 0$$

$$-[R(u^+) - R(u^-)]\xi - S(u^-) + S(u^+) = 0$$

$$i.e. \quad U_0 = \frac{d\xi}{dt} = \frac{S(u^+) - S(u^-)}{R(u^+) - R(u^-)} \quad (2.7.5)$$

where U_0 is the speed of shock.

(2.7.5) connects the propagating speed U_0 of discontinuity with the jump discontinuities in R and S. The shock wave defined by (2.7.5) represents the **weak solution of the initial value problem.** Since R(u) = u and $S(u) = u^2/2$, in the **initial value problem,**

$$U_0 = \frac{1}{2}\left[\frac{(u^+)^2 - (u^-)^2}{u^+ - u^-}\right] = \frac{1}{2}(u^+ + u^-) \quad (2.7.6)$$

2.8 ILLUSTRATING WEAK SOLUTION OF THE CAUCHY IVP

We solve the quasi-linear Cauchy problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \qquad x \in R, t > 0 \tag{2.8.1}$$

with the following Cauchy initial data:

(i) $u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$

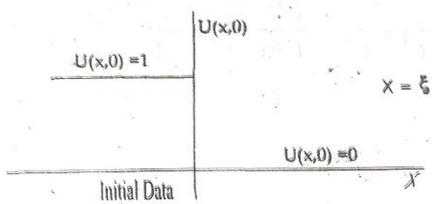
(ii) $u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$

In (i), $U_0 = \frac{1}{2}\left(U^- + U^+\right) = \frac{1}{2}$.

c(U) = U and thus U(x,0) is a decreasing function of x. Hence, no continuous solution of the initial value problem. However, the Weak solution is given by

$$U(x,0) = \begin{cases} 1 & x < U_0 t \\ 0 & x > U_0 t \end{cases} \quad , U_0 = 1/2$$

The solution has a jump discontinuity defined by $x = U_0 t$.



From the equation of the characteristics passing through $x = \xi$, we obtain

$$x = \xi + U(x,0)t$$

Thus,

$x = \xi + t$ when x < 0

$= \xi$ when x > 0

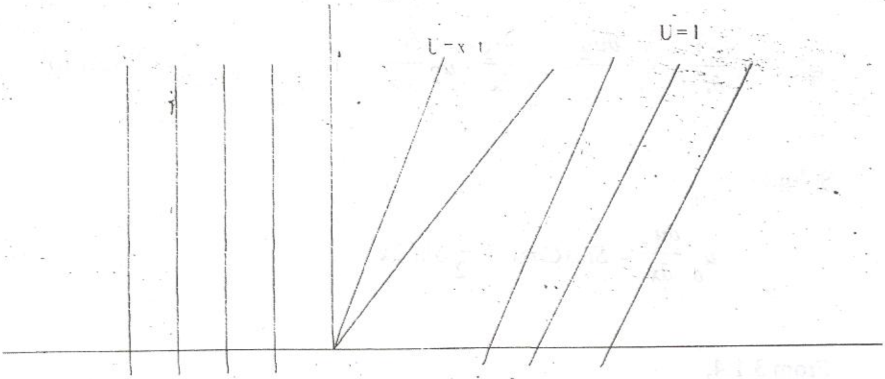


Figure 7. Characteristics and continuous solutions

3.0 SECULAR TERMS

Suppose we seek a solution for the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \text{ with } \quad U_0(x,0) = U_0 = Sinx, U_n(x,0) = 0 \text{ for } n = 1,2,3,\ldots \qquad (3.1.1)$$

in the form

$$u = u_0 + \in u_1 + \in^2 u_2 + \in^3 u_3 + \ldots \qquad (3.1.2)$$

Then, substituting 3.1.2 into 3.1.1 yields

$$\in \frac{\partial u_0}{\partial t} + \in^2 \frac{\partial u_1}{\partial t} + \in^3 \frac{\partial u_2}{\partial t} + \ldots + \left[\in u_0 + \in^2 u_1 + \in^3 u_2 + \ldots\right]\frac{\partial}{\partial x}\left[\in u_0 + \in^2 u_1 + \in^3 u_2 + \ldots\right]$$

$$L\in: \quad \frac{\partial u_0}{\partial t} = 0 \quad \therefore \quad u_0 = u_0(x) = Sinx \qquad (3.1.3)$$

$$L\in^2: \quad \frac{\partial u_1}{\partial t} + u_0 \frac{\partial u_0}{\partial x} = 0 \qquad (3.1.4)$$

$$L\epsilon^3: \quad \frac{\partial u_2}{\partial t} + u_o \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_o}{\partial x} = 0 \qquad (3.1.5)$$

$$L\epsilon^4: \quad \frac{\partial u_3}{\partial t} + u_o \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_o}{\partial x} = 0 \qquad (3.1.6)$$

Solutions:

$$u_o \frac{\partial u_0}{\partial x} = SinxCosx = \frac{1}{2} Sin\ 2x$$

From 3.1.4,

$$\frac{\partial u_1}{\partial t} = -\frac{1}{2} Sin\ 2x \text{ or } u_1 = \frac{t}{2} Sin\ 2x$$

$$= \frac{t}{2}\left[Sin\ 3x - Sinx\right] + \frac{t}{4}\left[Sin\ 3x + Sinx\right]$$

$$= \frac{3t}{2} Sin\ 3x - \frac{t}{4} Sinx$$

$$= \frac{t}{2}[\ 3Sin\ 3x - \frac{1}{2} Sinx]$$

From 3.1.6,

$$\frac{\partial u_2}{\partial t} = -\ [u_0 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_o}{\partial x}]$$

$$= \frac{t}{4} Sinx - \frac{3t}{2} Sin\ 3x$$

$$= \frac{t}{2}[\frac{1}{2} Sinx - 3\ Sin\ 3x]$$

$$u_2 = \frac{t}{4}[\frac{1}{2} Sinx - 3\ Sin\ 3x]$$

The solution is unbounded as $t \to \infty$. This is because no force is introduced to damping the nonlinear growth effect of the term $u\frac{\partial u}{\partial x}$. The terms involving t are called secular terms.

Illustration:

Consider a model case given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = x \text{ with } \quad u(x,0) = f(x) = x$$

The characteristics equations are

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{x} = k\left(say\right)$$

But,

$dx = ku, \quad du = kx$

$$\therefore dx + du = k(x+u) = d(x+u)$$

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{x} = \frac{d(x+u)}{(x+u)}$$

$$t = In(x+u) + InC$$

$$\Rightarrow x + u = c_1 e^t$$

$$\text{i.e. } (x+u)e^{-t} = c_2 \qquad (3.1.7)$$

Again, $xdx - udu = 0$

$$\Rightarrow u^2 - u^3 = c_3$$

$$\Rightarrow (u+x)(u-x) = c_3$$

$$\Rightarrow (u-x)c_2 e^t = c_3 \qquad (1.1.8)$$

But $\quad u = x \text{ when } t = 0$

3.1.7 gives $2x = c_2$ and 3.1.8 gives $c_3 = 0$

$$\therefore x = u = 2xe^t$$

$$\Rightarrow u = x\left(2e^t - 1\right)$$

illustrating an explosive solution as t → ∞.

But if f(x) = 1 = u(x,0)

3.1.7 → $x + 1 = c_2$

3.1.8 → $1 - x^2 = c_3 \Rightarrow x^2 = 1 - c_3$.

But $C_2^2 = x^2 + 2x + 1 = x^2 + 2(x+1) - 1 = 1 - C_3 + 2(C_2) - 1$

i.e. $C_2^2 = 2C_2 - C_3$

$$\Rightarrow c_3 = 2c_2 - c_2^2$$

$$u^2 - x^2 = 2(x+u)e^{-t} - (x+u)^2 e^{-2t} = (u+x)(u-x)$$

$$\Rightarrow u - x = 2e^{-t} - (x+u)e^{-2t}$$

$$\Rightarrow u\left(1 + e^{-2t}\right) = 2e^{-t} - xe^{-2t} + x$$

$$= 2e^t + x\left(1 - e^{-2t}\right)$$

Multiplying through by e^t we get,

$$u\left(e^t + e^{-t}\right) = 2 + x\left(e^t - e^{-t}\right)$$

$$\therefore u = \frac{2}{e^t + e^{-t}} + x \tanh t$$

= sech t + x tanh t

which yield the same conclusion

Applications:

1. Consider the nonlinear initial value problem

$$\frac{du}{dt} + u\frac{du}{dx} = 0, \quad t > 0$$

$$with \quad u\left(x,0\right) = \begin{cases} \left(a^2 - x^2\right), & |x| < a \\ 0, & |x| \geq a \end{cases}$$

Equation of characteristics are given by

$$\frac{du}{dt} = 0, \qquad \frac{dx}{dt} = u \text{ along a characteristics.}$$

Thus,

$$u = \left(x,t\right) = u\left(x\right) = a^2 - \xi^2, \quad |x| < a$$

$$x = \xi + tu\left(\xi\right) = \xi + t\left(a^2 - \xi^2\right)$$

From which,

$$\xi = \frac{1}{2t}[1 \pm \sqrt{1 - 4t\left(x - ta^2\right)}], \quad t \neq 0$$

Therefore, the required solution is

$$u\left(x,t\right) = \begin{cases} a^2 - \xi^2 \quad \frac{1}{2t^2}[2xt \pm \sqrt{\left(1 - 4xt + 4a^2t^2\right)}], & t \neq 0, |\xi| < a \\ 0, & |\xi| > 0 \end{cases}$$

For a = 1, $\xi = x - t(1 - \xi^2)$

When $\xi = 0$, $x = t$

$\xi = 0.5, \quad x = \dfrac{8 + 12t}{16} \quad or\ 0.5 + 0.75t$

$\xi = 0.75, \quad x = \dfrac{12 + 7t}{16} \quad or\ 0.75 + 0.4375t$

$\xi = 1.0, \quad x = 1$

$\xi = 1.5, \quad x = \dfrac{24 - 20t}{16} \quad or\ 1.5 - 1.25t$

$\xi = 1.75, \quad x = \dfrac{28 - 33t}{16} \quad or\ 1.75 - 2.0625t$

$\xi = 2, \quad x = 2 - 3t$

for which we obtain the following characteristics curves (Figure 8).

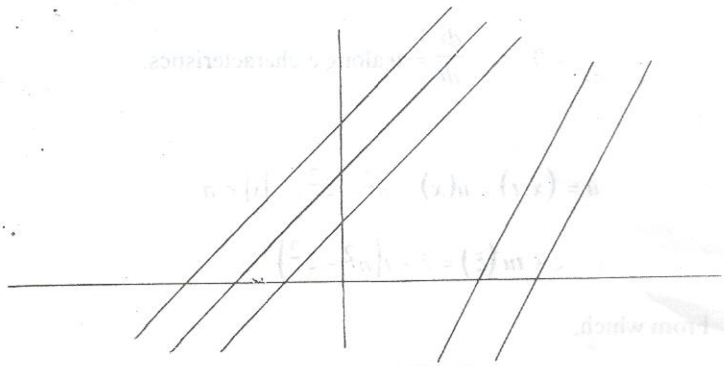


Figure 8. Characteristic curves for the equation
$\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 0$ with $u(x,0) = (a^2 - x^2)$ for $|x| < a$ and $= 0$ for $|x| \geq a$

2. Consider the nonlinear initial problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad t > 0, x \in R$$

$$with\ u(x,0) = x^2, \quad x \in R$$

The equation of characteristics are

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0} = 0$$

Thus,

$$\frac{dx}{dt} = u, \qquad \frac{du}{dt} = 0$$

i.e. $u = u(x) = \xi^2$ along the characteristic line originating from $x = \xi$

$$x = A + t\xi^2$$

$$x = \xi \quad when\ t = 0 \quad \therefore \quad x = \xi + t\xi^2$$

For which

$$\xi - = \frac{1}{2t}[-1 \pm \sqrt{1 + 4xt}], \qquad t \neq 0.$$

$$\xi^2 = \frac{1}{4t^2}[(1 + 2xt) \pm 2\sqrt{1 + 4xt}]$$

$$= \frac{1}{2t^2}[(1 + 2xt) \pm \sqrt{1 + 4xt}], \qquad t \neq 0.$$

$$\therefore U(x,t) = \frac{1}{2t^2}[(1 + 2xt) \pm \sqrt{1 + 4xt}]$$

The initial condition $\lim_{t \to o} u(x\ \ t) = u(x) = x^2$ must be satisfied. With positive sign, this limit does not exist.

$$\therefore u(x.t) = \frac{1}{2t^2}[(1 + 2xt) - \sqrt{1 + 4xt}], \qquad t \neq 0.$$

This solution is valid if $4xt + 1 > 0 \quad or \quad xt > -\frac{1}{4}$ (see Figure 9 below)

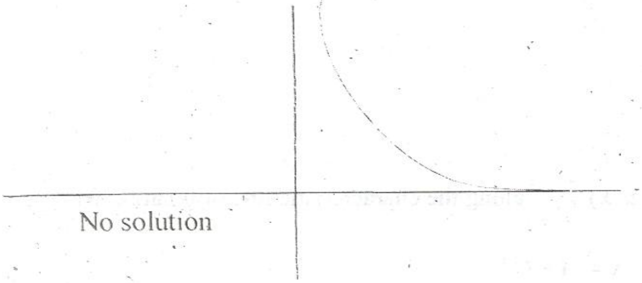


Figure 9. Characteristic curves for the equation. $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$ *with* $u(x,0) = x^2$

Wave Equation with Dissipation

Consider the equation with dissipation whose solution gives the symmetric graph below (Figure 11)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + k\frac{\partial^2 u}{\partial x^2} = 0 = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}[u^2] + k\frac{\partial^2 u}{\partial x^2} = 0 \qquad (4.11)$$

$$\theta = x - u_0 t$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial \theta}$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \theta^2}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial t} = -u_0\frac{\partial}{\partial \theta}$$

$$-u_0\frac{\partial}{\partial \theta}u + \frac{1}{2}\frac{\partial}{\partial \theta}u^2 + k\frac{\partial^2 u}{\partial \theta^2} = 0$$

Integrating, we have

$$-u_0 u + \frac{1}{2}u^2 + k\frac{\partial u}{\partial \theta} = 0$$

$$k\frac{\partial u}{\partial \theta} = u(u_0 - \frac{u}{2}) = \frac{u}{2}(2u_0 - u)$$

$$2k\frac{\partial u}{\partial \theta} = u(2u_0 - u) \; or \; \frac{du}{u(2u_0 - u)} = \frac{d\theta}{\cdot 2k}$$

$$\frac{1}{u(2u_0 - u)} = \frac{A}{u} + \frac{B}{2u_0 - u} \; or \; 1 = A(2u_0 - u) + Bu$$

$$u = 2u_0 \rightarrow 1 = 2u_0 B \; or \; B = \frac{1}{2u_0}$$

$$u = 0 \rightarrow A = \frac{1}{2u_0}$$

$$\therefore \frac{d\theta}{2k} = \frac{1}{2u_0}[\frac{1}{u} + \frac{1}{2u_0 - u}]du \; or \; \frac{\theta}{k} = \frac{1}{u_0}[In\, u - In(2u_0 - u)]$$

$$\frac{1}{u_0} In\,(\frac{u}{2u_0 - u}) = \frac{\theta}{k}$$

$$\frac{R_0 u}{2u_0 - u} = e^{\frac{\theta u_o}{k}} = k_0$$

$$R_0 u = k_0[2u_0 - u]$$

i.e. $(R_0 + k_0)u = 2k_0 u_0$

$$u = \frac{2k_0 U_0}{R_0 + k_0} = \frac{2k_0 u_0}{R_0 + e^{\frac{\theta u_0}{k}}} = \frac{2u_o e^{\frac{\theta u_0}{k}}}{R_0 + e^{\frac{\theta u_0}{k}}}$$

$$= \frac{2U_0}{1 + R_0 e^{\frac{\theta u_0}{k}}} \begin{cases} \rightarrow 2U_0 \quad as\ \theta \rightarrow \infty \\ \rightarrow \frac{2U_0}{1 + R_0} \; as\ k \rightarrow \infty \quad i.e.\ \text{infinity dissipative coefficient} \end{cases}$$

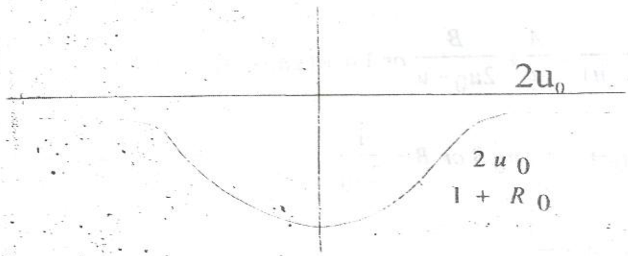


Figure 10. Graph for the solution of wave equations with dissipation.

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