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ON THE PROGRESSIVE OSCILLATIONS IN A SOLID WITH INTERNAL FRICTION

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SUMMARY

A theoretical consideration is presented. It concerns the progressive waves significantly modified by the dissipation induced by internal friction in the solid. The frictional term is consequently introduced in the formulation through a generalization of the elastic constants. These constants are then assumed to be proportional to the n[th] power of the frequency related to the dominant mode of the solid.

In the foregoing consideration, the crucial term through which all parameters depend is the Poisson's ratio υ. This ratio is also expressed in terms of the same viscous parameters associated with the solid.

Calculations were performed using a range of positive values of n. The case $n = 0$ corresponds to that of purely elastic solid. For $n \geq 1$, the range of values within which υ gives progressive wave solution shrinks with increasing n. Interestingly, this result completely agrees with that which we obtained by using sturm's theorem method.

1. INTRODUCTION

In general, an unbounded elastic solid supports two types of propagating wave forms. These are pressure wave (P – wave) and shear wave (S – wave). However, if the solid is bounded and semi-infinite with free surface, it supports an additional propagating wave called Rayleigh surface wave (R – wave) named after Lord Rayleigh who formulated the wave form (1885).

The respective velocity of each of the P -, S - and R – wave is distinct and a function of the parameters characterising the medium. In turn, these parameters depend strongly on the depth below the free-surface. Incidentally, these waves are not frequency dependent and consequently are non-dispersive.

However, the equation governing the propagation of Rayleigh wave type was firstly derived by Brownwich (Newland 1954) for the simple case of perfectly elastic solid. Later, the evolutional processes associated with the wave propagation in the infinite plane homogeneous solid were reformulated. As usual, P – and S – waves were present. However, with some elements of absorption introduced in the formulation, a dispersive Rayleigh wave was similarly observed. It is then obvious that the absorption induced by internal friction essentially is closely linked with the dispersive wave propagation

phenomenon in a solid. This development appears quite a promising one; for in the process of seismic waves propagating through solid materials (for example, Earth), they are attenuated leading to the energy loss. Essentially, this is the result of geometrical enlargement of the wave fronts
following the intrinsic material absorption in the solid.

2. THE OPERATIONS OF INTERNAL WAVE DISSIPATION.

This consideration is pertinent because, to realistically model wave trains as recorded with seismometer in the far field, the entire process whereby the seismic signals arrive the recording station should be taken into consideration. Consequently, it is not altogether realistic to treat the propagation of seismic surface waves as if they are completely governed by the equations of linear elasticity.

Consequently, Okeke (1985) introduced a damping term into the equations of the elastodynamics. This term was designed to be proportional to the time rate of change of the displacement of the solid materials. Even though the end results of the theory were quite reasonable, its generalisation (Okeke 1972; Newlands, 1958), appeared more promising from the physical consideration.

In the generalised approach, the Lame's elastic parameter λ and μ are replaced by $\lambda + \lambda^1 \partial/\partial t$ and $\mu + \mu^1 \partial/\partial t$ respectively in the equation for the equilibrium of the solid. λ^1 and μ^1 are the parameters for the internal frictions. Newlands (1958), consequently, introduced the following:

$$\Omega_\alpha = \frac{\rho\alpha^2}{\lambda^1 + 2\mu^1} = \rho\alpha^2 / \left(S_n G_n |\omega|^n\right),$$

$$\Omega_\beta = \rho\beta^2 / \mu^1 = \rho\beta^2 / \left(G_n |\omega|^n\right)$$

where α and β are respectively the speed of P – and S – wave defined by $\rho\alpha^2 = (\lambda + 2\mu)$, $\rho\beta^2 = \mu$, ρ is the density of the solid. ω is the dominant mode frequency of the same solid; n is an integer which will be determined during the subsequent calculations. S_n and G_n are constants which are n dependent such that Ω_α and Ω_β are dimensionally the same with ω.

3. THE EQUATION FOR RAYLEIGH WAVE SPEED

We introduce the following symbols:

$$R_\alpha = 1 + i\omega/\Omega_\alpha, \quad R_\beta = 1 + i\omega/\Omega_\beta \tag{3.1}$$

using equation (3.1), the Rayleigh wave speed C satisfies the equation

$$\left(2 - C^2/C_\beta^2\right)^2 = 4\left[\left(1 - C^2/C_\alpha^2\right)\left(1 - C^2/C_\beta^2\right)\right]^{1/2} \tag{3.2}$$

$$\text{where } C_\beta = \beta R_\beta, \quad C_\alpha = \alpha R_\alpha \tag{3.3}$$

(3.3) now defines the speed of the pressure and shear waves in dissipative medium.

Square both sides of (3.2), and rearrange, then

$$\left(2-C^2\big/C_\beta^2\right)^4-16\left(1-C^2\big/C_\alpha^2\right)\left(1-C^2\big/C_\beta^2\right)=0 \qquad (3.4)$$

Also the following symbols are introduced:

$$\zeta_\beta=C^2\big/C_\beta^2,\quad C^2\big/C_\alpha^2=P\zeta_\beta,\qquad q=1-\frac{C_\beta^2}{C_\alpha^2}=1-P=\frac{1}{2(1-\nu)}$$

Thus, $P=\dfrac{1-2\nu}{2(1-\nu)}$, υ is the Poisson ratio.

(3.4) takes the form

$$(2-\zeta_\beta)^4-16(1-P\zeta_\beta)(1-\zeta_\beta)=0 \qquad (3.5)$$

Thus, from (3.5)

$$[1+(1-\zeta_\beta)]^4=1+4(1-\zeta_\beta)+6(1-\zeta_\beta)^2+4(1-\zeta_\beta)^3+(1-\zeta_\beta)^4=16(1-P\zeta_\beta)(1-\zeta_\beta)$$

(3.5) gives finally,

$$(1-\zeta_\beta)^4+4(1-\zeta_\beta)^3+6(1-\zeta_\beta)^2+(16P\zeta_\beta-12)(1-\zeta_\beta)+1=0 \qquad (3.6)$$

let $\zeta_\beta = 1 - f$; then,

$$f^4+4f^3+6f^2-16Pf^2+16f+1=0 \qquad (3.7)$$

i.e. $f^4+4f^3+2(3-8P)f^2+16f+1=0$ (3.8)

Finally, (3.8) reduces to the following standard form

$$Af^4+4bf^3-6cf^2+4df+e=0 \qquad (3.9)$$

Where a = b = e = 1, $c=1-\dfrac{8P}{3}$, d = 4P, P = P(υ).

4. CUBIC EQUATION FOR (3.9)

$$\text{If } K_1=\frac{1}{2}[4+9c+8P] \qquad (4.0)$$

$$K_2=\frac{1}{8}\left[167+94.5c+238P+24cP+9c^2+16P^2\right] \qquad (4.1)$$

then,

$$f^3-\tfrac{3}{2}f^2+K_1f-K_2=0 \qquad (4.2)$$

The standard form of the cubic equation (4.2) does not usually contain f^2, thus, take

$$H = \frac{K_1}{3} - \frac{1}{4}(3+c)^2 \tag{4.3}$$

$$G = \frac{1}{2}(c+3)K_1 - K_2 - \frac{1}{4}(3+c)^3 \tag{4.4}$$

Equation (4.2) reduces to

$$f^3 + 3Hf + G = 0 \tag{4.5}$$

The three roots of (4.5) is related to the four roots of (3.9) through (4.3) and (4.4). Further details concerning such relationship are explained in any textbook of elementary algebra such as Archbold (1960).

As functions of P, we compute the following representations:

$$K_1(P) = \frac{1}{2}(13 - 16P) \tag{4.6}$$

$$K_2(P) = \frac{1}{8}(270.5 - 60P + 75P^2) \tag{4.7}$$

$$H(P) = -\frac{1}{6}\left(11 - 16P + \frac{32}{3}P^2\right) \tag{4.8}$$

$$G(P) = 4.7P^3 - 12.975P^2 + 9.1P - 41.8 \tag{4.9}$$

To study (4.5), we use the definition

$$V(f^3) = f^3 g(f - H/f) = f^6 + Gf^3 - H^3 = 0 \tag{4.10}$$

where $g(f) = f^3 + 3Hf + G = 0$ (4.11)

(4.10) is a quadratic equation in f^3. If f_1 and f_2 are the roots, then

$f_1 + f_2 = -G$, $f_1 f_2 = -H^3$. Thus, the solutions are

$$f_1 = \frac{1}{2}\left(-G + \sqrt{G^2 + 4H^3}\right), \qquad f_2 = \frac{1}{2}\left(-G - \sqrt{G^2 + 4H^3}\right) \tag{4.12}$$

The roots of (4.6) are now

$f_1 + f_2$, $wf_1 + w^2 f_2$, $w^2 f_1 + wf_2$,

where $w = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$, $w^2 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ are the two of the cube roots of unity.

If the end result of this study is to model a free and self induced surface wave of Rayleigh type, then, in equation (4.5), the quantity $G^2 + 4H^3 > 0$. In this case, f_1 and f_2 are real and one root of (4.2) is correspondingly real. This is the

Rayleigh wave phase speed. Thus, in a perfectly elastic solid, this property is satisfied if $-1 < \upsilon < \frac{1}{2}$ corresponding to $\frac{1}{4} < q < \frac{5}{6}$ (Hudson, 1980) and $0 < P < \frac{3}{4}$ in this study. These conclusions are also correct if n = 0 indicating the complete absence of viscosity. If n = 1, $-0.82 < \upsilon < .42$ corresponding to $.14 < P < .73$, if n = 2, then $\upsilon \in [-0.63, 0.33]$ and $P \in [0.25, 0.69]$. If n = 5, then, this interval is vanishingly small. Hence the larger n is, the smaller is the relevant values of υ. Thus, the analysis suggests that the growth and the decay of the surface elastic waves depend on the viscosity of the solid.

5. THE APPLICATION OF THE STURM'S FUNCTIONS

In equation (4.12), $G^2 + 4H^3 > 0$ in the following consideration: $\upsilon \in (-0.82, 0.42)$ for which n = 1 and $\upsilon \in (-0.63, 0.33)$ for which n = 0; $\omega = \frac{2\pi}{T}$, T = 8 seconds. This data is indeed within the range associated with seismic activities. Consequently, these developments suggest that (4.2) has, at least, a real root. Within this frame work, therefore, wave propagation in a possibility. However, there is still an element of probability in this conclusion. Thus, for further clarification of this case, we apply Sturm's theorem to equation (3.6). The associated functions are derived from (3.6) as follows:

$$F(\zeta_\beta) = \zeta_\beta^4 + 8\zeta_\beta^3 + 8(3-2P)\zeta_\beta^2 - (28-16P)\zeta_\beta + 12 \quad (5.1)$$

$$F_1(\zeta_\beta) = \zeta_\beta^3 + 6\zeta_\beta^2 + 4(3-2P)\zeta_\beta + (4P-7) \quad (5.2)$$

$$F_2(\zeta_\beta) = 3\zeta_\beta^2 - 3\zeta_\beta + 1 \quad (5.3)$$

$$F_3(\zeta_\beta) = (53-24P)\zeta_\beta - (37-12P) \quad (5.4)$$

$$F_4(\zeta_\beta) = (53-24P) - 12(4-3P)(37-12P) \quad (5.5)$$

ω/Ω_β	-2	-1	0	1	2
$f(\zeta_\beta)$	+	-	-	+	+
$f_1(\zeta_\beta)$	-	+	-	+	+
$f_2(\zeta_\beta)$	+	+	+	+	+
N	2	1	1	0	0

Table 1, n = 2

The number N shown in table 1 gives the number of changes of signs in the sequence of Sturm's functions (equations 5.1 to 5.5) for assigned values of the ratio ω/Ω_β. In particular, the essential property of the function $F_2(\zeta_\beta)$ is that it has no real root and hence remains invariably of the same sign for all real values of ζ_β. Thus, the functions $F_3(\zeta_\beta)$ and $F_4(\zeta_\beta)$ will not be involved in the subsequent deductions.

The analysis of Sturm's functions as shown in table I appears to confirm that there is one real root of $F(\zeta_\beta) = 0$ between 0 and 1, another root between – 2 and – 1; whilst the other two are complex. The root between 0 and 1 is of seismological value in that it is related to the Rayleigh dispersive wave in agreement with the preceeding results. Thus, the result obtained from the reducing cubic equation (4.5) agrees with that of the biquadratic equation (3.6) from which it is derived.

Identical calculations corresponding to n = 3, 4, 5 give similar results with that of n = 2. As n becomes very large, all the roots of $F(\zeta_\beta) = 0$ are found to be complex with large imaginary parts. This suggest the existence of a large but finite n for which all progressive oscillation in a solid are annihilated by viscous forces. This development will be of current interest by which we shall use the related Maple file as in Enns and Mcguire (2000).

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